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luizno.bjp@gmail.com

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Neutron Stars in Teleparallel Gravity

S. C. Ulhoa · P. M. M. Rocha

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Abstract In this paper, we deal with neutron stars, which are described by a perfect fluid model, in the context of the teleparallel equivalent of general relativity. We use numerical computations (based on the RNS code) to find the relation between the angular momentum of the field and the angular momentum of the source. One such relation was established for each stable star resulting from the numerical computation with an equation of state, the central energy density, and the ratio between the polar and equatorial radii as input, the central energy density and the ratio between polar and equatorial radii. We also find a regime in which the relation between gravitational angular momentum and moment of inertia (as well as the angular velocity of the fluid) is linear. We give the spatial distribution of the gravitational energy and show that it depends linearly on the squared angular velocity of the source.

Keywords Teleparallelism · Torsion tensor · Neutron stars · Astrophysics

1 Introduction

Most of the known stars are accurately described by Newtonian physics; such compact objects as neutron stars with masses of the order of one solar mass, by contrast, are not. The gravitational field of such objects is so strong that only general relativity is able to explain their properties [1]. Given that, due to the very nature of Einstein's equations, analytical methods are very difficult to implement, most interesting results come from nonalgebraic methods. In particular, to deal with rotating neutron stars, one usually resorts to numerical procedures [2, 3].

While the existence of such exotic objects was theoretically predicted in the 1930s [4, 5], the first observations took place in the 1960s [6–8]. In neutron stars, the mechanism preventing gravitational collapse is a repulsive interaction due to the quantum nature of the particles in the star, which compensates the gravitational pressure, so that neutron stars exist in very compact spatial volumes. Neutron stars with masses around one solar mass and radii around 10 km are known [9], as are pulsars rotating with periods as low as a millisecond and very high surface magnetic fields [10, 11]. These very unusual conditions make them perfect candidates to test any new predictions of a gravitational theory.

We intend to treat rapidly rotating neutron stars from the viewpoint of teleparallel gravity. From the dynamical point of view, teleparallel gravity and general relativity make identical predictions. On the other hand, teleparallel gravity allows for the definition of physically interesting quantities, such as the well-behaved gravitational field energy-momentum and angular momentum tensors [12].

In general relativity, the definitions of these quantities are still under development. The first attempt to define gravitational energy was based on pseudo-tensors that were

S. C. Ulhoa (✉)
Faculdade Gama, Universidade de Brasília, Setor Leste (Gama),
72444-240 Brasília, Federal District, Brazil
e-mail: sc.ulhoa@gmail.com; ulhoa@ualberta.ca

S. C. Ulhoa · P. M. M. Rocha
Theoretical Physics Institute, Physics Department,
University of Alberta, T6G 2J1 Alberta, Canada

P. M. M. Rocha
Instituto de Física, Universidade de Brasília, 70910-900 Brasília,
Federal District, Brazil
e-mail: paulomarciano@gmail.com; marciano@ualberta.ca

not invariant under coordinate transformations. This effort was followed by the Komar integrals [13] and then by the formalism known as the ADM formulation [14], which is based on the (3+1)-dimensional Hamiltonian formulation of general relativity, an approach using constraints to define energy, momentum, and angular momentum. The difficulty with this formalism is that such quantities are only well-defined asymptotically, at the spatial infinity. In addition, no expression obtained in the realm of general relativity is dependent on the reference frame—clearly a feature that is undesirable for energy, momentum, and angular momentum.

The expressions for the energy-momentum and angular momentum of the gravitational field, in the context of the teleparallelism equivalent to general relativity (TEGR), are invariant under transformations of the coordinates of the three-dimensional space-like surface. They are also dependent on the frame of reference, as one would expect. Over the years, they have been consistently applied to many different systems [15–19].

The frame dependence is an expected condition for any expression due to the field since, in special relativity, the energy of a particle for a stationary observer is m , but it is γm for an observer under a Lorentz boost, where $c = 1$ and γ is the Lorentz factor. There is no reason to abandon this feature when dealing with the gravitational field. Similar considerations apply to the momentum and angular momentum.

Our goal is to establish a relation between the angular momentum of the gravitational field, which is only predicted by the teleparallel theory of gravity, and the angular momentum, the angular velocity of the source, and the moment of inertia. We hope to identify the significance of the field quantities when related to matter quantities. In particular, it is known that in the final stage of the coalescence between two black holes in a binary system, the remnant black hole will acquire linear momentum [20, 21]. As a matter of fact, the field has to gain the exact amount needed to cancel it in order to preserve momentum conservation [22]. This example highlights the importance of knowing the relation between the field and source. We also intend to describe the behavior of the energy over the entire space-time and to relate it to other features of neutron stars.

This paper is organized as follows. In Section 2, we briefly discuss certain ideas in teleparallel gravity. In Section 3, we describe the space-time of a neutron star and fix the frame that will be used to compare features only due to the gravitational field with others due to the source. In Section 3.1, we discuss the field angular momentum and its relation to certain features of the source. In Section 3.2, we relate the gravitational energy to the angular velocity of the fluid. Plots of the numerical results are presented to describe the distribution of gravitational

energy in space. Finally, the last section contains our concluding remarks.

In our notation, space-time indices μ, ν, \dots and $\text{SO}(3,1)$ indices a, b, \dots run from 0 to 3. Time and space indices are indicated as $\mu = 0, i, \quad a = (0), (i)$. The metric tensor $g_{\mu\nu}$ raises and lowers space-time indices, while the Minkowski metric, $\eta^{ab} = \text{diag}(-+++)$, acts on $\text{SO}(3,1)$ indices. The tetrad field is denoted by $e^a{}_\mu$, and the determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$. Unless otherwise stated, we adopt units such that $G = c = 1$.

2 Teleparallelism Equivalent to General Relativity

Teleparallel gravity has been investigated over the years as an alternative to the theory of general relativity (GR). Such a theory is entirely equivalent to GR as far as the field equations are concerned. Both theories are derived from Lagrangians that only differ by a total divergence and, therefore, have the same field equations. The main advantage of TEGR over GR is allowing us to construct consistent expressions for the energy, momentum, and angular momentum [12]. We will now present ideas and expressions developed in the realm of TEGR.

The dynamical variables of TEGR are the tetrad fields defined in the Weitzenböck space-time, also known as the Cartan space-time. In the Riemann space-time the dynamics is given by the metric tensor. The tetrad field and metric tensor are related by the equalities

$$g^{\mu\nu} = e^{a\mu} e_a{}^\nu. \quad (1)$$

Therefore, for each metric tensor, one can define infinitely many tetrads. It follows that the Weitzenböck geometry is less restrictive than the Riemann geometry. The arbitrariness in the choice of the tetrad field is only apparent, since it can be entirely determined by a more physical condition, namely the observer's reference frame.

As a matter of fact, the Weitzenböck and Riemann geometries are intrinsically related. Consider first the Cartan connection [24], $\Gamma_{\mu\lambda\nu} = e^a{}_\mu \partial_\lambda e_{av}$, defined in the Weitzenböck space-time. We can equally well write it as

$$\Gamma_{\mu\lambda\nu} = {}^0\Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu}, \quad (2)$$

where the ${}^0\Gamma_{\mu\lambda\nu}$ are the Christoffel symbols, and $K_{\mu\lambda\nu}$ is given by

$$K_{\mu\lambda\nu} = \frac{1}{2} (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}). \quad (3)$$

$K_{\mu\lambda\nu}$ is the contortion tensor defined in terms of the torsion tensor $T_{\mu\lambda\nu}$, which is derived from $\Gamma_{\mu\lambda\nu}$. Indeed, we have

$$T^a{}_{\lambda\nu} = \partial_\lambda e^a{}_\nu - \partial_\nu e^a{}_\lambda. \quad (4)$$

The tetrad field transforms Lorentz indices into space-time ones and vice versa, thus $T_{\mu\lambda\nu} = e_{a\mu} T^a{}_{\lambda\nu}$.

The curvature of the Cartan space-time vanishes identically, as it follows from substituting $\Gamma_{\mu\lambda\nu}$ in (2), which yields the following identity:

$$eR(e) \equiv -e \left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) + 2\partial_\mu (e T^\mu), \quad (5)$$

where $T^\mu = T^b{}_{b\mu}$, and $R(e)$ is the Riemannian scalar curvature constructed out of the tetrad field.

The density Lagrangian of GR, which leaves the field equations invariant under coordinate transformations, is precisely the left-hand side of (5). Given that any divergence term in a Lagrangian density makes no contribution to the field equations, we drop out the last term on the right-hand side of (5) and define the teleparallel Lagrangian density as

$$L(e_{a\mu}) = -\kappa e \left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - L_M \equiv -\kappa e \Sigma^{abc} T_{abc} - L_M, \quad (6)$$

where $\kappa = 1/(16\pi)$, L_M is the Lagrangian density of the matter fields, and Σ^{abc} is defined by the equality

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c), \quad (7)$$

with $T^a = e^a{}_\mu T^\mu$.

The equivalence between TEGR and GR is now easily understood. Since they have the same Lagrangian density, the two theories yield the same dynamics. This does not mean that the predictions are identical. For instance, there is no quantity equivalent to the gravitational energy-momentum vector, which will be defined below, in the realm of general relativity. In fact, every attempt to define the gravitational energy in the context of GR, from Einstein's equations, has failed. This approach has only led to pseudo-tensors in GR.

The field equations can be obtained from the variational differentiation of the Lagrangian density with respect to $e^{a\mu}$. After a few algebraic manipulations, the resulting equality reads

$$e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e \left(\Sigma^{bv}{}_a T_{bv\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd} \right) = \frac{1}{4\kappa} e T_{a\mu}, \quad (8)$$

where $T_{a\mu} = e_a{}^\lambda T_{\mu\lambda} = \frac{1}{e} \frac{\delta \mathcal{L}_M}{\delta e^{a\mu}}$ is the energy-momentum tensor of the matter fields.

Explicit calculations show that (8) is equivalent to the Einstein equations [25].

The field equations can be rewritten as

$$\partial_\nu (e \Sigma^{a\lambda\nu}) = \frac{1}{4\kappa} e e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}), \quad (9)$$

where $t^{\lambda\mu}$ is defined by the relation

$$t^{\lambda\mu} = \kappa \left(4 \Sigma^{bc\lambda} T_{bc}{}^\mu - g^{\lambda\mu} \Sigma^{bcd} T_{bcd} \right). \quad (10)$$

$\Sigma^{a\lambda\nu}$ is a skew-symmetric tensor in the last two indices. This symmetry leads to the following local conservation law

$$\partial_\lambda (e t^{a\lambda} + e T^{a\lambda}) = 0 \quad (11)$$

from which it is possible to write down the continuity equation

$$\begin{aligned} \frac{d}{dt} \int_V d^3x e e^a{}_\mu (t^{0\mu} + T^{0\mu}) \\ = - \oint_S dS_j \left[e e^a{}_\mu (t^{j\mu} + T^{j\mu}) \right]. \end{aligned}$$

We, therefore, interpret $t^{\lambda\mu}$ as the energy-momentum tensor of the gravitational field [26]. With this concept in mind, we can define the total energy-momentum vector in a three-dimensional spatial volume V in a familiar way, as

$$P^a = \int_V d^3x e e^a{}_\mu (t^{0\mu} + T^{0\mu}). \quad (12)$$

The above-defined energy-momentum vector is frame-dependent and -independent of the choice of coordinates. In order to analyze the features of gravitational field, it is therefore mandatory to set up the reference frame first. There are countless tetrad fields that match a given metric tensor. In other words, the same physical system can be seen under the optics of as many observers as desired.

In the Hamiltonian formulation of TEGR, the constraint equations are interpreted as energy, momentum, and angular momentum equations for the gravitational field [27]. The four-angular momentum of the gravitational field can be shown to have the expression

$$L^{ab} = 4\kappa \int_V d^3x e (\Sigma^{a0b} - \Sigma^{b0a}). \quad (13)$$

The P^a and L^{ab} obey the algebra of the Poincaré group [28]. The above definition, like (12), is coordinate independent and changes with the observer.

Although obtained in a Hamiltonian formulation, (13) is readily interpretable. The angular momentum is, of course, the vector product between the momentum and coordinate. Since the tetrad fields are the dynamical variables, it is possible to understand the meaning of (11) from the perspective of the Lagrangian formalism. The tetrad fields play the role of coordinates, when contracted with the total energy-momentum tensor, yielding the above equation.

As explained before, the choice of the tetrad field is not random. It is, in fact, intimately linked to the observer's frame. For a given metric tensor, an infinity of possible frames exists, each of which completely characterized by

the tetrad field. In order to fix the kinematical state of the observer in the three-dimensional space, we have to specify six components of the tetrad field, the other ten components being related to the metric tensor. To this end, we consider the acceleration tensor [29, 30]

$$\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}] , \quad (14)$$

here written in terms of the torsion tensor.

Given a set of tetrad fields, the translational acceleration of the frame along a world-line C then follows from $\phi_{(0)}^{(i)}$, and the angular velocity, from $\phi_{(i)(j)}$. Consequently, the acceleration tensor is a suitable candidate to geometrically describe an observer in space-time. It does not contain any dynamical features dependent on field equations and has been tested for teleparallel gravity in many situations [31–33].

3 The Space-Time of a Rotating Neutron Star

The most general form of the metric tensor describing the space-time generated by a configuration with axial symmetry is represented by the line element [34]

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2g_{03}d\phi dt , \quad (15)$$

where *all* metric components depend on r and θ : $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$. If g denotes the determinant of the metric tensor, we have that $\sqrt{-g} = [g_{11}g_{22}(g_{03}g_{03} - g_{00}g_{33})]^{1/2}$.

For a rotating neutron star, with arbitrary angular velocity about the z -axis, we have the following components of the metric [35, 36]:

$$\begin{aligned} g_{00} &= -\exp(\gamma + \rho) + r^2\omega^2 \sin^2 \theta \exp(\gamma - \rho) , \\ g_{03} &= -\omega r^2 \sin^2 \theta \exp(\gamma - \rho) , \\ g_{11} &= \exp 2\alpha , \\ g_{22} &= r^2 \exp 2\alpha , \\ g_{33} &= r^2 \sin^2 \theta \exp(\gamma - \rho) . \end{aligned} \quad (16)$$

Here, α , γ , ρ , and ω are metric potentials, which are functions of r and θ . The matter inside the neutron star being described by a perfect fluid, the energy-momentum tensor has the form

$$T^{\mu\nu} = (\epsilon + p)U^\mu U^\nu + pg^{\mu\nu} , \quad (17)$$

where ϵ , p , and U^μ are the energy density, the pressure, and the fluid four-velocity field, respectively [35]. Our assumption of axial symmetry makes U^μ proportional to the time and angular Killing vectors. Therefore, $U^\mu \propto (1, 0, 0, \Omega)$, where Ω is the angular velocity of the fluid, measured at infinity. In this model, to define the metric components in (16), we have to specify an equation of state (EOS) [37]

relating the energy density to the pressure and the angular velocity of the fluid.

In order to analyze an axisymmetrical space-time within TEGR, we choose a stationary observer, who has to satisfy $\phi_{ab} = 0$. Adapted to this frame, the tetrad field reads

$$e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & -C \\ 0 & \sqrt{g_{11}} \sin \theta \cos \phi & \sqrt{g_{22}} \cos \theta \cos \phi & -Dr \sin \theta \sin \phi \\ 0 & \sqrt{g_{11}} \sin \theta \sin \phi & \sqrt{g_{22}} \cos \theta \sin \phi & Dr \sin \theta \cos \phi \\ 0 & \sqrt{g_{11}} \cos \theta & -\sqrt{g_{22}} \sin \theta & 0 \end{pmatrix} . \quad (18)$$

The functions A , C , and D are given by the equalities

$$\begin{aligned} A &= (-g_{00})^{1/2} , \\ C &= -\frac{g_{03}}{(-g_{00})^{1/2}} , \\ D &= \left[\frac{-\delta}{(r^2 \sin^2 \theta)g_{00}} \right]^{1/2} , \end{aligned} \quad (19)$$

while δ is defined by the relation $\delta = g_{03}g_{03} - g_{00}g_{33}$.

We use a numerical method to solve the Einstein equations producing the metric tensor. Our purpose is to establish a relation between the features of the gravitational field, such as its angular momentum, and some experimentally measurable intrinsic attribute of the neutron star, such as the moment of inertia. Over the years, a convenient numerical method has been developed [38], improved [39, 40], and modified [41–45]. We use the RNS code, available at <http://www.gravity.phys.uwm.edu/rns/>, here modified to deal with such teleparallel quantities as the energy-momentum vector and gravitational angular momentum. Our calculations use the nondimensional quantities listed in (4)–(13) of Ref. [39] and references therein. Here, however, we use \sqrt{K} , with $K = c^4/G\epsilon_c$, as the fundamental length scale of the system, instead of $K'^{N/2}$, where K' is the polytropic constant, and N is related to the adiabatic index, which would be more suitable to deal with polytropic stars. This assumes the matter to have no meridional motion, and the angular velocity Ω , as seen by an observer at rest at spatial infinity, to be constant.

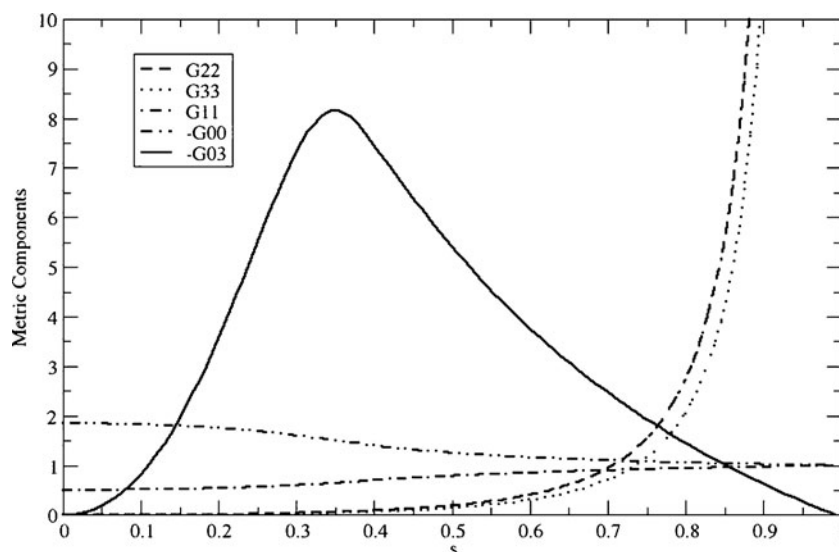
The program uses compact coordinates, μ and s , defined by the equality

$$\mu = \cos \theta , \quad r = R_e \left(\frac{s}{1-s} \right) , \quad (20)$$

where R_e is r at the equator of the star. Thus $s = 0.5$ represents a point on the equatorial surface of the star, while $s = 1$ represents spatial infinity.

Figure 1 depicts an illustrative example of the output of the code. We have chosen the central energy $\epsilon_c/c^2 = 10^{15} \text{ g/cm}^3$ and adopted the equation of state in Ref. [46], here denoted by the acronym EOSA (dense neutron matter). The metric components indicated in the figure recover the Minkowski space-time in spherical coordinates at spatial infinity.

Fig. 1 Example of the metric behavior for $\mu = 0$, using EOSA. Here, $G_{00} = g_{00}$, $G_{11} = g_{11}$, $G_{22} = g_{22}$, $G_{33} = g_{33}$, and $G_{03} = g_{03} \cdot 10^{-3}$



The angular momentum and energy–momentum of the gravitational field will be discussed in the following sections.

3.1 The Gravitational Angular Momentum of Neutron Stars

This section uses the results developed in Ref. [47]. Equation (40) of that paper, which relates the z -component of the gravitational angular momentum to the metric, reads

$$L^{(1)(2)} = -2k \oint_{s \rightarrow \infty} d\theta d\phi \left(\frac{g_{03} \sqrt{g_{22}} \sin \theta}{\sqrt{-g_{00}}} \right). \quad (21)$$

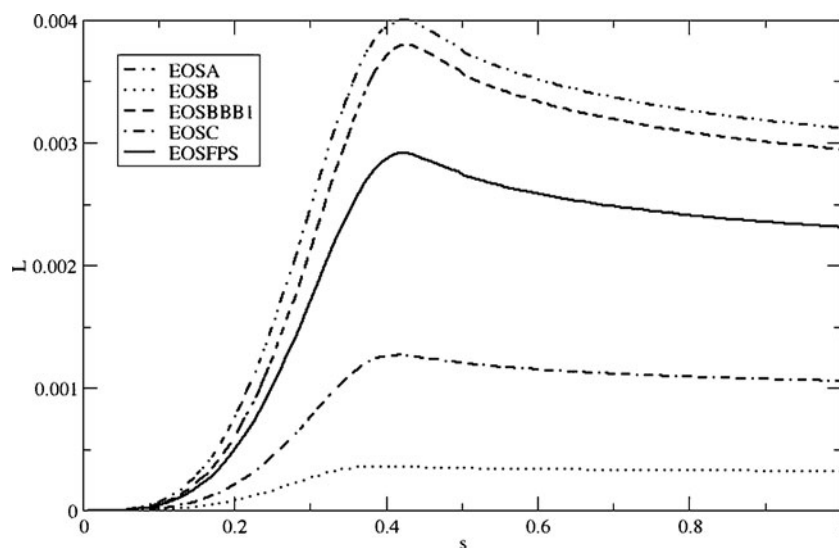
Equation (21) describes an axisymmetrical space-time, only the z -component of the angular momentum being nonzero, and the analysis was restricted to slowly, rigidly rotating neutron stars. Under these conditions, the angular momentum of the field was found to be proportional to

the (very small) angular momentum of the source. Given that these conditions are unrealistic, we now want to study rapidly rotating stars in more complex configurations with a view to determining how the field angular momentum depends on source features. We will use the shorthand L for $L^{(1)(2)}$, which will be dimensionless, unless otherwise stated. To transform back to the MKS system, all we have to do is to let $L \rightarrow \frac{c^3 K}{G} L$, where K is the fundamental length scale used in the numerical simulation.

We now have a computational tool to simulate rapidly rotating neutron stars and are able to investigate the behavior of the gravitational angular momentum. We will consider conditions that are more general than those in Ref. [47].

Figure 2 shows L as a function of s . The ratio between the polar and equatorial radii is 0.6. The curves peak approximately at the surface of the star, then decay smoothly as

Fig. 2 Spatial distributions of the gravitational angular momentum for the indicated equations of state. The central energy is $\varepsilon_c/c^2 = 10^{15} \text{ g/cm}^3$, and the equations of state are as follows: EOSA [46] (dense neutron matter), EOSB [48] (hyperonic matter), EOSBBB1 [49] (asymmetric nuclear matter, derived from the Brueckner–Bethe–Goldstone many-body theory), EOSC [50] (dense hyperonic matter), and EOSFPS [51] (neutron matter for an improved nuclear Hamiltonian)



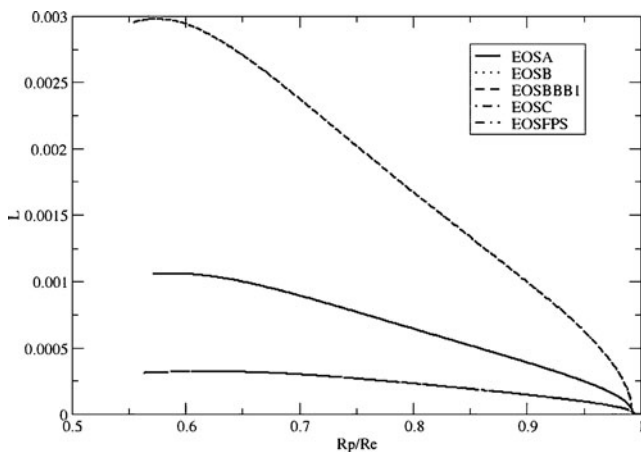


Fig. 3 Gravitational angular momentum as a function of the ratio between the polar and equatorial radii. The legend follows the convention of Fig. 2

$s \rightarrow 1$, a behavior in agreement with physical intuition, since we expect L to be maximum near the surface of the star.

The ratio between the polar radius (R_p) and the equatorial radius (R_e) defines the shape of the star. The larger this ratio, the smaller the gravitational angular momentum should be, since a unitary ratio, the highest possible value, corresponds to a spherical, static star. Figure 3 shows L as a function of the ratio R_p/R_e . The plots for EOSB, EOSC, and EOSFPS are nearly coincident. In the intermediate regime, with R_p/R_e neither too small nor too close to unity, L is nearly linearly related to R_p/R_e .

We are also interested in comparing the dependences of the field (L) and source (J) on the ratio R_p/R_e . We hence plot L/J as a function of R_p/R_e in Fig. 4. Again, the curves for EOSB, EOSC, and EOSFPS are superimposed, a feature

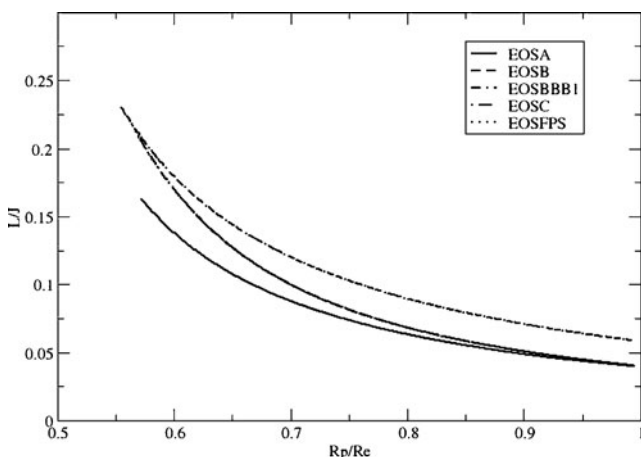


Fig. 4 Ratio between the gravitational angular momentum and the angular momentum of the source at the spatial infinity as a function of the ratio between the polar and equatorial radii. The legend follows the convention of Fig. 2

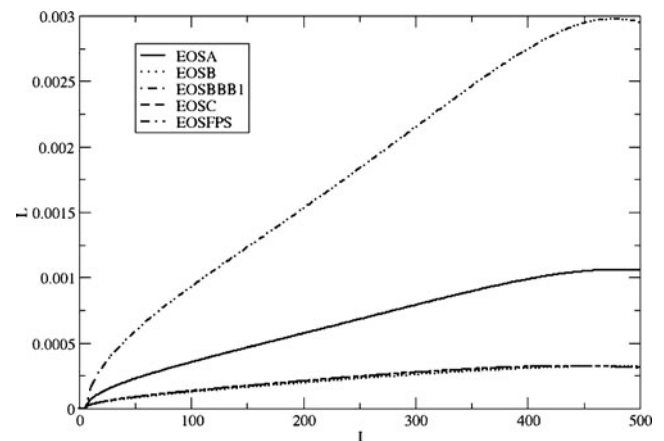


Fig. 5 Gravitational angular momentum as a function of the moment of inertia. The legend follows the convention of Fig. 2

also found in Figs. 5 and 6. Figure 4 shows that in rapidly rotating neutron stars, the gravitational angular momentum is comparable to the angular momentum of the source. The gravitational field can therefore be detected, since the angular momentum of the source is experimentally accessible. We have found that the ratio L/J is proportional to the inverse of R_p/R_e . For a given stable star, L is therefore proportional to $R_e/R_p J$.

Our computations make no reference to binary systems. Nonetheless, the conclusion that L and J have comparable magnitudes shows that the apparent nonconservation of angular momentum in such systems is not necessarily linked to the intensity of the emitted gravitational waves. While beyond the scope of this paper, additional investigation along this line of reasoning seems undoubtedly interesting.

Figures 5 and 6 present numerical results for the dependence of L on two other parameters of the star, the moment of inertia and the angular velocity of the fluid, respectively. Figure 5 shows that the gravitational angular momentum grows linearly with the moment of inertia, except at

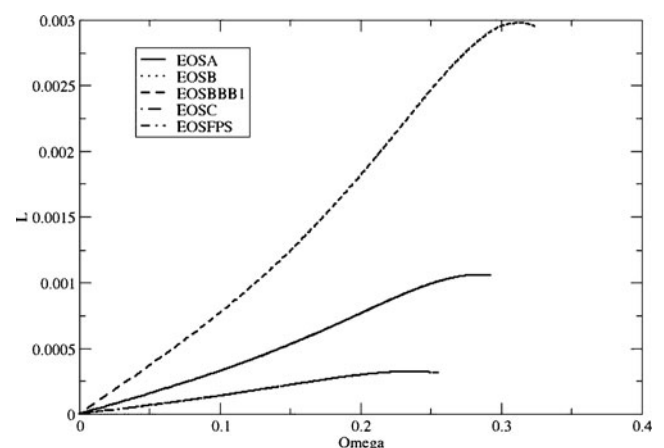


Fig. 6 Gravitational angular momentum as a function of the angular velocity of the fluid. The legend follows the convention of Fig. 2

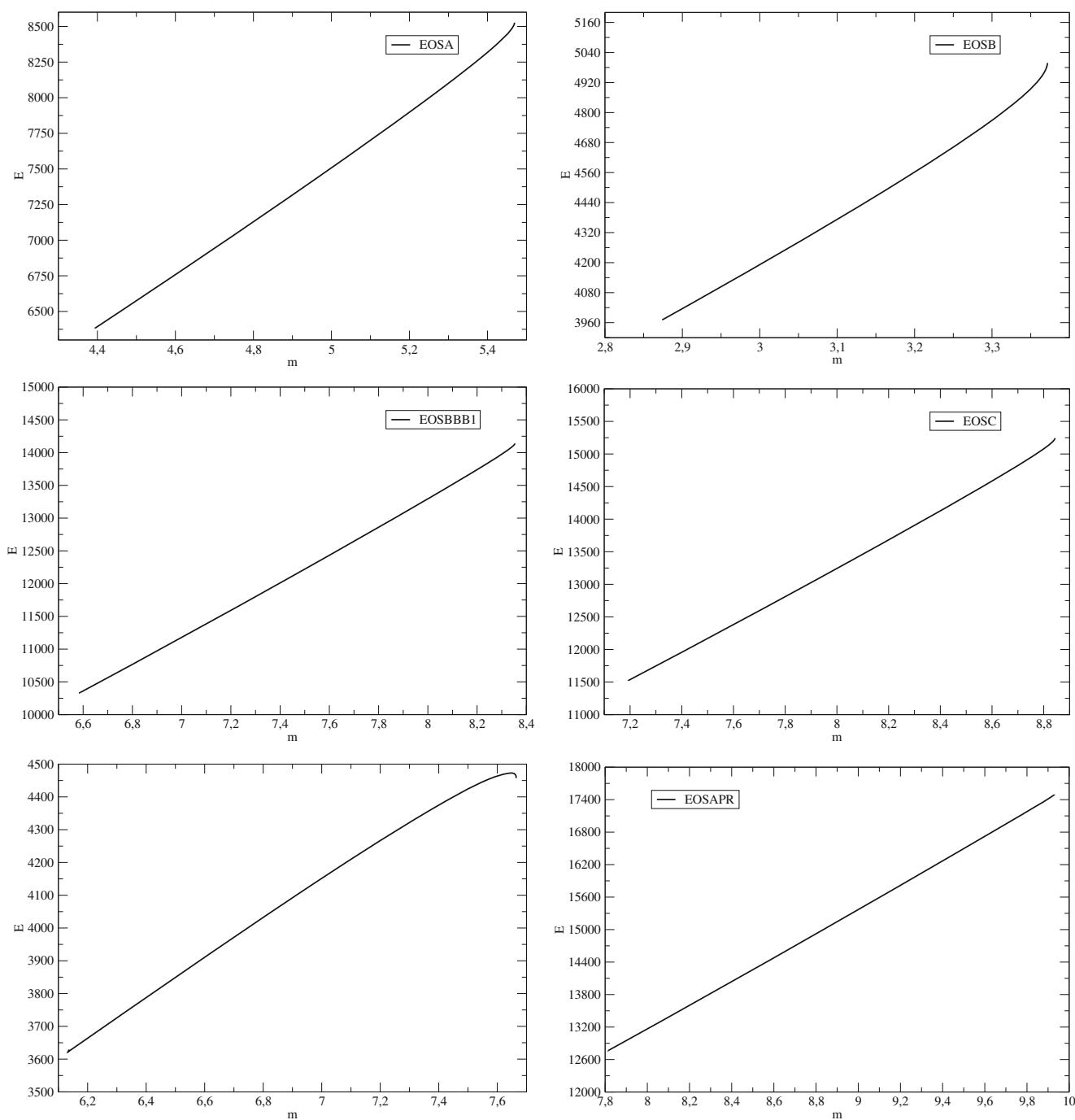


Fig. 7 Gravitational energy as a function of the total mass at the spatial infinity. In this figure $m = 10^{11} M$, where M is the total mass of the star. The last panel, labeled EOSAPR, has been constructed from the equation of state in Ref. [52], which includes relativistic corrections.

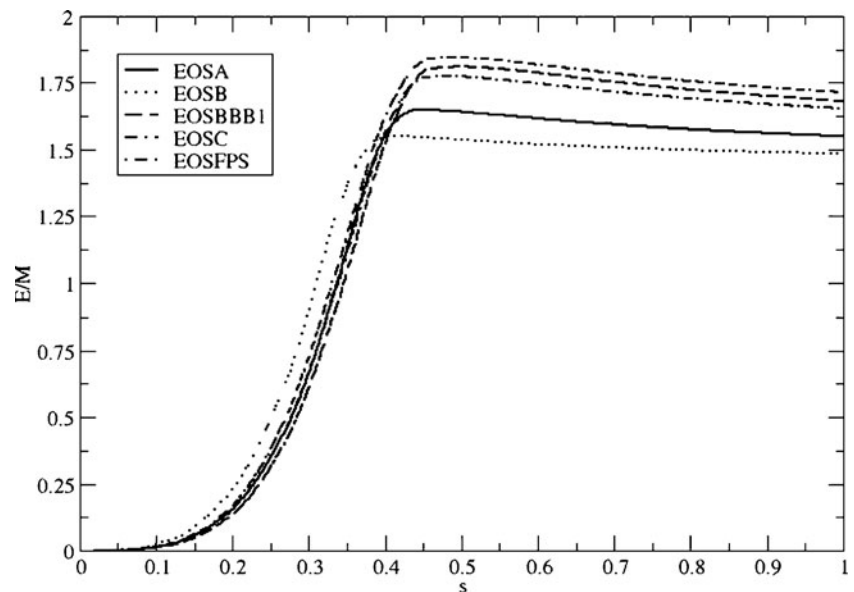
the edges of the figure. This is what we expect on intuitive grounds. The edges are nonlinear because they depict regions where I is maximally dependent on Ω .

In Fig. 6, by contrast, the relation between the gravitational angular momentum and the angular velocity of the fluid is nonlinear. This is also reasonable, since the angular momentum varies as the angular velocity of the source grows, because the rotation is not rigid.

3.2 The Energy-Momentum Vector of Neutron Stars

For completeness, we now present numerical results for the energy and momentum. Given that the metric is stationary, it is reasonable to expect no energy flux. Even if there is an initial flux, the star will emit gravitational waves until stability is reached, at which point our method becomes trustworthy. The combination of the tetrad field in

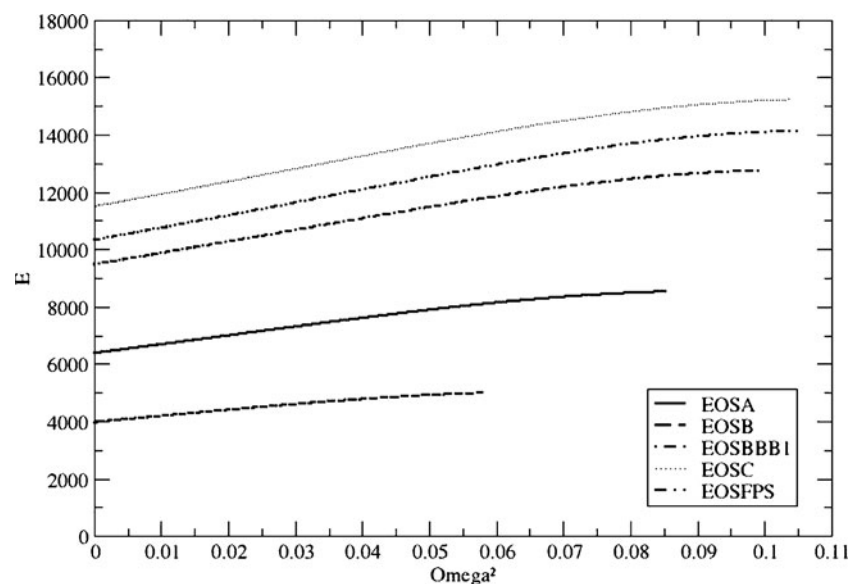
Fig. 8 Spatial distribution of the ratio between the gravitational energy and the total mass. The legend follows the convention of Fig. 2



(18) with (7) leads, after some algebraic manipulation, to the result

$$\begin{aligned}
 4e \Sigma^{(0)01} = & \frac{1}{\sqrt{(-g_{00}\delta)} g_{22}g_{11}} \left[2g_{11}g_{22}\delta - \delta \left(\frac{\partial g_{22}}{\partial r} \right) \right. \\
 & \times \sqrt{g_{22}g_{11}} - (g_{22})^{3/2} \sqrt{g_{11}} g_{03} \left(\frac{\partial g_{03}}{\partial r} \right) \\
 & + 2\sqrt{-g_{00}} (g_{22})^{3/2} \sqrt{\delta} g_{11} \sin \theta \\
 & \left. + g_{00} (g_{22})^{3/2} \sqrt{g_{11}} \left(\frac{\partial g_{33}}{\partial r} \right) \right]. \quad (22)
 \end{aligned}$$

Fig. 9 Gravitational energy as a function of the squared angular velocity of the source. The legend follows the convention of Fig. 2



Next, on the basis of (12), we examine the energy as a function of s . We work with dimensionless quantities and let E denote $P(0)$. To recover the energy from the dimensionless variable, we let $E \rightarrow (c^4 \sqrt{K}/G)E$, where \sqrt{K} is the fundamental length scale.

First, a simple check on the code. Figure 7 shows the total energy, i.e., the energy in a hypersurface of infinite radius, as a function of the mass. Although straight over a large fraction of the displayed range, the solid lines in the first five panels bend upwards or downwards at the highest masses. That the curvatures signal the breakdown of the nonrelativistic equations of state is shown by the rectilinear plot in the last panel of Fig. 7, which contrasts with the other

panels because the EOS in Ref. [52] includes relativistic corrections.

Figure 8 shows the ratio between the energy and mass as a function of s . For each equation of state, E/M rises up to the surface of the star and then decreases towards a constant as $s \rightarrow 1$, showing that the energy at infinity is proportional to the mass—a conclusion that cannot be directly inferred from Fig. 7. The slow decay beyond $s \approx 0.45$ shows that the rotational energy is very small.

Figure 9 shows the gravitational energy as a function of the squared angular velocity of the source for each of the indicated equations of state. As the angular velocity grows, the initially straight plots bend downwards. While, as already explained, the deviations from linearity in Fig. 7 reflect the inaccuracy of the nonrelativistic equations of state, those in Fig. 9 have physical origin: they indicate that the moment of inertia, which relates the energy of the field to the angular velocity, depends on the angular velocity of the fluid.

We have also computed the momentum of the gravitational field, using the spatial indices in (12). The relevant components are

$$\frac{4e \Sigma^{(1)01}}{\sin \phi} = \left[\frac{g_{03} \left(2 g_{22} g_{11} - \left(\frac{\partial g_{22}}{\partial r} \right) \sqrt{g_{22} g_{11}} \right) - (g_{22})^{3/2} \sqrt{g_{11}} \left(\frac{\partial g_{03}}{\partial r} \right)}{\sqrt{-g_{00} g_{22} g_{11}}} \right], \quad (23)$$

$$\frac{4e \Sigma^{(2)01}}{\cos \phi} = \left[\frac{g_{03} \left(-2 g_{22} g_{11} + \left(\frac{\partial g_{22}}{\partial r} \right) \sqrt{g_{22} g_{11}} \right) + (g_{22})^{3/2} \sqrt{g_{11}} \left(\frac{\partial g_{03}}{\partial r} \right)}{\sqrt{-g_{00} g_{22} g_{11}}} \right], \quad (24)$$

and $\Sigma^{(3)01} = 0$.

The momentum is therefore zero, a result that can only be mathematically understood if we recall that the right-hand side of (12) has been integrated over the coordinate ϕ . Physically, we expect the rotating star to transfer no momentum to the field.

4 Conclusion

We have analyzed the physics of neutron stars from the point of view of teleparallel gravity. We have determined the energy and angular momentum as functions of the compact spatial coordinate. In particular, we have computed the ratio between the field and source angular momenta. For

rapidly rotating neutron stars, the magnitudes of the two angular momenta are comparable. Figure 6 showed that, as expected, the ratio vanishes in the limit of slow rotations. As also shown by Fig. 6, at high angular velocities, the field angular momentum ceases to be proportional to the source angular momentum, an indication that the moment of inertia has become strongly dependent on the angular velocity of the field.

We have calculated the ratio between the gravitational energy and the mass as a function of the compact coordinate s . This ratio is maximized near the surface of the star. Beyond the surface, it decays to a constant at infinity, a value that is weakly dependent on the equation of state. We are lead to conclude that, as one would expect, each type of neutron star is characterized by a universal linear relation between the gravitational energy and mass at spatial infinity. Although we have found small deviations from linearity for sufficiently large masses, the comparison in Fig. 7 has linked those deviations to inaccuracies in the nonrelativistic equations of state.

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