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CONDENSED MATTER

Sine-Gordon Model: Renormalization Group Solution and Applications

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Abstract The sine-Gordon model is discussed and analyzed within the framework of the renormalization group theory. A perturbative renormalization group procedure is described, in which the sine-Gordon field is decomposed into slow and fast modes. An effective theory for the slow modes is derived and rescaled to yield the flow equations for the model. The resulting Kosterlitz–Thouless phase diagram is obtained and discussed in detail. The gap in this theory is estimated in terms of the sine-Gordon model parameters. The mapping between the sine-Gordon model and one-dimensional interacting-electron models, such as the g-ology and Hubbard models, is discussed. On the basis of the results borrowed from previous renormalization-group results for the sine-Gordon model, different aspects of Luttinger liquid systems are described, such as the nature of the excitations and phase transitions. The calculations are thoroughly and pedagogically described, to even reach the reader with no previous experience with the sine-Gordon model or the renormalization group approach.

Keywords Sine-Gordon model · Renormalization group theory · Kosterlitz–Thouless phase transition · g-ology model · Hubbard model · Luttinger liquids

1 Introduction

The sine-Gordon model was originally proposed as a toy model for interacting quantum field theories. It has been intensively investigated since then. In low dimensionalities, it often describes systems with strong pinning effects stemming from non-quadratic interactions. In the framework of the model, the physics of pinning is governed by the competition between the quadratic momentum, which promotes fluctuations, and the sine-Gordon potential, which tends to lock the model field in one of its minima.

The model has a broad range of applications. The sine-Gordon solitons are equivalent to the fermions of the Thirring model, of which the sine-Gordon Hamiltonian is the S-dual [1, 2]; to the magnetic vortices of the XY model [3]; to the kinks and dislocations propagating in crystalline structures [4, 5]; and to the correlation functions in the neutral Coulomb gas problem [6], to cite but a few examples.

The model is particularly useful to the describe strongly correlated electronic systems in one dimension, the so-called Luttinger liquids. A bosonization procedure [7] maps different Luttinger liquid Hamiltonians, such as those of the g-ology and Hubbard models, on the sine-Gordon model, for which a large stock of analytical results is available.

Exact results for the spectrum of the sine-Gordon model and S-matrix have long been known [8–12]. Results for form factors [13–17] and a finite-size correction to the spectrum [18–21] are also available in the literature. The free energy of the model and the specific heat have been computed by means of a Bethe ansatz [22].

Nevertheless, a number of open questions regarding the sine-Gordon model persist. Except for a specific coupling constant [23], exact results for the correlation functions are still missing. In this context, the renormalization group theory is an important analytical tool, for it leads to the understanding of the phase transition in the sine-Gordon model and determines the energy scale at which it occurs.

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2 The Model

The sine-Gordon model is defined by the Hamiltonian

$$H[\Pi,\varphi]=\int dx\,\left[\frac{v}{2}\left(\Pi^2+(\partial_x\varphi)^2\right)-\tilde{g}\cos(\beta\varphi)\right] \quad (1)$$

where $\varphi=\varphi(x,t)$ and $\Pi=\Pi(x,t)$ are canonically conjugated fields, that is,

$$\partial_t\varphi(x,t)=v\Pi(x,t). \quad (2)$$

The Lagrangian model reads

$$L[\varphi]=\int dx\,\{\,\partial_t\varphi\,.\,\Pi\,-\,H[\Pi,\varphi]\,\},$$

or

$$L[\varphi]=\int dx\,\left[\frac{1}{2v}(\partial_t\varphi)^2-\frac{v}{2}(\partial_x\varphi)^2+\tilde{g}\cos(\beta\varphi)\right]. \quad (3)$$

After integration by parts, the action

$$S[\varphi]=\int dt\,L[\varphi]$$

takes the following form:

$$S[\varphi]=\int\int dtdx\,\left[\frac{v}{2}\varphi\partial_x^2\varphi-\frac{1}{2v}\varphi\partial_t^2\varphi+\tilde{g}\cos(\beta\varphi)\right],$$

where we have assumed that the φ-field goes to zero at the boundaries of the integration plane.

For a reason that will become clear in Section 3.2, it is more convenient to work in imaginary time $t\to -it$, i.e., with the Euclidean action $iS\to S$, which reads

$$S[\varphi]=S_0[\varphi]+S_I[\varphi] \quad (4)$$

where

$$S_0[\varphi]=\int dx\,\frac{1}{2}\varphi\nabla_x^2\varphi \quad \text{with} \quad \nabla_x^2=\partial_x^2+\frac{1}{v^2}\partial_t^2 \quad (5)$$

and

$$S_I[\varphi]=\int dx\,l_I[\varphi] \quad \text{with} \quad l_I[\varphi]=g\cos(\beta\varphi). \quad (6)$$

Here, $x\to(x,vt)$ and $g=\tilde{g}/v$.

3 Renormalization Group Treatment

3.1 Conceptual Overview of Renormalization Group Theory

The renormalization group (RG) is essentially a theory of scale invariance and symmetries. A scale operation is a transformation depicting the appearance of the system and its behavior at different scales, which may be scales of length, energy, or any typical physical quantity in the system. The system is said to be scale invariant under a certain scale transformation if it looks the same in all scales.

In RG theory, the microscopical physical quantities, such as the mass, charge, and interaction parameters, depend on the scale at which the system is observed. The scale-invariant properties, which emerge from the macroscopic structure of the system, are related to the degree of order. The globally invariant properties are represented by physical quantities called order parameters.

Many states of matter are characterized by scale-invariant properties and order parameters. Examples are the lattice structure in a crystalline solid, which translates into a periodic density of particles, the spins orientation in a ferromagnetic material, which results in a net spontaneous magnetization, and the localization of charge in an insulator, which is described by charge-density waves.

When, for some reason, an originally scale-invariant system becomes scale dependent, it is said to have undergone a phase transition. The phase transition, i.e., the loss of invariance, takes place at a certain critical scale that is typical of each system. The critical scale defines an energy, called the gap—or mass, in quantum field theory language—, which gauges the disturbance in the order parameter. The critical point of the phase transition is identified by the parameters of the system at the critical scale.

As an example, picture a perfect crystalline solid at $T=0$. As the temperature rises, the translational invariance of the lattice is eventually lost at a certain critical length scale set by the characteristics of the material. At this scale, the lattice correlations lose the competition against the thermal fluctuations, and the solid structure melts into a fluid.

Here, we are interested in the scaling behavior of the sine-Gordon model, or rather, of physical systems that can be described by the sine-Gordon model. The following three subsections present an overview of the RG procedure. In Section 3.2, the decomposition of a generic quantum field theory into slow and fast modes is presented. The purpose of Section 3.3 is to express the so-called residual action, which mixes slow and fast modes, in terms of Green's functions. In Section 3.4, an effective action for the slow modes is derived, the fast modes being averaged out. The last two sections are dedicated to the application of the general formalism to the sine-Gordon model. In Section 3.5, the effective action for the slow modes is evaluated. The resulting effective theory is then renormalized, a rescaled sine-Gordon model resulting. The flow equations for the model are derived in Section 3.6.

3.2 General Procedure I: Decomposition into Slow and Fast Modes

The RG procedure as it is presented in this section follows Kenneth G. Wilson's development in the late 1960's [24],

for which he was awarded the 1982 Nobel Prize in Physics. Nowadays, this formulation is routinely called "Wilsonian approach" to the RG theory.

The procedure is based on splitting the field $\varphi(x)$ of the theory into two components corresponding to different momentum–frequency regions of the original Fourier decomposition. Mathematically, we have that the field

$$\varphi(x) = \int \frac{dq}{(2\pi)^2}\,\varphi(q)e^{iqx}$$

is divided as follows:

$$\varphi(x) = \int_{\text{bulk}} \frac{dq}{(2\pi)^2}\,\varphi(q)e^{iqx} + \int_{\text{shell}} \frac{dq}{(2\pi)^2}\,\varphi(q)e^{iqx}$$

where $x \to (x, vt)$, $q \to (q, \omega/v)$, $qx \to (qx + \omega t)$, $|q|^2 \to q^2 + \omega^2/v^2$, and

$$\text{bulk} \equiv |q| < \frac{\Lambda}{s},$$

while

$$\text{shell} \equiv \frac{\Lambda}{s} < |q| < \Lambda,$$

with $s \gtrsim 1$, where Λ is a momentum–frequency cutoff.

In the original covariant space, the momentum–frequency shell would correspond to the unbounded surface between the two hyperbolae $q^2 - \omega^2/v^2 = \Lambda^2/s^2$ and $q^2 - \omega^2/v^2 = \Lambda^2$. Although this surface imposes a cutoff in the magnitude $|q|$, the individual coordinates q and ω remain unbounded. The integration of a function $f(q, \omega)$ over the shell will therefore diverge unless $f(q, \omega)$ decays sufficiently fast. The purpose of the imaginary time rotation just above (4) is none other than to avoid complications that might arise from an unbounded shell.

In a more compact form, we may write the φ-field as

$$\varphi(x) = \varphi^s(x) + \delta\varphi(x) \tag{7}$$

where

$$\varphi^s(x) = \int_{\text{bulk}} \frac{dq}{(2\pi)^2}\,\varphi(q)e^{iqx}, \tag{8}$$

and

$$\delta\varphi(x) = \int_{\text{shell}} \frac{dq}{(2\pi)^2}\,\varphi(q)e^{iqx}. \tag{9}$$

The φ^s-field contains the so-called slow modes of the original φ-field while the $\delta\varphi$-field contains the fast modes.

The idea is to obtain the action for the theory, written for the φ-field in the full momentum–frequency space, in terms of slow and fast mode fields, and take its average with respect to the unperturbed ground state of the fast modes. The result of this averaging is an effective action for the slow modes. A *renormalized* theory is thus obtained from the effective one through a scale transformation, or renormalization, of the momentum–frequency cutoff. The RG statement, based on the assumed scale invariance of the theory, is that the original and renormalized theories are equal, i.e., that the effective theory of the slow modes, defined in the bulk, is equivalent to a scale renormalization of the full original theory in the entire momentum–frequency space. This equivalence allows the derivation of the RG flow equations for the theory, which comprise the final outcome of the RG approach.

Let us proceed by rewriting the action $S[\varphi]$ in terms of the φ^s- and $\delta\varphi$-fields. Since

$$\begin{aligned}\int dx\,\delta\varphi\nabla_x^2\varphi^s &= \int dx\,\varphi^s\nabla_x^2\delta\varphi = \\ &= \int dx \int_{\text{bulk}} \frac{dq}{(2\pi)^2} \int_{\text{shell}} \frac{dq'}{(2\pi)^2}\,\varphi(q) \\ &\quad\times\left(-q'^2 - \frac{\omega'^2}{v^2}\right)\varphi(q')e^{i(q+q')x},\end{aligned}$$

computing the space–time integral followed by a momentum–frequency integration over the shell leads to

$$\begin{aligned}\int dx\,\delta\varphi\nabla_x^2\varphi^s &= \int dx\,\varphi^s\nabla_x^2\delta\varphi = \\ &= \int_{\text{bulk}} \frac{dq}{(2\pi)^2}\,\varphi(q)\left(-q^2 - \frac{\omega^2}{v^2}\right)\varphi(-q) \\ &\quad\times\Theta\left(|q| - \frac{\Lambda}{s}\right) = 0,\end{aligned}$$

where the Heaviside step function $\Theta\left(|q| - \frac{\Lambda}{s}\right)$ inserted in the previous equation guarantees that the momentum–frequency coordinates that emerge from the integration over the shell are indeed inside the shell boundaries. Finally, the momentum–frequency integral over the bulk conditioned by the Heaviside step function vanishes.

Substitution of (7) into (5) therefore gives the result

$$S_0[\varphi] = S_0[\varphi^s] + S_0[\delta\varphi]. \tag{10}$$

The interaction contribution to the action, that is, $S_I[\varphi]$ in (6), will be treated via perturbation theory around the slow-mode φ^s-field, a procedure that is similar to the usual saddle-point expansion around a fixed classical field configuration. Up to second order in the perturbation, i.e., to second order in the fast-mode $\delta\varphi$-field, we have that

$$\begin{aligned}S_I[\varphi] &= S_I[\varphi^s] + \int dx\,a^s(x)\delta\varphi(x) \\ &\quad + \int dx dx'\,\delta\varphi(x)b^s(x,x')\delta\varphi(x')\end{aligned} \tag{11}$$

where the coefficients $a^s(x)$ and $b^s(x)$ are given by

$$a^s(x) = \frac{\delta I_I[\varphi]}{\delta\varphi(x)}\Big|_{\varphi^s} \tag{12}$$

and

$$b^s(x,x') = \frac{1}{2}\frac{\delta^2 I_I[\varphi]}{\delta\varphi(x)\delta\varphi(x')}\Big|_{\varphi^s}. \tag{13}$$

Substitution of (10) and (11) into (4) yields the following form for the full action $S[\varphi]$:

$$S[\varphi] = S[\varphi^s] + \delta S[\varphi^s, \delta\varphi] \tag{14}$$

with

$$\delta S[\varphi^s,\delta\varphi] = S_0[\delta\varphi] + \int dx\, a^s(x)\delta\varphi(x) + \int dx dx'\, \delta\varphi(x) b^s(x,x')\delta\varphi(x').$$

We then substitute (5) for $S_0[\delta\varphi]$ on the right-hand side to obtain the result

$$\delta S[\varphi^s,\delta\varphi] = \int dx dx'\, \delta\varphi(x)\left[\delta(x-x')\frac{1}{2}\nabla^2_{x'} + b^s(x,x')\right] \times\delta\varphi(x') + \int dx\, a^s(x)\delta\varphi(x). \tag{15}$$

We see from (14) that the full action $S[\varphi]$ splits into two contributions: the action $S[\varphi^s]$ for the slow modes and a residual piece $\delta S[\varphi^s,\delta\varphi]$ mixing slow and fast modes. A simple field transformation eliminating the linear term in $\delta\varphi(x)$ then yields, to second order in perturbation theory around the slow modes, a residual action equivalent to a quadratic theory for the fast modes with a mass-like term given by $b^s(x,x')$ that encodes the influence of the slow modes as well as that of the interactions.

To derive an effective theory for the slow modes, we will average the residual action $\delta S[\varphi^s,\delta\varphi]$ over the unperturbed ground state of the fast mode $\delta\varphi$-field operators, so that $\delta S[\varphi^s,\delta\varphi]$ will be reduced to $\delta S_{\rm eff}[\varphi^s]$ and $S[\varphi]$ to $S_{\rm eff}[\varphi^s]$. As already explained, the RG procedure corresponds to obtaining $S[\varphi]$ back from $S_{\rm eff}[\varphi^s]$ via a renormalized theory $S_R[\varphi]$ by means of a scale renormalization of the cutoff $\Lambda/s \to \Lambda$.

To compute the average of the residual action, it is convenient to express $\delta S[\varphi^s,\delta\varphi]$ in terms of Green's functions.

3.3 General Procedure II: Green's Function Expression for δS

The Green's function $G_0(x,x')$ for the free residual action is defined by the equalities

$$\begin{cases} G_0^{-1}(x,x') = \delta(x-x')\,\frac{1}{2}\nabla^2_{x'} \\ \int dx''\, G_0^{-1}(x,x'')G_0(x'',x') = \delta(x-x'). \end{cases} \tag{16}$$

From the definition, it follows that

$$\frac{1}{2}\nabla^2_x G_0(x,x') = \delta(x-x'), \tag{17}$$

which implies that $G_0(x,x') = G_0(x-x')$, so that the Fourier transform of (17) reads

$$\frac{1}{2}\nabla^2_x \int \frac{dq}{(2\pi)^2}\, G_0(q) e^{iq(x-x')} = \int \frac{dq}{(2\pi)^2}\, e^{iq(x-x')},$$

from which it follows that

$$\int \frac{dq}{(2\pi)^2}\, G_0(q)\frac{1}{2}\left(-q^2 - \frac{\omega^2}{v^2}\right) e^{iq(x-x')} = \int \frac{dq}{(2\pi)^2}\, e^{iq(x-x')} \tag{18}$$

and we find that

$$G_0(q) = G_0(q,\omega) = -\frac{2}{q^2+\omega^2/v^2}. \tag{19}$$

The Green's function $G(x,x')$ for the full residual action is defined by the equations

$$\begin{cases} G^{-1}(x,x') = G_0^{-1}(x,x') - \Sigma(x,x') \\ \int dx''\, G^{-1}(x,x'')G(x'',x') = \frac{1}{(2\pi)^2}\delta(x-x') \end{cases} \tag{20}$$

where $\Sigma(x,x')$ is the self-energy of the theory, which accounts for the corrections to the free Green's function due to interactions and external fields.

From the definition (16), it then follows that

$$\frac{1}{2}\nabla^2_x G(x,x') - \int dx''\, \Sigma(x,x'')G(x'',x') = \frac{1}{(2\pi)^2}\delta(x-x'),$$

the Fourier transform of which yields the equality

$$\int \frac{dq}{(2\pi)^2}\frac{dq'}{(2\pi)^2}\, G(q,q')\frac{1}{2}\left(-q^2-\frac{\omega^2}{v^2}\right) e^{iqx+iq'x'} - \int dx'' \int \frac{dq}{(2\pi)^2}\frac{dk}{(2\pi)^2}\frac{dk'}{(2\pi)^2}\frac{dq'}{(2\pi)^2}\, \Sigma(q,k)G(k',q')e^{iqx+i(k+k')x''+iq'x'} = \int \frac{dq}{(2\pi)^4}\, e^{iq(x-x')}. \tag{21}$$

Equation (21) can be written in the form

$$\int \frac{dq}{(2\pi)^2}\frac{dq'}{(2\pi)^2}\left[G(q,q')G_0^{-1}(q) - \int \frac{dq''}{(2\pi)^2}\,\Sigma(q,q'')G(-q'',q')\right]e^{iqx+iq'x'} = \int \frac{dq}{(2\pi)^2}\frac{dq'}{(2\pi)^2}\,\delta(q+q')e^{iqx+iq'x'}, \quad (22)$$

from which it follows that

$$G(q,q') = G_0(q)\delta(q+q') + G_0(q)\int \frac{dq''}{(2\pi)^2}\,\Sigma(q,q'')G(-q'',q'). \quad (23)$$

Equation (23) is the Dyson equation, which as usual includes the full self-energy. In our perturbation theory around the slow modes, correct to second order in the fast modes, analogous to a second-order saddle-point expansion, the self energy is simply given by the expression

$$\Sigma(q,q') = -b^s(q,q'), \quad \Sigma(x,x') = -b^s(x,x'). \quad (24)$$

Thus, (23) can be rewritten as the relation

$$G(q,q') = G_0(q)\delta(q+q') - G_0(q)\int \frac{dq''}{(2\pi)^2}\,b^s(q,q'')G(-q'',q'). \quad (25)$$

We can now use (25) to expand $G(q,q')$ in powers of the interaction coupling constant g. To zero order in g, we have that

$$G^{(0)}(q,q') = G_0(q)\delta(q+q'). \quad (26)$$

To first order in g, we have that

$$G^{(1)}(q,q') = G_0(q)\delta(q+q') - G_0(q)\int \frac{dq''}{(2\pi)^2}\,b^s(q,q'')\times G_0(-q'')\delta(-q''+q')$$

which reduces to the equality

$$G^{(1)}(q,q') = G_0(q)\delta(q+q') - \frac{1}{(2\pi)^2}G_0(q)b^s(q,q')G_0(-q'). \quad (27)$$

To second order, we find that

$$G^{(2)}(q,q') = G_0(q)\delta(q+q') - G_0(q)\int \frac{dq''}{(2\pi)^2}\,b^s(q,q'')\times\left[G_0(-q'')\delta(-q''+q') - \frac{1}{(2\pi)^2}G_0(-q'')b^s(-q'',q')G_0(-q')\right]$$

which can be rewritten in a form analogous to (27):

$$G^{(2)}(q,q') = G_0(q)\delta(q+q') - \frac{1}{(2\pi)^2}G_0(q)b^s(q,q')G_0(-q') + \frac{1}{(2\pi)^2}G_0(q)\int \frac{dq''}{(2\pi)^2}\,b^s(q,q'') \times G_0(-q'')b^s(-q'',q')G_0(-q'). \quad (28)$$

The sequence defined by (26), (27), and (28) can be extended to an arbitrary order.

We now go back to the residual action $\delta S[\varphi^s,\delta\varphi]$ and substitute (16), (20), and (24) into (15) to find the result

$$\delta S[\varphi^s,\delta\varphi] = \int dx dx'\,\delta\varphi(x)G^{-1}(x,x')\delta\varphi(x') + \int dx\,a^s(x)\delta\varphi(x).$$

Finally, we can obtain a quadratic expression in the fast-mode $\delta\varphi$-fields by means of a simple field transformation. Let

$$\delta\varphi(x) = \bar\varphi(x) + r(x). \quad (29)$$

Then,

$$\delta S[\varphi^s,\delta\varphi] = \delta S[\varphi^s,\bar\varphi,r] = \int dx dx'\Big\{\bar\varphi(x)G^{-1}(x,x')\bar\varphi(x') + \bar\varphi(x)\left[2G^{-1}(x,x')r(x') + a^s(x')\delta(x-x')\right] + r(x)\left[G^{-1}(x,x')r(x') + a^s(x')\delta(x-x')\right]\Big\}. \quad (30)$$

Now, if

$$\int dx'\,G^{-1}(x,x')r(x') = -\frac{1}{2}a^s(x),$$

i.e., if

$$r(x) = -\frac{1}{2}\int dx'\,G(x,x')a^s(x'), \quad (31)$$

then

$$\delta S[\varphi^s,\bar\varphi] = \int dx dx'\left[\bar\varphi(x)G^{-1}(x,x')\bar\varphi(x') - \frac{1}{4}a^s(x)G(x,x')a^s(x')\right]. \quad (32)$$

We are now in position to average $\delta S[\varphi^s, \bar{\varphi}]$ over the ground state of the fast modes.

3.4 General Procedure III: Average over the Fast-Mode Ground State

Equations (31), (20), and (24) imply that $r(x) = r[\varphi^s(x)]$. Since from (7) and (29),

$$\varphi(x) = \varphi^s(x) + r(x) + \bar{\varphi}(x),$$

we can redefine the slow modes to include the field $r(x)$ via the transformation $\varphi^s(x) + r(x) \to \varphi^s(x)$ and write

$$\varphi(x) = \varphi^s(x) + \bar{\varphi}(x). \tag{33}$$

We now follow the argument leading to (10) to find that the Hamiltonian

$$H_0[\Pi, \varphi] = \int dx \, \frac{v}{2} \left(\Pi^2 + (\partial_x \varphi)^2 \right)$$

can be divided in two parts:

$$H_0[\Pi, \varphi] = H_0[\Pi^s, \varphi^s] + H_0[\bar{\Pi}, \bar{\varphi}]. \tag{34}$$

Let $|0\rangle^{\varphi}$, $|0\rangle^{\varphi^s}$, and $|0\rangle^{\bar{\varphi}}$ be the unperturbed ground states of the full, slow, and fast modes, respectively. From (34), we have that

$$|0\rangle^{\varphi} = |0\rangle^{\varphi^s} |0\rangle^{\bar{\varphi}}. \tag{35}$$

It is now our goal to rewrite the full unperturbed Green's function

$$G_0(x, x') \equiv {}^{\varphi}\langle 0 | \varphi(x) \varphi(x') | 0 \rangle^{\varphi}$$

in the subspace of the fast modes. With the help of (33), we have that

$$\begin{aligned} G_0(x, x') = {}& {}^{\varphi}\langle 0 | \varphi^s(x) \varphi^s(x') | 0 \rangle^{\varphi} + {}^{\varphi}\langle 0 | \bar{\varphi}(x) \bar{\varphi}(x') | 0 \rangle^{\varphi} \\ & + {}^{\varphi}\langle 0 | \varphi^s(x) \bar{\varphi}(x') | 0 \rangle^{\varphi} + {}^{\varphi}\langle 0 | \bar{\varphi}(x) \varphi^s(x') | 0 \rangle^{\varphi}. \end{aligned} \tag{36}$$

Because the field operators inside the brackets act on different subspaces, the last two terms on the right-hand side of (36) vanish. Scale invariance requires the first two terms to be equal, that is, the space–time correlations to be independent of the momentum–frequency scale of the fields.

We therefore have that

$$G_0(x, x') = 2 \, {}^{\varphi}\langle 0 | \bar{\varphi}(x) \bar{\varphi}(x') | 0 \rangle^{\varphi},$$

and in view of (35), it follows that

$$G_0(x, x') = 2 \, {}^{\bar{\varphi}} \left\langle 0 \left| {}^{\varphi^s}\langle 0 | \bar{\varphi}(x) \bar{\varphi}(x') | 0 \rangle^{\varphi^s} \right| 0 \right\rangle^{\bar{\varphi}},$$

and since ${}^{\varphi^s}\langle 0 | 0 \rangle^{\varphi^s} = 1$, we can see that

$$G_0(x, x') = 2 \, {}^{\bar{\varphi}}\langle 0 | \bar{\varphi}(x) \bar{\varphi}(x') | 0 \rangle^{\bar{\varphi}}. \tag{37}$$

The Fourier transforms of the unperturbed Green's function and of the delta functions are properly redefined in the subspace of the fast modes as follows:

$$G_0(x, x') = G_0(x - x') = \int_{\text{shell}} \frac{dq}{(2\pi)^2} \, G_0(q) e^{iq(x - x')}, \tag{38}$$

$$\delta(x) = \int_{\text{shell}} \frac{dq}{(2\pi)^2} \, e^{iqx}, \tag{39}$$

and

$$\delta(|q| - \Lambda/s) = \int dx \, e^{-iqx}. \tag{40}$$

i.e., with integrals constrained to the high momentum–frequency shell.

The effective residual contribution of the slow modes to the action, which we will call $\delta S_{\text{eff}}[\varphi^s]$, is obtained as the average of the residual $\delta S[\varphi^s, \bar{\varphi}]$ over the fast modes:

$$\delta S_{\text{eff}}[\varphi^s] \equiv {}^{\bar{\varphi}}\langle 0 | \delta S[\varphi^s, \bar{\varphi}] | 0 \rangle^{\bar{\varphi}}. \tag{41}$$

Substituting (32) and (37) into (41), we obtain the expression

$$\begin{aligned} \delta S_{\text{eff}}[\varphi^s] = \int dx dx' \Bigg[& \frac{1}{2} G_0(x, x') G^{-1}(x, x') \\ & - \frac{1}{4} a^s(x) G(x, x') a^s(x') \Bigg]. \end{aligned}$$

From (5), (16), (20), (24), and (37), it then follows that

$$\begin{aligned} \delta S_{\text{eff}}[\varphi^s] = \int dx dx' \Bigg\{ & \frac{1}{2} G_0(x, x') \left[G_0^{-1}(x, x') + b^s(x, x') \right] \\ & - \frac{1}{4} a^s(x) G(x, x') a^s(x') \Bigg\}, \end{aligned}$$

which we write in the form

$$\begin{aligned} \delta S_{\text{eff}}[\varphi^s] = {}& {}^{\bar{\varphi}}\langle 0 | S_0[\bar{\varphi}] | 0 \rangle^{\bar{\varphi}} \\ & + \int dx dx' \Bigg[\frac{1}{2} G_0(x, x') b^s(x, x') \\ & - \frac{1}{4} a^s(x) G(x, x') a^s(x') \Bigg]. \end{aligned}$$

The first term on the right-hand side, a constant, can be absorbed through a trivial redefinition of $\delta S_{\text{eff}}[\varphi^s]$, which is then expressed by the relation

$$\begin{aligned} \delta S_{\text{eff}}[\varphi^s] = \int dx dx' \Bigg[& \frac{1}{2} G_0(x, x') b^s(x, x') \\ & - \frac{1}{4} a^s(x) G_0(x, x') a^s(x') \Bigg]. \end{aligned} \tag{42}$$

Here, we have substituted $G_0(x, x')$ for $G(x, x')$ in the last term in the integrand on the right-hand side of (41), to keep terms only up to second order in the g-coupling.

We substitute $\delta S_{\text{eff}}[\varphi^s]$, i.e., the right-hand side of (42), for $\delta S[\varphi^s, \delta\varphi]$ on the right-hand side of (14) to write down

the following expression for the full slow-mode effective action:

$$S_{\text{eff}}[\varphi^s] = S[\varphi^s] + \delta S_{\text{eff}}[\varphi^s]. \tag{43}$$

To compute the contribution of $\delta S_{\text{eff}}[\varphi^s]$ to the effective theory, we have to write an explicit relation for the φ^s-field dependence of the integrand on the right-hand side of (42). Each quantum field theory has its specific relation. Section 3.5 finds the one pertaining to the sine-Gordon model, our ultimate target being the rescaled action for the model.

3.5 Application I: Rescaled Action for the Sine-Gordon Model

We start out by computing the first term on the right-hand side of (42):

$$F_1[\varphi^s] \equiv \int dx dx' \frac{1}{2} G_0(x, x') b^s(x, x'). \tag{44}$$

Recalling the redefinition of the Green's function in the fast-mode subspace in (38), we find that

$$\begin{aligned} F_1[\varphi^s] = & \frac{1}{2} \int dx dx' \int_{\text{shell}} \frac{dq}{(2\pi)^2} \\ & \times \int \frac{dq'}{(2\pi)^2} \frac{dq''}{(2\pi)^2} \\ & G_0(q) b^s(q', q'') e^{iq(x-x')+iq'x+iq''x'}, \end{aligned}$$

which is equivalent to the equality

$$F_1[\varphi^s] = \frac{1}{2} \int_{\text{shell}} \frac{dq}{(2\pi)^2} G_0(q) b^s(-q, q).$$

From (13) and (6), we have that

$$b^s(x, x') = -\frac{\beta^2}{2} g \cos(\beta\varphi^s)\delta(x-x') = -\frac{\beta^2}{2} l_I[\varphi^s]\delta(x-x'),$$

so that the Fourier transform

$$b^s(q, q') = \int dx dx' \, b^s(x, x') e^{iqx+iq'x'}$$

is given by the equalities

$$b^s(q, q') = \int dx \left(-\frac{\beta^2}{2} l_I[\varphi^s]\right) e^{i(q+q')x} = b^s(q + q') \tag{45}$$

from which it follows that

$$F_1[\varphi^s] = \frac{1}{2} b^s(q = 0) \int_{\text{shell}} \frac{dq}{(2\pi)^2} G_0(q). \tag{46}$$

Substitution of (19) for $G_0(q)$ on the right-hand side of (46) then yields the result

$$F_1[\varphi^s] = -b^s(q = 0) \int_{\text{shell}} \frac{dq}{(2\pi)^2} \frac{1}{q^2 + \omega^2/v^2}, \tag{47}$$

which can be rewritten in the form

$$F_1[\varphi^s] = -b^s(q = 0) \int_0^{2\pi} \int_{\Lambda/s}^{\Lambda} \frac{d\theta d|q|}{(2\pi)^2} \frac{1}{|q|}$$

and hence leads to the result

$$F_1[\varphi^s] = -\frac{1}{2\pi} b^s(q = 0) \ln(s). \tag{48}$$

Substitution of (45) for $b^s(q = 0)$ on the right-hand side of (48) then tells us that

$$F_1[\varphi^s] = \ln(s) \frac{\beta^2}{4\pi} \int dx \, l_I[\varphi^s].$$

Finally, using (6), we can see that

$$F_1[\varphi^s] = -g \left(\frac{dl\beta^2}{4\pi}\right) \int dx \, \cos(\beta\varphi^s) \tag{49}$$

where

$$dl \equiv -\ln(s). \tag{50}$$

Consider next the second term on the right-hand side of (42). To compute it, we expand $a^s(x')$ around $x' = x$ to show that, up to second order in the expansion, the expression

$$F_2[\varphi^s] \equiv -\int dx dx' \frac{1}{4} G_0(x, x') a^s(x) a^s(x'), \tag{51}$$

is equivalent to the relation

$$\begin{aligned} F_2[\varphi^s] = & -\frac{v^2}{4} \int dx dx' dt dt' \, G_0(x - x', t - t') \\ & \times \Big[(a^s(x, t))^2 + + a^s(x, t) \partial_x a^s(x, t)(x' - x) \\ & + a^s(x, t) \partial_t a^s(x, t)(t' - t) \\ & + \frac{1}{2} a^s(x, t) \partial_x^2 a^s(x, t)(x' - x)^2 \\ & + \frac{1}{2} a^s(x, t) \partial_t^2 a^s(x, t)(t' - t)^2 \\ & + a^s(x, t) \partial_x \partial_t a^s(x, t)(x' - x)(t' - t) \Big]. \end{aligned} \tag{52}$$

Equation (52) reduces to the simpler form

$$\begin{aligned} F_2[\varphi^s] = & -\frac{v^2}{4} \int dx dX dt dT \, G_0(X, T) \\ & \times \Big[(a^s(x, t))^2 - X a^s(x, t) \partial_x a^s(x, t) \\ & - T a^s(x, t) \partial_t a^s(x, t) + \frac{1}{2} X^2 a^s(x, t) \partial_x^2 a^s(x, t) \\ & + \frac{1}{2} T^2 a^s(x, t) \partial_t^2 a^s(x, t) \\ & + XT a^s(x, t) \partial_x \partial_t a^s(x, t) \Big]. \end{aligned} \tag{53}$$

Using the equality

$$G_0(q, \omega) = v \int dX dT \, G_0(X, T) e^{-i(qX+\omega T)}$$

together with (38) and (40), it follows that

$$\partial_q^n \partial_\omega^m G_0(q,\omega) = v \int dX dT (-i)^{n+m} X^n T^m G_0(X,T) e^{-i(qX+\omega T)},$$

$$X^n T^m G_0(X,T) = \frac{1}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} i^{n+m} \partial_q^n \partial_\omega^m G_0(q,\omega) e^{i(qX+\omega T)},$$

and

$$v \int dX dT\, X^n T^m G_0(X,T) = \frac{i^{n+m}}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} \partial_q^n \partial_\omega^m G_0(q,\omega) \delta(|q| - \Lambda/s). \quad (54)$$

Equation (54) suggests the shorthand

$$f_{\partial_q^n, \partial_\omega^m}(\Lambda/s) \equiv \frac{i^{n+m}}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} \partial_q^n \partial_\omega^m G_0(q,\omega) \delta(|q| - \Lambda/s), \quad (55)$$

so that we can write the expression

$$\begin{aligned} F_2[\varphi^s] = -\frac{v}{4} \int dx dt \Big[& f_{\partial_q^0 \partial_\omega^0}(\Lambda/s)(a^s(x,t))^2 \\ & - f_{\partial_q^1 \partial_\omega^0}(\Lambda/s) a^s(x,t) \partial_x a^s(x,t) \\ & - f_{\partial_q^0 \partial_\omega^1}(\Lambda/s) a^s(x,t) \partial_t a^s(x,t) \\ & + \frac{1}{2} f_{\partial_q^2 \partial_\omega^0}(\Lambda/s) a^s(x,t) \partial_x^2 a^s(x,t) \\ & + \frac{1}{2} f_{\partial_q^0 \partial_\omega^2}(\Lambda/s) a^s(x,t) \partial_t^2 a^s(x,t) \\ & + f_{\partial_q^1 \partial_\omega^1}(\Lambda/s) a^s(x,t) \partial_x \partial_t a^s(x,t) \Big]. \end{aligned}$$

From (6) and (12), we have the following relations:

$$\begin{aligned} a^s(x) &= -\beta g \sin(\beta\varphi^s), \\ \left(a^s(x)\right)^2 &= (\beta g)^2 \sin^2(\beta\varphi^s) \\ &= \frac{(\beta g)^2}{2} \left[1 - \cos(2\beta\varphi^s)\right], \\ a^s(x) \partial_{x,t} a^s(x) &= \beta(\beta g)^2 \sin(\beta\varphi^s)\cos(\beta\varphi^s) \partial_{x,t}\varphi^s \\ &= \frac{\beta^3 g^2}{2} \sin(2\beta\varphi^s) \partial_{x,t}\varphi^s, \end{aligned}$$

and

$$a^s(x) \partial_{x,t}^2 a^s(x) = \beta(\beta g)^2 \sin(\beta\varphi^s) \left[-\beta \sin(\beta\varphi^s)(\partial_{x,t}\varphi^s)^2 + \cos(\beta\varphi^s) \partial_{x,t}^2 \varphi^s \right],$$

which can also be written in the form

$$a^s(x) \partial_{x,t}^2 a^s(x) = -\frac{\beta^4 g^2}{2} [1 - \cos(2\beta\varphi^s)](\partial_{x,t}\varphi^s)^2 + \frac{\beta^3 g^2}{2} \sin(2\beta\varphi^s) \partial_{x,t}^2 \varphi^s,$$

and also the relation

$$a^s(x) \partial_x \partial_t a^s(x) = -\frac{\beta^4 g^2}{2} [1 - \cos(2\beta\varphi^s)](\partial_x\varphi^s)(\partial_t\varphi^s) + \frac{\beta^3 g^2}{2} \sin(2\beta\varphi^s) \partial_x \partial_t \varphi^s.$$

Since the oscillatory contributions to the integrals have zero average, we only have to keep the non-oscillatory contributions, which yield the relation

$$\begin{aligned} F_2[\varphi^s] = & -\frac{(\beta g)^2 V}{8} f_{\partial_q^0 \partial_\omega^0}(\Lambda/s) \\ & + \frac{\beta^4 g^2 v}{8} \int dx dt \Big[\frac{1}{2} f_{\partial_q^2 \partial_\omega^0}(\Lambda/s)(\partial_x\varphi^s)^2 \\ & + \frac{1}{2} f_{\partial_q^0 \partial_\omega^2}(\Lambda/s)(\partial_t\varphi^s)^2 \\ & + f_{\partial_q^1 \partial_\omega^1}(\Lambda/s)(\partial_x\varphi^s)(\partial_t\varphi^s) \Big], \end{aligned} \quad (56)$$

where $V \equiv v \int\int dx dt$ is the volume of the system, in space and time.

The following three expressions are special instances of (55):

$$f_{\partial_q^2 \partial_\omega^0}(\Lambda/s) = -\frac{1}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} \partial_q^2 G_0(q,\omega) \delta(|q| - \Lambda/s), \quad (57a)$$

$$f_{\partial_q^0 \partial_\omega^2}(\Lambda/s) = -\frac{1}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} \partial_\omega^2 G_0(q,\omega) \delta(|q| - \Lambda/s), \quad (57b)$$

and

$$f_{\partial_q^1, \partial_\omega^1}(\Lambda/s) = -\frac{1}{v} \int_{\text{shell}} \frac{dq d\omega}{(2\pi)^2} \partial_q \partial_\omega G_0(q,\omega) \delta(|q| - \Lambda/s). \quad (57c)$$

The derivatives in the integrands on the right-hand sides can be easily obtained from (19):

$$\partial_q^2 G_0(q,\omega) = \frac{4}{\left(q^2 + \omega^2/v^2\right)^2} \left(1 - \frac{4q^2}{q^2 + \omega^2/v^2}\right),$$

$$\partial_\omega^2 G_0(q,\omega) = \frac{4}{v^2 \left(q^2 + \omega^2/v^2\right)^2} \left(1 - \frac{4\omega^2/v^2}{q^2 + \omega^2/v^2}\right),$$

and

$$\partial_q \partial_\omega G_0(q,\omega) = -\frac{16 q\omega/v^2}{\left(q^2 + \omega^2/v^2\right)^3}.$$

In polar coordinates, (57) therefore reads

$$f_{\partial_q^2\partial_\omega^0}(\Lambda/s) = -\int_0^{2\pi}\int_{\Lambda/s}^{\Lambda}\frac{d\theta d|q||q|}{(2\pi)^2}\frac{4}{|q|^4} \times\Big(1-4\cos^2(\theta)\Big)\delta(|q|-\Lambda/s) = \frac{2s^3}{\pi\Lambda^3}, \tag{58a}$$

$$f_{\partial_q^0\partial_\omega^2}(\Lambda/s) = -\int_0^{2\pi}\int_{\Lambda/s}^{\Lambda}\frac{d\theta d|q||q|}{(2\pi)^2}\frac{4}{v^2|q|^4} \times\Big(1-4\sin^2(\theta)\Big)\delta(|q|-\Lambda/s) = \frac{1}{v^2}\frac{2s^3}{\pi\Lambda^3}, \tag{58b}$$

and

$$f_{\partial_q^1\partial_\omega^1}(\Lambda/s) = \int_0^{2\pi}\int_{\Lambda/s}^{\Lambda}\frac{d\theta d|q||q|}{(2\pi)^2}\frac{16}{v|q|^4} \times\cos(\theta)\sin(\theta)\delta(|q|-\Lambda/s) = 0. \tag{58c}$$

Armed with these results, we go back to (56), absorb the constant term on the right-hand side, and integrate the second term on the right-hand side by parts to show that

$$F_2[\varphi^s] = -\frac{\beta^4 g^2 s^3 v}{8\pi\Lambda^3}\int dxdt\left[\varphi^s\partial_x^2\varphi^s+\varphi^s\frac{1}{v^2}\partial_t^2\varphi^s\right], \tag{59}$$

or using the relation (5) for the Laplacian,

$$F_2[\varphi^s] = -\frac{\beta^4 g^2 s^3}{8\pi\Lambda^3}\int dx\varphi^s\nabla_x^2\varphi^s. \tag{60}$$

We can now take advantage of (50) and expand the coefficient multiplying the integral on the right-hand side of (60) around $dl = 0$ ($s = 1$), to find the result

$$F_2[\varphi^s] = -\left(\frac{\beta^4 g^2}{8\pi\Lambda^3}-\frac{dl3\beta^4 g^2}{8\pi\Lambda^3}\right)\int dx\varphi^s\nabla_x^2\varphi^s,$$

and since the first term within parentheses on the right-hand side vanishes in the limit of large momentum–frequency cutoff Λ, we can see that

$$F_2[\varphi^s] = \frac{dl3\beta^4 g^2}{8\pi\Lambda^3}\int dx\varphi^s\nabla_x^2\varphi^s. \tag{61}$$

Recall from (44) and (51) that $F_1[\varphi^s]$ and $F_2[\varphi^s]$ are the first and second terms on the right-hand side of (42). Substitution in the latter equation on the left-hand sides of (49) and (61) then yields the result

$$\delta S_{\text{eff}}[\varphi^s] = \int dx\left[\left(\frac{dl3\beta^4 g^2}{8\pi\Lambda^3}\right)\varphi^s\nabla_x^2\varphi^s -g\left(\frac{dl\beta^2}{4\pi}\right)\cos(\beta\varphi^s)\right].$$

According to (43) and (4)–(6), then, the full slow-mode effective action can be finally written as

$$S_{\text{eff}}[\varphi^s] = \int dx\left[\frac{1}{2}\left(1+\frac{dl3\beta^4 g^2}{4\pi\Lambda^3}\right)\varphi^s\nabla_x^2\varphi^s +g\left(1-\frac{dl\beta^2}{4\pi}\right)\cos(\beta\varphi^s)\right]. \tag{62}$$

To go back to the original scale, in which $\varphi^s\to\varphi$, we carry out the substitutions

$$(x, vt)\to\left(\frac{x}{s},\frac{vt}{s}\right),$$

so that in (62),

$$\int dx\to\frac{1}{s^2}\int dx;\quad \nabla_x^2\to s^2\nabla_x^2.$$

The rescaled action is therefore given by the expression

$$S_R[\varphi] = \int dx\left[\frac{1}{2}\left(1+\frac{dl3\beta^4 g^2}{4\pi\Lambda^3}\right)\varphi\nabla_x^2\varphi +gs^{-2}\left(1-\frac{dl\beta^2}{4\pi}\right)\cos(\beta\varphi)\right]. \tag{63}$$

Under the field transformation $\beta\varphi\to\varphi$, the rescaled and original actions become

$$S_R[\varphi] = \int dx\left[\frac{1}{2\beta^2}\left(1+\frac{dl3\beta^4 g^2}{4\pi\Lambda^3}\right)\varphi\nabla_x^2\varphi +gs^{-2}\left(1-\frac{dl\beta^2}{4\pi}\right)\cos(\varphi)\right] \tag{64}$$

and

$$S[\varphi] = \int dx\left[\frac{1}{2\beta^2}\varphi\nabla_x^2\varphi+g\cos(\varphi)\right], \tag{65}$$

respectively.

3.6 Application II: the Sine-Gordon Model Flow Equations

The RG statement regarding the scale invariance of the theory amounts to matching (64) to (65). The equivalence between these two expressions yields the following prescription for writing the renormalized parameters in terms of the bare ones and the rescaling variable dl:

$$\beta_R^{-2} = \beta^{-2}\left(1+\frac{dl3\beta^4 g^2}{4\pi\Lambda^3}\right), \tag{66}$$

and

$$g_R = gs^{-2}\left(1-\frac{dl\beta^2}{4\pi}\right). \tag{67}$$

Defining the differential of a parameter as $dX \equiv X_R - X$, we can rewrite (66) in differential form as

$$d\beta^{-2} = \frac{3\beta^2 g^2}{4\pi\Lambda^3} dl \tag{68}$$

equivalent to

$$\frac{d\beta^{-2}}{dl} = \frac{3\beta^2 g^2}{4\pi\Lambda^3} \tag{69}$$

and since

$$\frac{d\beta^{-2}}{dl} = -\left(\beta^2\right)^{-2} \frac{d\beta^2}{dl}, \tag{70}$$

we find that

$$\frac{d\beta^2}{dl} = -\frac{3\beta^6 g^2}{4\pi\Lambda^3}.$$

Likewise, we can rewrite (67) in the form

$$g_R = g(1+2dl)\left(1 - \frac{dl\beta^2}{4\pi}\right),$$

from which it follows that

$$dg = g\left(2 - \frac{\beta^2}{4\pi}\right) dl \tag{71}$$

or the differential equation

$$\frac{dg}{dl} = 2g\left(1 - \frac{\beta^2}{8\pi}\right).$$

The RG flow equations for the sine-Gordon model parameters are given in terms of the scale $l = -\ln(s) + l_0$ by the equalities

$$\begin{cases} \dfrac{du}{dl} = 2u(1-K), \\ \dfrac{dK}{dl} = -u^2 K^3, \end{cases} \tag{72}$$

where

$$K = \frac{\beta^2}{8\pi}, \tag{73a}$$

and

$$u = 4\sqrt{\frac{3\pi}{\Lambda^3}}\, g. \tag{73b}$$

4 Kosterlitz–Thouless Phase Diagram

4.1 Analysis of the Flow Equations

A solution of the system of first-order coupled differential (72) can be depicted as a path $(K(l), u(l))$ in the $K-u$ plane, where l is a parametric running variable. Each such path describes the flow of a point initially at $(K_0, u_0) \equiv (K(l_0), u(l_0))$ as the variable l starts at l_0 and runs towards the original scale of the model, i.e., to recover the full momentum–frequency space. The phase diagram of (72) is given by the collection of all possible paths $(K(l), u(l))$ in the $K-u$ plane.

To discuss such paths, we start out with the notion that (72) implies $du/dl = dK/dl = 0$ for $u = 0$. We say that the phase diagram has a line of fixed points at $u = 0$, meaning that the flow stops if and when it reaches that line for some $l = l^*$. The system parameters will take on the value $(K(l^*), 0)$ for any scale $l \geq l^*$. In particular and as expected, a system with bare parameters $(K_0, 0)$ will not flow at all, i.e., a free quadratic model will remain unrenormalized under a scale transformation.

Next, we see from the first of (72) that the flow of the coupling u (i) points upward inside the $K < 1$ half of the phase diagram and downward inside the $K > 1$ half, if $u > 0$; (ii) points downward inside the $K < 1$ half of the phase diagram and upward inside the $K > 1$ half, if $u < 0$; and (iii) points horizontally at $K = 1$.

Therefore, the fixed points along the $u = 0$ line are unstable for $K < 1$, since a point just above or just below the segment $\{K < 1, u = 0\}$ will flow away from it, and stable for $K > 1$, since a point just above or just below the segment $\{K > 1, u = 0\}$ will flow towards it.

The second of (72) implies that the flow of the parameter K (iv) points to the left inside the $K > 0$ half of the phase diagram and (v) points to the right inside the $K < 0$ half, regardless of the value of u, and (vi) stops as K reaches the $K = 0$ line. In this case, according to items (i) and (ii), u flows up (vertically) for $u > 0$ and down (vertically) for $u < 0$. In particular, this shows that no path can cross the $K = 0$ line, where the flow of K changes direction.

Combining the above conclusions, we identify three regions in the phase diagram:

1. *Strong-coupling regime*: The region of paths $(K(l), u(l))$ constrained to the $K < 1$ half of the phase diagram, which flow to the large-$|u|$ and small-$|K|$ regime. In this regime, the interaction, whose strength is proportional to $|u|$, is said to be relevant.
2. *Vanishing-coupling regime*: The region of paths $(K(l), u(l))$ constrained to the $K > 1$ half of the phase diagram, which flow to the regime of vanishing u and fixed K. In this regime, the interaction is irrelevant.
3. *Crossover regime*: The region of paths $(K(l), u(l))$ that go from the $K > 1$ into the $K < 1$ half of the phase diagram, thus initially flowing towards a minimum value of $|u|$ attained at $K = 1$, where $du/dl = 0$. Past this point, these paths turn into the regime of large $|u|$ and small positive K. In this case, the interaction is said to be marginal.

According to item (iv), there are no paths running in the opposite direction, i.e., from $0 < K < 1$ to $K > 1$.

Let us complement this discussion with a simple algebraic analysis. We focus on the region around the line $K=1$, where the interesting physics takes place. Then, letting

$$K = 1 + v, \tag{74}$$

we can rewrite (72), to first order in v, as

$$\begin{cases} \dfrac{du}{dl} = -2uv, \\ \dfrac{dv}{dl} = -u^2(1+3v). \end{cases}$$

These equalities can be rewritten, up to first order in v, in the form

$$\begin{cases} u\dfrac{du}{dl} = -2u^2v, \\ v\dfrac{dv}{dl} = -u^2v, \end{cases}$$

from which it immediately follows that

$$\begin{cases} \dfrac{du^2}{dl} = -4u^2v, \\ \dfrac{dv^2}{dl} = -2u^2v. \end{cases}$$

Multiplication of the bottom equation by 2 and subtraction then shows that

$$\frac{d}{dl}\left(u^2 - 2v^2\right) = 0, \tag{75}$$

or in view of (74),

$$\frac{d}{dl}\left(u^2 - 2(K-1)^2\right) = 0. \tag{76}$$

The difference $u^2 - 2(K-1)^2$ is therefore an invariant for each solution $(K(l), u(l))$, i.e.,

$$u^2(l) - 2(K(l)-1)^2 = c, \tag{77}$$

where, for a given solution, c is a constant determined, for example, by the initial conditions, $c = u_0^2 - 2(K_0-1)^2$.

Now, let $(K_f, |u| \to 0)$ be the extreme point of a path $(K(l), u(l))$ flowing to or from the line of fixed points $u=0$. Equation (77) implies that

$$K_f = 1 \pm \sqrt{\frac{-c}{2}}.$$

Unless $c \leq 0$, K_f cannot exist, implying that the path cannot flow to or from the fixed-point line. More specifically, we have three alternatives:

$c = 0$

$$u(l) = \pm\sqrt{2}(K(l)-1)$$
$$\text{fixed point:} \quad (K_f, u) = (1, 0)$$

$c < 0$

$$|u(l)| < \sqrt{2}|K(l)-1|$$
$$\text{fixed point:} \quad (K_f, u) = \left(1 \pm \sqrt{\tfrac{-c}{2}}, 0\right)$$

$c > 0$

$$|u(l)| > \sqrt{2}|K(l)-1|$$
$$\nexists \text{fixed points.}$$

On the basis of this analysis, we draw in Fig. 1 the well-known Kosterlitz–Thouless (K-T) phase diagram for the sine-Gordon model. The model undergoes a K-T phase transition [25, 26] when the pair (K_0, u_0) varies across the boundaries of the K-T phase diagram. The straight lines $u = \pm\sqrt{2}(K-1)$, given by the condition $c=0$, are called separatrices, because they define the boundaries between the strong-coupling, vanishing-coupling, and crossover regimes. The strong- and vanishing-coupling regimes consist of the family of hyperbolae defined by the condition $c < 0$, which can be seen as enclosed by the separatrices, while the crossover regime corresponds to the hyperbolae defined by $c > 0$—the region outside the separatrices.

4.2 Gap

In both the strong-coupling and crossover regimes, the flows are towards large $|u|$. At some critical scale along these flows, call it l_c, the interaction becomes so strong that the system undergoes a phase transition. At l_c, scale invariance is lost, and the RG statement, no longer valid. Based on the perturbative nature of the RG procedure, a reasonable estimate for l_c is the scale along the flow at which the $|u|$ becomes unitary. The critical correlation length ξ_c can be evaluated by means of the expression $\xi_c \propto \exp(l_c)$.

Since the gap Δ is proportional to ξ_c^{-1}, the equality

$$\Delta = \exp(-l_c)$$

gives an estimate for the gap—except for a multiplicative energy factor—opening up in a system that starts at $|u_0| < 1$ and flows to the large $|u|$-regime.

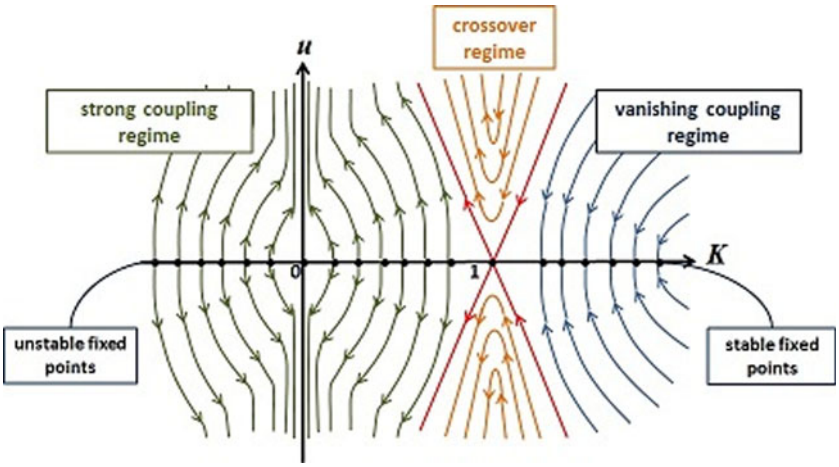

Fig. 1 Kosterlitz–Thouless phase diagram for the sine-Gordon model

We now have to determine l_c and therefore Δ, as a function of the sine-Gordon bare parameters K_0 and $|u_0| < 1$. A first approximation is to consider the perturbative RG only up to first order in the coupling constant g. A straightforward integration of the flow equation for u then determines l_c.

To first order in g, we are led to simplified flow equations of the form

$$\begin{cases} \dfrac{du}{dl} = 2u(1-K), \\ \dfrac{dK}{dl} = 0, \end{cases}$$

as can be seen directly from (72) with $u \to 0$.

Since $K = K_0$ is now a constant, we can write that

$$\int_{u_0}^{u} \frac{du'}{u'} = 2(1-K_0)\int_{l_0}^{l} dl',$$

from which it follows that

$$u(l) = u_0 \exp[2(1-K_0)(l-l_0)].$$

We see that, for $K_0 < 1$, $|u|$ grows without bound, while for $K_0 > 1$, $|u|$ decreases until it reaches the fixed-point line $u = 0$. First-order perturbative RG cannot capture crossover paths. In particular, at this level of approximation, $K = 1$ represents a line of fixed points. As an illustration, the sine-Gordon model phase diagram resulting from first-order perturbative RG is sketched in Fig. 2.

Back to the gap, for $K_0 < 1$ and $|u_0| < 1$, we can write that

$$1 = |u_0| \exp[2(1-K_0)(l_c - l_0)],$$

which is equivalent to the expression

$$l_c = l_0 + \ln\left(|u_0|^{1/2(K_0-1)}\right).$$

Therefore, to first order

$$\Delta = \Delta(K_0, u_0) = \begin{cases} c_0|u_0|^{1/2(1-K_0)} & \Big(K_0 \le 1(\text{and}|u_0| < 1)\Big), \\ 0 & (K_0 \ge 1). \end{cases} \tag{78}$$

As shown in Fig. 3, for given $|u_0| < 1$, the gap decreases with K_0 (since $|u_0| < 1$) until it reaches zero at the critical value $K_0^c = 1$. On the other hand, given $K_0 < 1$, the gap increases with $|u_0|$ until it reaches the maximum $c_0 = \exp(-l_0)$, corresponding to $|u_0| = 1$. This behavior of the gap with K_0 and $|u_0|$ reflects the shrinking of the critical correlation length as one goes deeper into the strong-coupling regime, i.e., as $|u_0|$ increases and K_0 decreases.

The line $K = 1$ in Fig. 2 defines the boundary between the gapless (vanishing-coupling regime) and gapped (strong-coupling regime) regions of the phase diagram. The system undergoes a phase transition between the gapless and gapped phases when the parameter K_0 crosses the line $K = 1$.

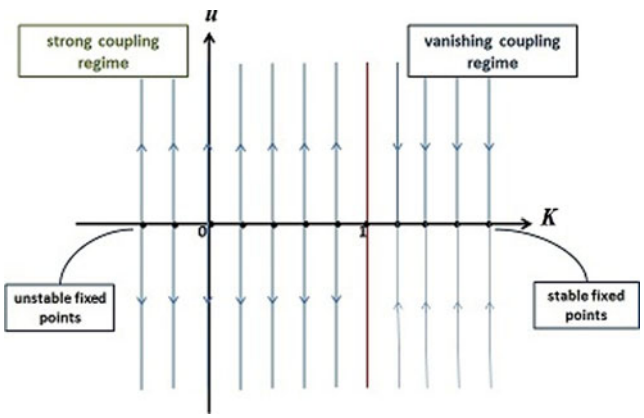

Fig. 2 Phase diagram for the sine-Gordon model resulting from first-order perturbative RG

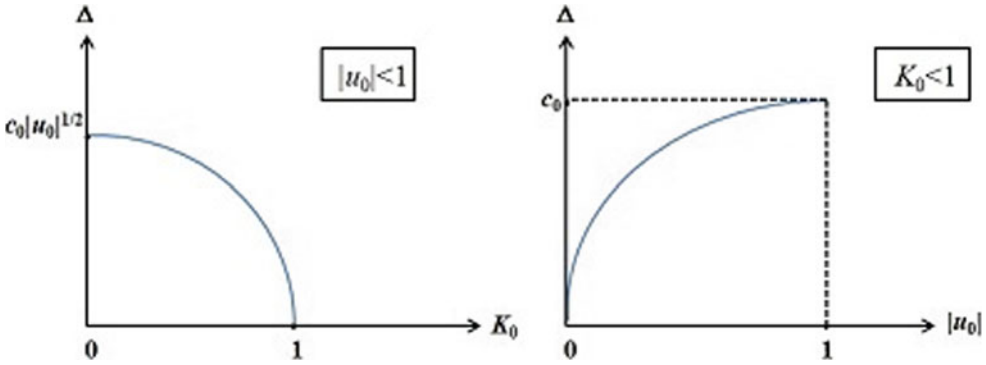

Fig. 3 First-order gap as a function of K_0 (*left panel*) and $|u_0|$ (*right panel*)

To describe the gap more accurately than the first-order result in (78) and Fig. 3, we expand the gapped region into the crossover regime of Fig. 1, where marginal paths may start in the region $K > 1$ but ultimately flow into the large $|u|$ regime. We must keep in mind that the border line in the K-T phase diagram is no longer given by $K = 1$, but by $K = 1 \pm u/\sqrt{2}$, where the upper (lower) sign corresponds to $u > 0$ ($u < 0$).

When the expansion is combined with the first-order results of Fig. 3, we are led to the qualitative picture for the gap, resulting from second-order perturbative RG, depicted in Fig. 4.

For given $|u_0| < 1$, the gap decreases with K_0 until reaching zero at the critical value $K_0^c = 1 + |u_0|/\sqrt{2}$. The dashed line in the panel on the left represents the first-order approximation. The difference between the two curves is due to the contribution of marginal paths to the gap. As indicated by the second equality in (72), for small K_0, the parameter K remains nearly constant along the second-order RG flow. In other words, close to the $K = 0$ line, the second-order flow is essentially vertical, like the first-order flow. For small K_0, the first- and second-order approximations should therefore give roughly the same gap, as shown by Fig. 4. As K_0 increases, the first- and second-order lines depart from each other and flow to distinct critical points.

For $K_0 < 1$, the dependence of the gap on $|u_0|$ should be similar to that in the first-order approximation. The same reasoning is even qualitatively inapplicable to $K_0 > 1$, because marginal flows starting with $K_0 > 1$ are nontrivial. In particular, since such flows are more extended, the critical scale l_c, at which a gap opens up, may exceed the cutoff, in practice implying the absence of phase transition. Nonetheless, deep inside the strong-coupling sector of the K-T phase diagram, where the gap is larger, the estimate $\Delta = c_0|u_0|^{1/2(1-K_0)}$ remains quantitatively reliable.

5 Applications to Condensed Matter Physics

The main motivation behind applications of the sine-Gordon model to condensed matter physics comes from its equivalence to the bosonized version of the fermionic g-ology or Hubbard models for one-dimensional interacting-electron systems.

As it has been known for several decades that interacting-electron systems in dimensions $D > 1$ are well described by the Landau–Fermi liquid theory [27–29]. In one dimension, however, the Fermi liquid picture fails due to the instability— the *Peierls instability* [30]— generated by $2k_F$ scattering processes, which become particulary important because the Fermi surface of one-dimensional systems comprises only two points. The failure of the Fermi liquid theory in $D = 1$ received first formal attention in 1950, when Tomonaga proposed that the fermionic excitations in one dimension should be understood as "quantized sound waves," that is, phonon-like bosonic excitations [31]. In

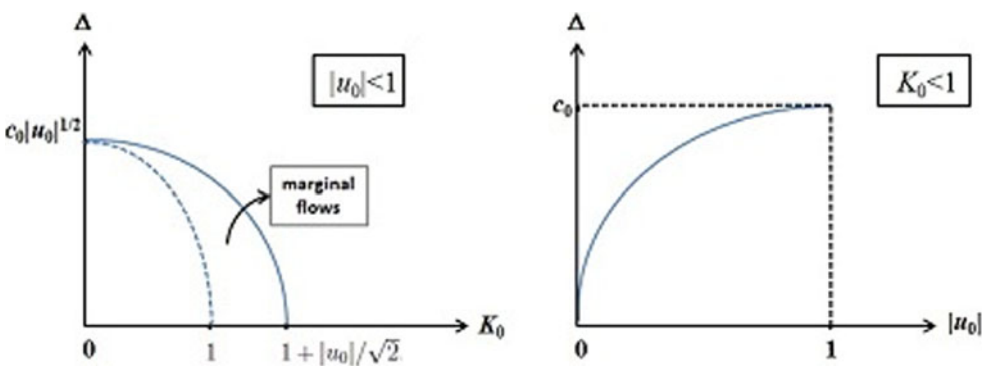

Fig. 4 Qualitative depiction of the second-order gap as a function of K_0 (*left panel*) and $|u_0|$ (*right panel*)

1963, Luttinger extended this idea and proposed a model, for which he obtained an incorrect solution [32]. The correct solution would came in 1965, with the work of Mattis and Lieb [33]. The term *Luttinger liquid*, now ordinarily applied to designate one-dimensional electronic systems, was only coined in 1981, by Haldane, who proposed a physical interpretation for the bosonic collective modes of single-particle excitations of fermions in one dimension [34].

Among the models and formalisms that have been proposed to investigate the special behavior of Luttinger liquids, the sine-Gordon model is of special interest because the studies focused on it have accumulated vast knowledge. Through a bosonization procedure, the sine-Gordon bare parameters u_0 and K_0 and the non-renormalized velocity v are connected to the original microscopic couplings defined for the fermionic models. A comprehensive review on bosonization methods can be found in Giamarchi's book *Quantum Physics in One Dimension* [7].

5.1 The g-ology Model

The following expressions relates the sine-Gordon bare parameters $\tilde{g}_0$, K_0, and v to the microscopic couplings of the one-dimensional g-ology model (see p. 53, Eqs. (2.105)–(2.106) adapted to the present notation, in [7]):

$$\tilde{g}_0 \to \tilde{g}_{0\nu} = \begin{cases} 0 & (\nu = c) \\ \dfrac{-2g_{1\uparrow\downarrow}}{(2\pi\alpha)^2} & (\nu = s) \end{cases}, \tag{79}$$

$$K_0 \to K_{0\nu} = \left[\frac{1 + y_{4\nu}/2 + y_\nu/2}{1 + y_{4\nu}/2 - y_\nu/2}\right]^{1/2}, \tag{80}$$

and

$$v \to v_\nu = v_F\left[(1 + y_{4\nu}/2)^2 - (y_\nu/2)^2\right]^{1/2}, \tag{81}$$

where

$$y_\nu \equiv \frac{g_\nu}{\pi v_F}, \tag{82}$$

$$g_\nu = g_{1\uparrow\uparrow} - g_{2\uparrow\uparrow} \mp g_{2\uparrow\downarrow}, \tag{83}$$

and

$$g_{4\nu} = g_{4\uparrow\uparrow} \pm g_{4\uparrow\downarrow}, \tag{84}$$

and the subindexes $\nu = c, s$ refer to the charge- and spin-separated sectors of the full bosonized Hamiltonian, respectively. In (83) and (84), the upper signs refer to c and the lower ones to s.

In the standard g-ology notation, the coupling g_4 corresponds to forward scattering between electrons of equal chirality, while g_2 and g_1 correspond to forward and backscattering, respectively, between electrons of different chiralities. The intensity of each g-scattering may depend on whether the spins of the two interacting electrons are parallel ($g_{\uparrow\uparrow}$) or antiparallel ($g_{\uparrow\downarrow}$). For spinless fermions, the g_2 and g_1 processes are identical, since one can exchange the indiscernible outgoing particles. Once the spin enters the picture, these two processes become intrinsically different and contribute to the bosonized theory in different ways, as can be seen from (79), (83).

Most frequently, when writing models for interacting electrons, one is concerned with the standard Coulomb repulsion between the particles. In the present context, the repulsion translates into positive g-couplings for all processes. However, electrons can sometimes interact attractively, e.g., through phonon-mediated coupling. To accommodate attractions, we allow for (certain) processes with negative g-couplings.

Equations (79)–(84) show that the Luttinger liquid separates into a charge sector, described by a model of free bosons with velocity v_c, and a spin sector mapping into a bosonic sine-Gordon model with parameters v_s, $\tilde{g}_{0s}$, and K_{0s}. Since $v_c \neq v_s$, charge and spin excitations travel independently.

From the point of view of the original electronic system, the massless charge sector represents electrons in a metallic phase. The behavior of the spin sector, albeit not as simple, can be understood on the basis of the sine-Gordon model phase diagram with bare parameters determined by (79)–(84). Figure 5 depicts this phase diagram.

A few general observations seem appropriate. First, note that (80) excludes the negative-K half of the full K-T phase diagram. In fact, from (73), negative K corresponds to imaginary β, which in turn introduces a hyperbolic cosine in (1). Therefore, although one might explore its mathematical implications, the case of $K < 0$ is of no physical interest.

Next, note that (80)–(83) imply $K_{0s} > 1$ for $g_{1\uparrow\uparrow} > 0$ and g_2-processes of comparable intensity, i.e., for $\tilde{g}_{0s} < 0$—we are assuming that $g_{1\uparrow\uparrow}$ and $g_{1\uparrow\downarrow}$ have the same sign—and vice versa. Therefore, for repulsive g_1-interaction, the only physically meaningful region of the full K-T phase diagram is the one bounded above by the line

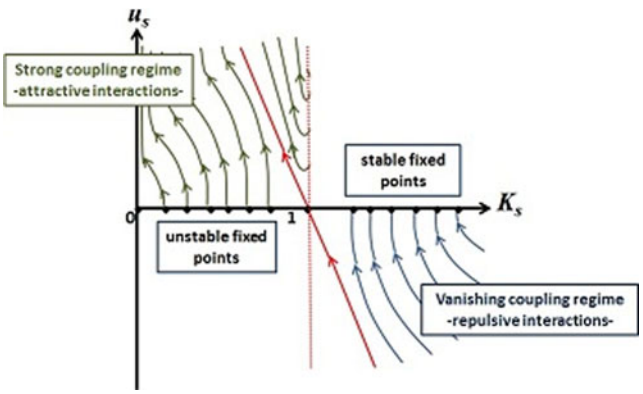

Fig. 5 Phase diagram for the spin sector of the g-ology model

$u_s = 0$ and to the left by the line $K_s = 1$. By contrast, an attractive g_1-interaction corresponds to the region bounded below by the line $u_s = 0$ and to the right and left by the lines $K_s = 1$ and $K_s = 0$, respectively.

Finally, note that half of the upper crossover regime of the full K-T phase diagram was incorporated in the strong-coupling regime, since u_s increases monotonically along the remaining part of the (now relevant) flows. The lower crossover regime, which would have to be artificially interrupted at the $K_s = 1$ line, can be altogether excluded on the basis of the weak-interaction (i.e., small-$|u_{0s}|$) argument.

A number of conclusions can be gleaned from the phase diagram. Repulsive backscattering processes in one-dimensional electronic systems are irrelevant and the resulting gapless spin excitations behave, in effect, as a collection of free bosons propagating with velocity v_s given by (81)–(84). Attractive backscattering processes flow to the regime of strong interactions and thus are relevant and open a gap in the spin sector. A gapped spin excitation implies trapping of the spin φ_s-field at a cosine minimum and order that breaks rotational symmetry. The Mermin–Wagner theorem [35] bars the breaking of a continuous symmetry in $1 + 1$ dimension, but not the breaking of a discrete symmetry. Assuming that the repulsive or attractive nature of the electronic interactions is a definite property of a given system, then it is impossible to drive a phase transition by varying the pair (K_{0s}, u_{0s}) across the point $(1, 0)$.

5.2 The g-ology Model at Commensurate Fillings—Umklapp Processes

In one-dimensional electron systems with commensurate fillings, a fourth type of interaction, *umklapp* scattering, arises. The correspondent coupling constant is termed g_3 in the g-ology dictionary. Best known is the case of half-filling, which corresponds to scattering of two left movers to the other side of the Fermi level through a momentum transfer of $4k_F$ from the lattice. For quarter-filling, an umklapp will result from the similar scattering of four particles with $8k_F$ momentum transfer.

In any case, given that the filling is commensurate, the bare parameter $\tilde{g}_{0c}$ of (79), no longer zero, is associated with a cosine perturbation of the type

$$+\tilde{g}_{0c} \cos\left(n.\sqrt{8\pi K_{0c}}\varphi - \delta x\right) \tag{85}$$

where

$$\tilde{g}_{0c} = \tilde{g}^n_{0c} = \frac{2g_{3,n}}{(2\pi\alpha)^2}. \tag{86}$$

Here, n is the order of the commensurability—for example, $n = 1$ corresponds to half-filling, and $n = 2$ to quarter-filling—which defines the amplitude and the wavelength of the cosine potential, and the parameter δ is the doping, which measures the deviation from commensurability.

Due to the phase shift δx, the perturbation will oscillate fast, and its spatial integral will vanish unless $\delta x \to 0$. In other words, away from a commensurate filling (finite δ), umklapp is absent, and we recover the previous picture of free bosonic charge excitations. But at commensurate filling ($\delta = 0$), the Luttinger liquid separates into two independent sine-Gordon models: one for the charge sector with parameters v_c, $\tilde{g}_{0c}$, and $\bar{K}_{0c} = n^2 K_{0c}$ and one for the spin sector with parameters v_s, $\tilde{g}_{0s}$, and K_{0s} (with v_ν, $\tilde{g}_{0\nu}$, and $K_{0\nu}$ given by (79)–(86)).

Figure 6 shows the phase diagram for the charge sector of the g-ology model at half-filling ($n = 1$), assuming a positive umklapp coupling $g_{3,n}$. For the charge sector, (80)–(83) imply that $K_{0c} > 1$ if $g_{1\uparrow\uparrow} > g_{2\uparrow\uparrow} + g_{2\uparrow\downarrow}$, with no implication on the value of $\tilde{g}_{0c}$, which is proportional to $g_{3,n}$ and therefore positive. This condition raises a number of possible scenarios. For repulsive interactions, it is satisfied for backscattering (between electrons having parallel spins) that is more than twice as intense as forward scattering. The opposite holds for attractive interactions. If backscattering is repulsive and forward scattering is attractive, the condition is always verified. In the opposite scenario, the condition is never verified. In general, it is possible to drive a phase transition in the charge sector of a one-dimensional commensurate electronic system by tuning the strength of the interactions so that the pair (K_{0c}, u_{0c}) moves across the separatrix $K_c = 1 + u_c/\sqrt{2}$. In the vanishing-coupling regime, umklapp is irrelevant, charge excitations are gapless, and the system is metallic. In the strong-coupling regime, umklapp becomes relevant, the charge excitations develop a gap, and the system becomes an insulator. In the crossover regime, umklapp is marginal.

Another way to drive a metal–insulator phase transition in a one-dimensional electronic system is to change the filling. Given a fixed (K_{0c}, u_{0c}) in the strong-coupling regime,

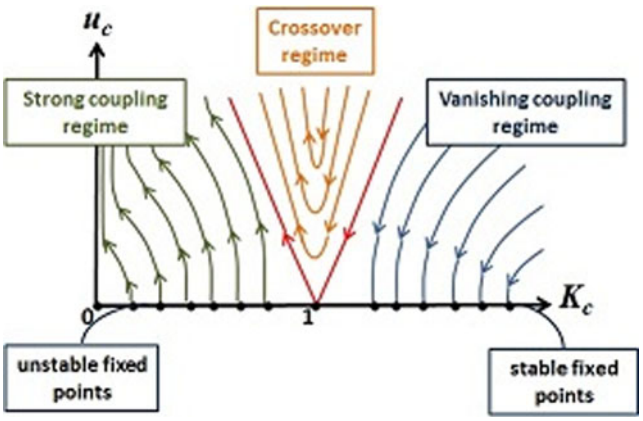

Fig. 6 Phase diagram for the charge sector of the g-ology model at half-filling

the system can undergo a metal–insulator phase transition by varying δ, i.e., the commensurability parameter. This phase transition is of the incommensurate–commensurate type, also known as Mott transition.

5.3 The Hubbard Model

The sine-Gordon bare parameters $\tilde{g}_0$, K_0, and v are related to the couplings of the Hubbard model by the equalities (see p. 202, Eq. (7.9) adapted to the present notation, in [7])

$$\tilde{g}_0 \to \tilde{g}_{0\nu} = \begin{cases} 0 & (\nu = c) \\ \dfrac{-2U}{(2\pi\alpha)^2} & (\nu = s) \end{cases}, \tag{87}$$

$$v K_0 \to v_\nu K_{0\nu} = v_F, \tag{88}$$

and

$$\frac{v}{K_0} \to \frac{v_\nu}{K_{0\nu}} = v_F \left(1 \pm \frac{U}{\pi v_F}\right), \tag{89}$$

where, as before, $\nu = c, s$, the upper sign refers to c and the lower one to s. In the Hubbard model, the coupling U represents an on-site interaction of the g_1 nature with the additional restriction that the (local) interaction can only take place between electrons with opposite spins, to satisfy the Pauli exclusion principle. The Hubbard model can be seen as a simplification of the g-ology model.

From (88) and (89) and the inequality $K > 0$, we find the following relation between K_{0s} and U:

$$K_{0s} = \frac{1}{\sqrt{1 - \dfrac{U}{\pi v_F}}}. \tag{90}$$

Also here, the Luttinger liquid separates into independent charge and spin excitations described by a free model and a sine-Gordon model, respectively, with their respective parameters. From (90), we have that $U > 0$ if and only if $K_{0s} > 1$. Therefore, for a system with a repulsive and sufficiently weak Hubbard interaction, the pair of bare parameters (K_{0s}, u_{0s}) lies inside the $u_s < 0$ vanishing-coupling, irrelevant regime of the full K-T phase diagram. Only the gapless phase is then accessible to the spin system, which consists of free bosonic excitations. Meanwhile, for attractive U, the pair of bare parameters (K_{0s}, u_{0s}) will fall into either the $u_s > 0$ strong-coupling regime or in the left half of the crossover regime, where the interaction becomes relevant. In either cases, the spin sector develops a gap and the spin field orders. The phase diagram of the spin sector is the same as that in Fig. 5, which was obtained in the context of the g-ology model. Here, again, for a metal in which the nature of the Hubbard on-site interaction is either repulsive or attractive, we expect no phase transition between the gapless and the gapped phases of the spin excitations.

If the system is at half-filling, the charge sector develops an umklapp interaction of the form

$$+\tilde{g}_{0c} \cos\left(\sqrt{8\pi K_{0c}}\varphi - \delta x\right), \tag{91}$$

where in the language of the Hubbard model,

$$\tilde{g}_{0c} = \frac{2U}{(2\pi\alpha)^2}. \tag{92}$$

For commensurate fillings other than 1/2, the umklapp interaction is described by similar expressions.

From (88) and (89), we can see that K_{0c} is related to U by the equality

$$K_{0c} = \frac{1}{\sqrt{1 + \dfrac{U}{\pi v_F}}}. \tag{93}$$

Thus, at commensurate filling, the Luttinger liquid separates into two independent sine-Gordon models: one for the charge and one for the spin sector, each with its own parameters. Figure 7 shows the phase diagram for the charge sector of the Hubbard model at half-filling.

From (93), we have that $U > 0$ implies $K_{0c} < 1$, the reciprocal statement also being true. For a repulsive interaction, the pair of bare parameters (K_{0c}, u_{0c}) therefore falls inside either the $u_c > 0$ strong-coupling regime or in the left half of the crossover regime. In both situations, the interaction is relevant, a gap opens, and the system becomes an insulator. On the other hand, a sufficiently weak attractive interaction places (K_{0c}, u_{0c}) inside the $u_c < 0$ vanishing-coupling regime, where the interaction is irrelevant. In this regime, the charge excitations are gapless, so that the system remains in the metallic phase. As before, one cannot drive a metal–insulator phase transition between the repulsive and attractive portions of the phase diagram in a system with interactions of definite nature.

In summary, the Hubbard model describes the following types of one-dimensional systems of interacting electrons: away from commensurability, the system is metallic,

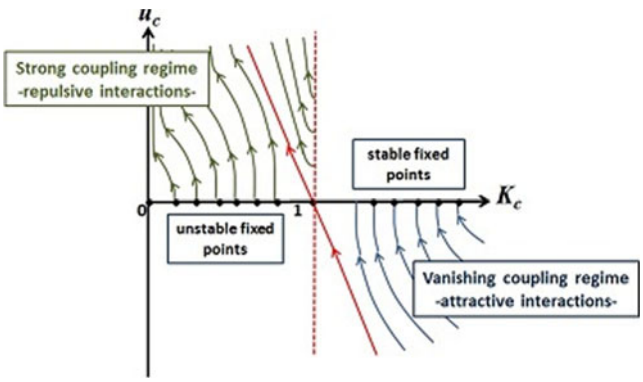

Fig. 7 Phase diagram for the charge sector of the Hubbard model at half-filling

described by gapless charge excitations, and if the Hubbard interaction is repulsive, gapless spin excitations preserving rotational symmetry. If the interaction is attractive, the system exhibits gapped, symmetry-breaking spin excitations. For commensurate filling and repulsive interaction, the system is an insulator with gapped charge excitations and gapless spin excitations. An attractive interaction leads to a metal with gapless charge excitations and gapped spin excitations.

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