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High-Speed Cylindrical Collapse of Type-I Matter

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Abstract In this paper, the cylindrical symmetric gravitational collapse with anisotropic pressure has been investigated using high-speed approximation scheme. The collapsing speed of the fluid is assumed to be very large. To see the effects of pressure, we have used the equations $\sqrt{p_R/\rho} = k$ and $\sqrt{p_T/\rho} = w$ of states for radial pressure and tangential pressure, respectively. It is observed that if the ratios of both pressures, that is, tangential and radial pressures, to energy density are bounded from below by some positive value, there arise two possibilities depending on whether $1 + k^2 - 2w^2 > 0$ or $1 + k^2 - 2w^2 < 0$. For $1 + k^2 - 2w^2 > 0$, the high-speed approximation scheme fails, while for $1+k^2-2w^2<0$, the high-speed approximation works. For vanishing w and k, the high-speed scheme does not break down, and, as a result, a naked singularity forms in this case. For $p_T = p_R = p$, all the results reduce to the perfect fluid case obtained by Nakao and Morisawa (Prog Theor Phys 113:73, 2005).

Keywords High-speed · Gravitational collapse

1 Introduction

The investigation of whether the spacetime singularity formed by the gravitational collapse will be visible to local or distant observers or not is one of the open problems in general relativity. The visible singularity is called naked, while the invisible one is called a black hole. This issue has attracted much attention of researchers after the formulation of the singularity theorems of Hawking and Penrose [2–4], which predict that the singularity is a generic property of spacetime in general relativity. However, these theorems do not specify the nature of the singularity, i.e., whether it will be a naked one or a black hole.

In 1969, Penrose [5] hypothesized the cosmic censorship conjecture about the visibility of spacetime singularity. According to this conjecture, the spacetime singularity cannot be naked. That is, the outcome of gravitational collapse must be a black hole. This conjecture is unproved yet. The existing work on gravitational collapse contains both types of singularities, depending upon the choice of initial data [6–15]. The cosmic censorship conjecture hypothesis provides inspiration for a detailed study of dynamically developing gravitational collapse models to get a correct form of the cosmic censorship.

Most of the work on gravitational collapse has been done by considering the spherically symmetric dust fluid model (see [6–15]). The advantage of this model is that Einstein field equations can be solved easily for exact solutions. The dust fluid is a restricted assumption because the effects of pressure cannot always be ignored in the formation of spacetime singularity. Thus, it is important to discuss this issue by including pressure. Spherically symmetric systems have no degree of gravitational radiation. Therefore, nonspherical perturbations to these systems have been considered in the study of generation of gravitational radiation [16–20]. The study of cylindrically symmetric gravitational collapse

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is important because it has a degree of gravitational radiation, and also a naked spacetime singularity exists in this system [21, 22].

Some numerical work on cylindrical gravitational collapse is also available [23–25]. But the results are inconsistent [26]. In any case, this work provides strong evidence for the generation of gravitational radiation. There also exists the work on cylindrical gravitational collapse which improves our understanding on nonspherical relativistic dynamics [27–32]. Nakao and Morisawa [1, 26] introduced an approximation scheme called high-speed approximation scheme to study the gravitational collapse. In this approximation scheme, the collapsing speed is assumed almost equal to the speed of light. Using this scheme, they investigated cylindrical dust collapse and cylindrical perfect fluid collapse with some interesting results.

The analysis of naked singularity formation by considering type-I matter and spherically symmetric system has been done by many researchers. Dwivedi and Joshi [33] investigated the spherical gravitational collapse of general type-I matter by the local analysis and showed that the initial data are crucial for whether the naked singularity forms at the symmetric center or not. Goswami and Joshi [34] extended this work to higher dimensional spacetimes. Various researchers have also investigated the spherically symmetric collapse with vanishing radial pressure and nonvanishing tangential pressure [35–40]. Herrera and Santos [41] have investigated cylindrical gravitational collapse with anisotropic pressure using matching conditions. Ahmad et al. [42] investigated high-speed spherical collapse of type-I matter with some interesting results. Motivated by these studies, we investigate the cylindrical gravitational collapse with anisotropic pressure using high-speed approximation in this paper. This work extends the work done by Nakao and Morisawa [1] for isotropic pressure. The aim of this work is to see the differences among the results when the pressure is anisotropic. It is verified that our results reduce to the perfect fluid case as obtained by Nakao and Morisawa [1] for $p_T = p_R = p$.

The plan of the paper is as follows. In Section 1, we write the Einstein field equations for the cylindrically symmetric spacetime with anisotropic pressure. The null dust solution is given in Section 2. Section 3 is devoted to the high-speed approximation scheme. The effects of pressures on the high-speed gravitational collapse are discussed in Section 4. Finally, the summary of the results is given in Section 5.

In this paper, the geometrized units in which $c = G = 1$ and the convention and notations in the textbook by Wald [43] are adopted.

2 Cylindrically Symmetric Systems with Anisotropic Pressure

We consider cylindrically symmetric spacetimes with whole-cylinder symmetry. The metric for such spacetime is given by [44]

$$ds^2 = e^{2(\gamma-\psi)}\left(-dt^2 + dr^2\right) + e^{2\psi}dz^2 + e^{-2\psi}R^2 d\varphi^2, \quad (2.1)$$

where γ, ψ, and R are functions of t and r. Then, Einstein equations are

$$\gamma' = \left(R'^2 - \dot{R}^2\right)^{-1}\left\{RR'\left(\dot{\psi}^2 + \psi'^2\right) - 2R\dot{R}\dot{\psi}\psi' + R'R'' - \dot{R}\dot{R}' - 8\pi G\sqrt{-g}\left(R'T^t_t + \dot{R}T^r_t\right)\right\}, \quad (2.2)$$

$$\dot{\gamma} = -\left(R'^2 - \dot{R}^2\right)^{-1}\left\{R\dot{R}\left(\dot{\psi}^2 + \psi'^2\right) - 2RR'\dot{\psi}\psi' + \dot{R}R'' - R'\dot{R}' - 8\pi G\sqrt{-g}\left(\dot{R}T^t_t + R'T^r_t\right)\right\}, \quad (2.3)$$

$$\dot{\gamma} - \gamma'' = \psi'^2 - \dot{\psi}^2 - \frac{8\pi G}{R}\sqrt{-g}T^\varphi_\varphi, \quad (2.4)$$

$$\ddot{R} - R'' = -8\pi G\sqrt{-g}\left(T^t_t + T^r_r\right), \quad (2.5)$$

$$\ddot{\psi} + \frac{\dot{R}}{R}\dot{\psi} - \psi'' - \frac{R'}{R}\psi' = -\frac{4\pi G}{R}\sqrt{-g}\left(T^t_t + T^r_r - T^z_z + T^\varphi_\varphi\right). \quad (2.6)$$

Here 'a dot' represents differentiation with respect to t and 'a prime' represents differentiation with respect to r.

We consider the energy–momentum tensor for the type-I matter defined in [45] by

$$T^\mu_\nu = \rho u^\mu u_\nu + p_R s^\mu s_\nu + p_T \Omega^\mu_\nu, \quad (2.7)$$

where

$$\Omega^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu - s^\mu s_\nu, \quad (2.8)$$

where ρ, p_R, p_T, u^μ, and s^μ are energy density, radial and tangential pressures, four-velocity vector, and radial vector, respectively. Here, we consider the matter field, with p_R, p_T, and ρ positive, to satisfy the dominant energy condition, i.e., $\rho \geq p_T$, $\rho \geq p_R$. One of the reasons to take this matter field is that it includes most of the known physical forms of matter, like dust, perfect fluid, etc. For $p_R = p_T = 0$, it reduces to dust matter and for $p_R = p_T$, it takes the form of perfect fluid matter. Therefore, it is more general when compared to the perfect fluid case. The four-velocity vector can be taken as

$$u^{\mu} = N\,(1, -1+V, 0, 0)\,, \tag{2.9}$$

and the four-velocity radial vector is

$$s_{\mu} = N\,(1-V, 1, 0, 0)\,, \tag{2.10}$$

where

$$N = \frac{e^{-(\gamma-\psi)}}{\sqrt{V\,(2-V)}}. \tag{2.11}$$

We introduce three new variables D, P_R, and P_T as

$$D = \frac{\sqrt{-g}N\,(\rho+p_R)}{\sqrt{V\,(2-V)}} = \frac{e^{(\gamma-\psi)}}{V\,(2-V)}R\,(\rho+p_R)\,, \tag{2.12}$$

$$P_R = \frac{\sqrt{-g}Np_R}{\sqrt{V\,(2-V)}} = \frac{e^{(\gamma-\psi)}R\;p_R}{V\,(2-V)}, \tag{2.13}$$

$$P_T = \frac{\sqrt{-g}Np_T}{\sqrt{V\,(2-V)}} = \frac{e^{(\gamma-\psi)}R\;\;p_T}{V\,(2-V)}. \tag{2.14}$$

For time like u^{μ}, V should be positive. Using (2.12)–(2.14), the energy–momentum tensor (2.7) takes the form

$$T^{\mu}_{\nu} = \frac{e^{-3(\gamma-\psi)}}{R}\left[Dk^{\mu}k_{\nu} + Ve^{2(\gamma-\psi)}\,(2-V)\right.$$
$$\left.\times\left\{P_R\delta^{\mu}_{\nu} + (P_T-P_R)\,\Omega^{\mu}_{\nu}\right\}\right], \tag{2.15}$$

where

$$k^{\mu} = \frac{u^{\mu}}{N} = (1, -1+V, 0, 0)\,, \tag{2.16}$$

and its associated tensor is

$$k_{\nu} = \frac{u_{\nu}}{N} = e^{2(\gamma-\psi)}\,(-1, -1+V, 0, 0)\,. \tag{2.17}$$

The nonzero components of energy–momentum tensor are given by

$$\sqrt{-g}T^{t}_{t} = e^{\gamma-\psi}\left[-D + V\,(2-V)\,P_T - (P_T-P_R)\right.$$
$$\left.\times\left\{1-(1-V)^2\,e^{-4(\gamma-\psi)}\right\}\right], \tag{2.18}$$

$$\sqrt{-g}T^{t}_{r} = e^{\gamma-\psi}\,(1-V)\left[-D-(P_T-P_R)\left\{1-e^{-4(\gamma-\psi)}\right\}\right]$$
$$= -\sqrt{-g}T^{r}_{t}, \tag{2.19}$$

$$\sqrt{-g}T^{r}_{r} = e^{\gamma-\psi}\left[D\,(1-V)^2 + V\,(2-V)\,P_T + (P_T-P_R)\right.$$
$$\left.\times\left\{(1-V)^2 - e^{-4(\gamma-\psi)}\right\}\right], \tag{2.20}$$

$$\sqrt{-g}T^{z}_{z} = e^{\gamma-\psi}\,V\,(2-V)\,P_T = \sqrt{-g}T^{\varphi}_{\varphi}, \tag{2.21}$$

where g is the determinant of the metric (2.1).

The law of conservation of energy–momentum

$$T^{\mu\nu}_{\;\;;\,\nu} = 0, \tag{2.22}$$

becomes

$$\partial_u D = -\frac{1}{2}\,(DV)' + \frac{1}{2}\left[V\,(2-V)\,P_T - (P_T-P_R)\right.$$
$$\left.\times\left\{1-(1-V)\,e^{-4(\gamma-\psi)}\right\}\right]^{\cdot}$$
$$+\left[(1-V)\,(P_T-P_R)\left(1-e^{-4(\gamma-\psi)}\right)\right]'$$
$$+\frac{V}{2}\,(2-V)\,P_T\left(\dot{\psi}-\dot{\gamma}-\frac{\dot{R}}{R}\right) + \frac{D}{2}\,(1-V)$$
$$\times\left[2\partial_u\,(\gamma-\psi) - V\left(\dot{\gamma}-\dot{\psi}\right)\right] + \frac{(P_T-P_R)}{2}$$
$$\times\left[e^{-4(\gamma-\psi)}\left\{2\partial_u\,(\gamma-\psi) + V\left(\gamma'-\psi'\right)\right\}\right.$$
$$\left.+\,(V-1)\left\{2\partial_u\,(\gamma-\psi) - V\left(\dot{\gamma}-\dot{\psi}\right)\right\}\right], \tag{2.23}$$

$$D\partial_u V = (1-V)\,\partial_u D - \frac{D}{2}\left\{2\partial_u\,(\psi-\gamma) - V\left(\dot{\gamma}-\dot{\psi}\right)\right\}$$
$$+\left\{(1-V)\,(P_T-P_R)\left(1-e^{-4(\gamma-\psi)}\right)\right\}^{\cdot}$$
$$+\frac{1}{2}\left[V\,(1-V)\,D - V\,(2-V)\,P_T\right.$$
$$\left.+\,(P_T-P_R)\left\{(1-V)^2 - e^{-4(\gamma-\psi)}\right\}\right]'$$
$$+\frac{1}{2}P_T V(2-V)\left(\gamma'-\psi'+\frac{R'}{R}\right) + \frac{(P_T-P_R)}{2}$$
$$\times\left[2\partial_v\,(\gamma-\psi) - e^{-4(\gamma-\psi)}\left\{2\partial_v\,(\gamma-\psi)\right.\right.$$
$$\left.-\,2\partial_u\,(\gamma-\psi)\,V + V\,(1-V)\left(\gamma'-\psi'\right)\right\}$$
$$\left.-\,V\left(\dot{\gamma}-\dot{\psi}\right)\right], \tag{2.24}$$

where $u = t - r$ is retarded time and $v = t + r$ is advanced time.

To observe the energy flux of gravitational radiation, we take C-energy and the energy flux vector defined in [44]. The C-energy $E = E\,(t, r)$ is given by

$$E = \frac{1}{8}\left\{1 + e^{-2\gamma}\left(R^{\cdot 2} - R'^{2}\right)\right\}, \tag{2.25}$$

and the energy flux is given by

$$\sqrt{-g}J^{\mu} = \frac{1}{2\pi G}\left(\frac{\partial E}{\partial r}, -\frac{\partial E}{\partial t}, 0, 0\right). \tag{2.26}$$

The nonvanishing components of the energy flux are given by

$$\sqrt{-g}J^{t} = \frac{e^{-2\gamma}}{8\pi G}\left\{RR'\left(\dot{\psi}^2 + \psi'^2\right) - 2R\dot{R}\dot{\psi}\psi' - 8\pi G\sqrt{-g}\left(R'T^t_t + \dot{R}T^r_t\right)\right\}, \tag{2.27}$$

and

$$\sqrt{-g}J^{r} = \frac{e^{-2\gamma}}{8\pi G}\left\{R\dot{R}\left(\dot{\psi}^2 + \psi'^2\right) - 2RR'\dot{\psi}\psi' - 8\pi G\sqrt{-g}\left(R'T^r_t - \dot{R}T^r_r\right)\right\}. \tag{2.28}$$

3 Null Dust Solution

Here, we find the solution of the field equations in the limit $V \to 0$. In this limit keeping $D,\ P_R,\ P_T$ finite, the energy–momentum tensor (2.15) becomes

$$T^{\mu}_{\nu} = \frac{De^{-3(\gamma-\psi)}}{R}k^{\mu}k_{\nu}, \tag{3.1}$$

where

$$k^{\mu} = (1, -1, 0, 0), \tag{3.2}$$

is the null vector. This shows that for the case of very large collapsing velocity, i.e., for $V \approx 0$, the system under consideration is approximated by a null dust system. For this purpose, we treat the deviation V of the four-velocity from null as a perturbation and apply linear perturbation method.

The solution for collapsing null dust is easily obtained as

$$\psi = 0, \tag{3.3}$$

$$R = r, \tag{3.4}$$

$$\gamma = \gamma_B(v), \tag{3.5}$$

$$8\pi DGe^{\gamma} = \frac{d\gamma_B}{dv}, \tag{3.6}$$

where $\gamma_B(v)$ is an arbitrary function. This solution was first given by Morgan [27]. Later on, it was studied by Letelier and Wang [29] and Nolan [30]. We take this solution as a background solution for perturbation analysis. It is obvious from (3.1) and (3.4) that the energy–momentum tensor diverges at the symmetry axis $r = 0$ if D does not vanish simultaneously. This gives rise to a naked singularity.

4 High-speed Approximation Scheme

We introduce a small parameter ϵ and assume that both V and ψ are of $O(\epsilon)$. Then , we write $\gamma,\ R$ and D as

$$e^{\gamma} = e^{\gamma_B}\left(1 + \delta_{\gamma}\right), \tag{4.1}$$

$$R = r\left(1 + \delta_R\right), \tag{4.2}$$

$$D = D_B\left(1 + \delta_D\right), \tag{4.3}$$

where $\delta_{\gamma},\ \delta_R,\ \delta_D$ are of $O(\epsilon)$ and

$$D_B = \frac{1}{8\pi Ge^{\gamma_B}}\frac{d\gamma_B}{dv}. \tag{4.4}$$

Using the equations above, (2.2)–(2.6) and (2.23) up to $O(\epsilon)$ can be written as

$$\delta'_{\gamma} = (r\delta_R)'' + 8\pi GD_Be^{\gamma_B} \times\left\{\delta_{\gamma} - \psi + \delta_D - 2\frac{P_TV}{D_B} - 2\partial_v(r\delta_R) + V(P_T - P_R)\left(1 + 4e^{-4\gamma_B}\right)\right\}, \tag{4.5}$$

$$\dot{\delta}_{\gamma} = \left(r\dot{\delta_R}\right)^{\prime} + 8\pi GD_Be^{\gamma_B} \times\Big[\delta_{\gamma} - \psi + \delta_D - 2\partial_v(r\delta_R) - (P_T - P_R) \times\left\{2\left(1 - 4e^{-4\gamma_B}\right)\partial_u(r\delta_R) - (1 - V)\left(1 - 4e^{-\gamma_B}\right) + 4e^{-4\gamma_B}\left(\delta_{\gamma} + \psi\right)\right\}\Big], \tag{4.6}$$

$$\ddot{\delta}_{\gamma} - \delta''_{\gamma} = -\frac{16\pi G}{r}e^{\gamma_B}VP_T, \tag{4.7}$$

$$r\ddot{\delta}_R - (r\delta_R)'' = 16\pi GVe^{\gamma_B} \times\left\{\left(D_B - 2P_T\right) + \left(P_T - P_R\right)\left(1 - e^{-4\gamma_B}\right)\right\}, \tag{4.8}$$

$$\ddot{\psi} - \psi'' - \frac{1}{r}\psi' = 8\pi GVe^{\gamma_B} \times\left\{\left(D_B - 2P_T\right) + \left(P_T - P_R\right)\left(1 - e^{-4\gamma_B}\right)\right\}, \tag{4.9}$$

$$\partial_u(\delta_D+\delta_\gamma-\psi)$$
$$=-\frac{1}{2D_B}\frac{\mathrm{d}D_B}{\mathrm{d}v}V-\frac{1}{2}\left(V'-\frac{\mathrm{d}\gamma_B}{\mathrm{d}v}V\right)$$
$$+\frac{1}{D_B}\left\{(P_TV)^{\cdot}-P_TV\frac{\mathrm{d}\gamma_B}{\mathrm{d}v}\right\}$$
$$+\frac{1}{2D_B}(P_T-P_R)\left[\left\{e^{-4\gamma_B}\left(2\partial_u(\delta_\gamma-\psi)+V\frac{\mathrm{d}\gamma_B}{\mathrm{d}v}\right)\right.\right.$$
$$\left.\left.-2\partial_u(\delta_\gamma-\psi)+V\frac{\mathrm{d}\gamma_B}{\mathrm{d}v}2\partial_u(\delta_\gamma-\psi)\right\}+V\frac{\mathrm{d}\gamma_B}{\mathrm{d}v}\right]. \tag{4.10}$$

Using (4.10), the (2.24) up to $O(\epsilon)$ becomes

$$\partial_u\{(D_B-2P_T)V\}=\frac{P_TV}{r}+(P_T-P_R)e^{-4\gamma_B}\partial_v(\delta_\gamma-\psi). \tag{4.11}$$

The energy flux (2.25) up to $O(\epsilon)$ is given by the equality

$$E=\frac{1}{8}\left[1-e^{-2\gamma_B}+2e^{-2\gamma_B}\left\{\delta_\gamma-(r\delta_R)'\right\}\right]. \tag{4.12}$$

Equation (4.4) shows that γ_B is constant in the vacuum region, i.e., for $D_B=0$. Therefore, in the vacuum region, (4.5)–(4.6) become

$$\delta_\gamma'=(r\delta_R)'', \tag{4.13}$$

or

$$\delta_\gamma-(r\delta_R)'=\text{constant}. \tag{4.14}$$

Thus, in the vacuum region, the energy flux E, given in (4.12), is constant up to $O(\epsilon)$. Therefore, up to $O(\epsilon)$, the energy flux vector J^μ vanishes. To see the generation of gravitational waves, from (2.27)–(2.28), the components of energy flux up to $O(\epsilon^2)$, can be written as

$$\sqrt{-g}J^t=\frac{r}{8\pi G}\left(\dot{\psi}^2+\psi'^2\right), \tag{4.15}$$
$$\sqrt{-g}J^r=-\frac{r}{4\pi G}\dot{\psi}\psi'. \tag{4.16}$$

respectively. It is worth mentioning here that all the results reduce to the perfect fluid case [1] for $p_T=p_R=p$.

5 Effects of Pressure on High-Speed Collapse

To see the effects of pressure on high-speed collapse, we use the equations of state for both radial and tangential pressures as

$$\sqrt{\frac{p_R}{\rho}}=k\,(t,r), \tag{5.1}$$
$$\sqrt{\frac{p_T}{\rho}}=w\,(t,r). \tag{5.2}$$

Here, p_T, p_R, and ρ are positive. The dominant energy condition $\rho\geq p_T, \rho\geq p_R$ implies that $k\leq 1$, $w\leq 1$.

It is not easy to solve (4.11) for V in terms of k and w because of the term $\partial_v(\delta_\gamma-\psi)$. We are interested in seeing the behavior of the perturbation velocity V near the background singularity $r=0$. For naked singularity formation, the perturbation velocity V should be finite at the background singularity $r=0$. Therefore, we assume that

$$\partial_v(\delta_\gamma-\psi)=0, \tag{5.3}$$

so that we are left with

$$\partial_u\{(D_B-2P_T)V\}=\frac{P_TV}{r}. \tag{5.4}$$

From (2.12), (2.14), (5.1), and (5.2), it follows that

$$P_T=\frac{w^2}{1+k^2}D. \tag{5.5}$$

Using (5.5) in (5.4), we get

$$\partial_u\left[\left(1-\frac{2w^2}{1+k^2}\right)V\right]=\frac{w^2}{\left(1+k^2\right)r}V. \tag{5.6}$$

The above equation can be rewritten as

$$\partial_u\ln\left[\left(\frac{1+k^2-2w^2}{1+k^2}\right)V\right]=\frac{w^2}{\left(1+k^2-2w^2\right)r}. \tag{5.7}$$

Integrating this equation, we get

$$V=C(v)\,\frac{1+k^2}{1+k^2-2w^2}\exp\left[\int_{U(v)}^{u}\frac{2w^2(x,v)}{\{1+k^2(x,v)-2w^2(x,v)\}(v-x)}\mathrm{d}x\right], \tag{5.8}$$

Here, $C(v)$ has the same support as the background density D_B and $U(v)$ is an arbitrary function of advanced time v. The last equation implies that for $k=1, w=1$, the velocity perturbation V is not defined. Thus, the high-speed approximation is not applicable to this case. This result is consistent with [1]. We examine further the following cases.

5.1 Bounded Case

If w and k are bounded from below by some positive values b_1 and b_2, respectively, i.e.,

$$w \geq b_1, \tag{5.9}$$

$$k \geq b_2, \tag{5.10}$$

then these inequalities yield two cases:

$$\frac{2w^2}{1+k^2-2w^2} = {m_1}^2, \tag{5.11}$$

$$\frac{2w^2}{1+k^2-2w^2} = -{m_2}^2, \tag{5.12}$$

depending upon $1+k^2-2w^2 > 0$ and $1+k^2-2w^2 < 0$, respectively. Here, m_1 and m_2 are nonvanishing constants.

Then, from (5.8) and (5.11), we get

$$V = \text{Const} \times \exp\left({m_1}^2 \int \frac{1}{v-x} \mathrm{d}x\right), \tag{5.13}$$

or

$$V = \text{Const} \times r^{-{m_1}^2}. \tag{5.14}$$

This equation shows that the perturbed value of V diverges as the fluid element approaches to symmetry axis, i.e., at $r = 0$, $V \to \infty$. Thus, high-speed fails for this case. This result is consistent with [1].

For the second case (5.12), from (5.8), we get

$$V = \text{Const} \times r^{{m_2}^2}. \tag{5.15}$$

The above equation shows that the perturbed value of V remains finite and has value zero as fluid element approaches to symmetry axis, i.e., at $r = 0$. Thus, the high-speed approximation is valid for this case. This is not interesting because, for naked singularity formation, the perturbed value of V should be positive. This result is different from [1]. It is worth mentioning here that for $k = w$, $1+k^2-2w^2 > 0$ and therefore, the second case does not exist.

5.2 Vanishing Case

Here, we consider the case when w and k vanishes for the high energy density limit, i.e., $\rho \to \infty$ (since $\rho \geq p_T, \rho \geq p_R$). The energy density will become larger and larger when the fluid elements approach the background singularity, i.e., $r = 0$, (the region of $D_B \neq 0$). For this case, we consider asymptotic expansions of w and k as

$$w^2(u,v) \approx W^2(t)\, r^i = W^2(v-r)\, r^i \approx W^2(v)\, r^i, \tag{5.16}$$

$$k^2(u,v) \approx K^2(t)\, r^j = K^2(v-r)\, r^j \approx K^2(v)\, r^j. \tag{5.17}$$

Here, W, K are arbitrary functions and i, j are two positive constants. Using these asymptotic expansions in (5.7), we get

$$V \approx C(v) \frac{1+K^2(v)\, r^j}{1+K^2(v)\, r^j - W^2(v)\, r^i} \exp \times \left[\int_{U(v)}^{u} \frac{2W^2(v)\, r^i}{\{1+K^2(v)\, r^j - 2W^2(v)\, r^i\}(v-x)} \mathrm{d}x\right]. \tag{5.18}$$

For $r = 0$, $V \approx C(v)$. This shows that the perturbed value of velocity remains finite in this case. Thus, a naked singularity forms at the background singularity.

5.3 Generation of gravitational waves

Equation (4.9) shows that the larger value of velocity perturbation increases the potential function ψ. In the asymptotically flat region, the C-Energy flux is given in (4.16). This implies that the large value of velocity perturbation leads to a large amount of gravitational waves. This is similar to the perfect fluid case [1].

6 Summary and Discussion

Many researchers [33–41] have investigated gravitational collapse with anisotropic pressure. In this paper, we have investigated the cylindrically symmetric gravitational collapse with type-I matter, using high-speed approximation scheme. This work is an extension of the work done by Nakao and Morisawa [1] for perfect fluid to type-I matter. Type-I matter includes most of the known physical forms of matter like dust, perfect fluid, etc. For $p_R = p_T = 0$, it reduces to dust matter, and for $p_R = p_T$, it takes the form of perfect fluid matter. Therefore, it is more general when compared to perfect fluid case. To examine the effects of both tangential and radial pressures, we have assumed energy density and both pressures to be positive. We have considered equation of state for both radial and tangential pressures as $p_R = k^2\rho$ and $p_T = w^2\rho$. The detail of the results obtained are as follows.

In our analysis, we have assumed $\partial_v(\delta_\gamma - \psi) = 0$. That is, $\delta_\gamma - \psi$ is function of u only. For $k = w = 1$, the perturbation velocity V is not defined, and the high-speed approximation is not possible. Furthermore, we have examined the following two cases: bounded and vanishing cases of k and w. The bounded case is further divided into two cases, depending upon $1+k^2-2w^2>0$ or $1+k^2-2w^2<0$. For case $1+k^2-2w^2>0$, the high-speed approximation scheme fails and for case $1+k^2-2w^2<0$, the high-speed approximation works but the perturbed value of V has value zero as the fluid element approaches the symmetry axis, i.e., at $r=0$. This is not interesting because for naked singularity formation, the perturbed value of V should be positive. This result is different from [1]. It is worth mentioning here that for $k=w$, $1+k^2-2w^2>0$ and, therefore, the second case does not appear. For the vanishing case, it is demonstrated that as energy density $\rho\to\infty$, the high-speed collapse works well until a naked singularity forms.

Equation (4.9) shows that a large velocity perturbation V implies a large amount of gravitational radiation. These results are consistent with the perfect fluid case [1]. These results are the same as obtained by Nakao and Morisawa [1]. The difference is that these results are valid with the assumption $\partial_v(\delta_\gamma-\psi)=0$, while in [1], these results are valid without that assumption. Additional work will be necessary to investigate (4.11) without taking that assumption. This may provide different results.

It is verified that for $p_T=p_R=p$, all the mathematical results reduce to the perfect fluid case [1]. Thus, this work is a generalization of [1].

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References

1. K. Nakao, Y. Morisawa, Prog. Theor. Phys. **113**, 73 (2005)
2. R. Penrose, Phys. Rev. Lett. **14**, 57 (1965)
3. S.W. Hawking, Proc. R. Soc. Lond. **A300**, 187 (1967)
4. S.W. Hawking, R. Penrose, Proc. R. Soc. Lond. **A314**, 529 (1970)
5. R. Penrose, Nuovo Cimento **1**, 252 (1965)
6. D.M. Eardley, L. Smarr, Phys. Rev. **D19**, 2239 (1979)
7. D. Christodoulou, Commun. Math. Phys. **93** 171 (1984)
8. R.P.A.C. Newman, Class. Quantum Grav. **3**, 527 (1986)
9. B. Waugh, K. Lake, Phys. Rev. **D38**, 1315 (1988)
10. J.P.S. Lemos, Phys. Lett. **A158**, 271 (1991)
11. I.H. Dwivedi, P.S. Joshi, Class. Quantum Grav. **9**, L69 (1992)
12. P.S. Joshi, T.P. Singh, Gen. Relat. Gravit. **27**, 921 (1995)
13. P.S. Joshi, T.P. Singh, Phys. Rev. **D51**, 6778 (1995)
14. P.S. Joshi, I.H. Dwivedi, Phys. Rev. **D47**, 5357 (1993)
15. P.S. Joshi, *Gravitational Collapse and Spacetime Singularities* (Cambridge University Press, 2008).
16. H. Iguchi, T. Harada, K. Nakao, Phys. Rev. **D57**, 7262 (1998)
17. H. Iguchi, T. Harada, K. Nakao, Prog. Theor. Phys. **101**, 1235 (1999)
18. H. Iguchi, T. Harada, K. Nakao, Prog. Theor. Phys. **103**, 53 (2000)
19. K. Nakao, H. Iguchi, T. Harada, Phys. Rev. **D63**, 084003 (2001)
20. T. Harada, H. Iguchi, K. Nakao, Prog. Theor. Phys. **107**, 449 (2002)
21. K.S. Thorne, in *Magic Without Magic*; John Archibald Wheeler, ed. by J. Klauder (Frieman, San Francisco, 1972), p. 231
22. S.A. Hayward, Class. Quantum Grav. **17**, 1749 (2000)
23. T. Piran, Phys. Rev. Lett. **41**, 1085 (1978)
24. F. Echeverria, Phys. Rev. **D47**, 2271 (1993)
25. T. Chiba, Prog. Theor. Phys. **95**, 321 (1996)
26. K. Nakao, Y. Morisawa, Class. Quantum Grav. **21**, 2101 (2004)
27. T.A. Morgan, Gen. Relat. Gravit. **4**, 273 (1973)
28. T. Apostolatos, K.S. Thorne, Phys. Rev. **D46**, 2435 (1992)
29. P.S. Letelier, A. Wang, Phys. Rev. **D49**, 5105 (1994)
30. B.C. Nolan, Phys. Rev. **D65**, 104006 (2002)
31. P.R.C.T. Pereira, A. Wang, Phys. Rev. **D62**, 124001 (2000)
32. B.C. Nolan, L.V. Nolan, Calss. Quantum Grav. **21**, 3693 (2004)
33. I.H. Dwivedi, P.S. Joshi, Commun. Math. Phys. **166**, 117 (1994)
34. R. Goswami, P.S. Joshi, Phys. Rev. D **76**, 084026 (2007)
35. G. Magli, Class. Quantum Grav. **14**, 1937 (1997); G. Magli, Class. Quantum Grav. **15**, 3215 (1998)
36. S.M.C.V. Goncalves, S. Jhingan, G. Magli, Phys. Rev. D **65**, 064011 (2002)
37. T. Harada, H. Iguchi, K. Nakao, Phys. Rev. D **58**, 041502 (1998)
38. T. Harada, K. Nakao, H. Iguchi, Class. Quantum Grav. **16**, 2785 (1999)
39. R. Goswami, P.S. Joshi, Class. Quantum Grav. **19**, 5229 (2002)
40. A. Mahajan, T. Harada, P.S. Joshi, K. Nakao, Prog. Theor. Phys. **118**, 865 (2007)
41. L. Herrera, N.O. Santos, Class. Quantum Grav. **22**, 2407 (2005)
42. Z. Ahmad, T. Harada, K. Nakao, M. Sharif, Class. Quantum Grav. **26**, 035007 (2009)
43. R.M. Wald, *General Relativity* (The University of Chicago Press, 1984)
44. K.S. Thorne, Phys. Rev. **138**, B251 (1965)
45. S.W. Hawking, G.F.R. Ellis, *The Large ScaleStructure of Spacetime* (Cambridge University Press, Cambridge, 1973)