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STATISTICAL

How Physicists Made Stable Lévy Processes Physically Plausible

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Abstract Stable Lévy processes have very interesting properties for describing the complex behaviour of non-equilibrium dissipative systems such as turbulence, anomalous diffusion or financial markets. However, although these processes better fit the empirical data, some of their statistical properties can raise several theoretical problems in empirical applications because they generate infinite variables. Econophysicists have developed statistical solutions to make these processes physically plausible. This paper presents a review of these analytical solutions (truncations) for stable Lévy processes and how econophysicists transformed them into data-driven processes. The evolution of these analytical solutions is presented as a progressive research programme provided by (econo)physicists for theoretical problems encountered in financial economics in the 1960s and the 1970s.

Keywords Truncation · Stable Lévy processes · Econophysics

1 Introduction

Stable Lévy processes have very interesting properties for describing the complex behaviour of non-equilibrium dissipative systems such as turbulence, anomalous diffusion or financial markets.[1]

However, although stable processes better fit the empirical data, some of their statistical properties can raise several theoretical problems in empirical applications because they generate infinite variables. Of course, every sample is, by definition, finite, meaning that we always have finite empirical observations implying a 'natural truncation' for the data. The large majority of works dealing with distribution truncation presents this technique as an empirical evidence because all physical systems have finite parameters. In the 1990s, some physicists reversed this usual argument by showing that truncations are not only empirical limitations due to the finiteness of samples but they also can be theoretically defined in order to describe finite physical systems because these truncations can be based on a physically plausible meaning. In this perspective, the truncation techniques can be seen as an analytical tool of understanding physical systems. This theoretical formulation of physically plausible truncation techniques is relatively new since it has been developed in physics in the 1990s [1].

The following pages will present the different analytical solutions provided by physicists in order to make stable Lévy processes physically plausible. The truncation issue presents an interdisciplinary dimension since it has been developed by what we call 'econophysicists', who are physicists applying their knowledge to economics and finance. More precisely, 'econophysics' (for the contraction of economics and physics) refers to the application of models and concepts coming from condensed matter physics to economic and financial

Research highlights The main categories of truncated stable Lévy processes, problems related to infinite variance and asymptotic reasoning, and analytical solutions proposed by econophysicists presented as a theoretical and historical justification of econophysics.

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[1] Some examples will be mentioned in Section 2.2. See [1, 2] or [23] for a specific literature review dedicated to these applications of stable Lévy processes in physics.

phenomena. This article will review the new truncation techniques associated with the emergence of econophysics. Technically speaking, the material presented in this review is often well known by the majority of physicists working on financial market problems, but the real contribution of this paper is, on the one hand, to review the analytical tools developed by econophysicists and, on the other hand, to present this knowledge as an analytical solution to a very old problem encountered by financial economists in the 1960s.

Moreover, this article shows, in contrast to works presenting econophysics as a fragmented and non-unified approach, that this field is a coherent and progressive research program with a real methodological evolution. This paper also deals with connections between mathematics and physics since it explains how a strictly mathematical technique (truncation of stable Lévy processes) requires a physical meaning in order to be effectively used by physicists.

The following two sections will define stable Lévy processes and will mention all physical subfields in which they have been observed. The third section will present the main arguments calling the physical relevance of these processes into question. The fourth part will introduce the analytical solutions developed by econophysicists in order to make Lévy processes physically plausible. Finally, the last section will remind that the non-applicability of stable Lévy processes is a very old issue in financial economics. I will then explain why solutions have been developed by physicists rather than by financial economists.

2 Stable Lévy Processes

2.1 Definition

In order to describe the complex behaviour of dissipative systems without using the Gaussian distribution, physicists used sophisticated processes called stable Lévy processes, which are a specific category of Lévy processes. These Lévy processes refer to a class of infinitely divisible distributions that can take into account skewness and exceed kurtosis (i.e. leptokurticity). These processes have been developed, in the 1920s, by the famous French statistician Paul Lévy who defined a large category of processes that have right-continuous paths with a left limit whose increments are independent and time-homogeneous. The class of Lévy processes is very important in finance because "they are simplest statistical family whose paths consist of continuous motion interspersed with jump discontinuities of random size appearing at random time" [2] (p. 1337). In his work, Lévy demonstrated that the invariance of the distribution form with respect to the addition of independent variables was not specific to Gaussian distribution. Lévy's work led to the identification of specific stable distributions, which appear as a generalization of three well-known distributions: Gaussian, Cauchy[2] and Lévy.[3] They are the only three stable distributions with a closed formula for probability density function. For a long time, this absence of a closed formula defining the probability density had been the main problem[4] with the use of stable Lévy distributions. This problem is now solved since density functions are estimated with specific computer programs.

Although stable distributions are a specific type of Lévy process, they appear as fixed points of a convolution operation.[5] In other words, there are asymptotic attractors for a distribution convoluted with itself a large number of times since it converges towards a stable law.[6] Stable Lévy processes are also infinitely divisible random processes, and they have the property of scaling (self-similarity), meaning that financial variables (daily, weekly and monthly) can be studied through a stable distribution of exactly the same form for each level of scale. Since there is no closed-form formula for

[2] We can find some applications of this framework in the study of resonance in physics [3] or in spectroscopy [4].

[3] This distribution is mainly used to characterize the frequencies of geomagnetic reversals or the length of the path followed by a photon in a turbid medium [4].

[4] The second drawback of these distributions relates to their theoretical properties, which imply that only certain statistical moments exist [5]. It can be shown that the variance does not exist when $\alpha<2$ and that the mean does not exist when $\alpha\leq 1$. More generally, the p^{th} moment exists if and only if $p<\alpha$ [4]. In contrast to this theoretical result, all statistical moments computed from a sample exist because all samples are finite. However, as Nolan [6] and Taleb [7] emphasized it, the problem is that the sample does not tell you much about stable laws because sample variance does not converge to a well-defined moment. This indeterminate variance is a very significant drawback of stable laws, which are therefore often considered as useless for empirical applications.

[5] These asymptotic distributions depend on the fatness of the tails. Crudely speaking, if the distribution has a minimum and a maximum, the limiting distribution will be a Gumbel distribution. If the tails of the distribution decrease quickly (like normal or exponential distributions), the asymptotic distribution will be a Weibull distribution; finally, if tails decrease slowly (like stable Lévy distributions), the limiting distribution will be a Frichet distribution, which takes the form of a power law. See Feller [8, 9].

[6] This is a very interesting property for the statistical analysis of stock returns. As Fama [10] (p. 424) points out, this property is responsible for much of the appeal of stable distributions as descriptions of empirical distributions of price changes. The price change in a speculative series for any time interval can be regarded as the sum of the changes from transaction to transaction during the interval. If the changes between transactions are independent, identically distributed stable variables, daily, weekly and monthly changes will follow stable distributions of exactly the same form, except for origin and scale.

densities, a stable Lévy distribution is often described by its characteristic function $\varnothing(t)$,whose most popular form is given[7] by

$$\log\varnothing(t) = \begin{cases} -\sigma^{\alpha}|t|^{\alpha} & \left\{1 - i\beta \mathrm{sign}(t)\tan\frac{\pi\alpha}{2}\right\} + iut\,,\ \alpha \neq 1 \\ -\sigma\,|t| & \left\{1 - i\beta \mathrm{sign}(t)\frac{2}{\pi}\log|t|\right\} + iut, \alpha = 1 \end{cases} \quad (1)$$

Stable Lévy distributions require four parameters to describe. Depending on the value of the four parameters evoked above, two categories of Lévy processes can then be identified: stable Lévy and non-stable Lévy. More precisely, a random variable X is said to be stable[8] if there are specific values for the parameters (α, β, γ, δ), such as $0<\alpha\leq 2$, $\gamma\geq 0$, $-1\leq\beta\leq 1$, $\gamma\in\Re$. All the stable distributions (Gaussian, Cauchy, Poisson, etc.) can then be found as a function of the value of each parameter (α, β, γ, δ).[9]

The parameterization of these factors generated a lot of theoretical debates because it directly depends on the way of generating the process. There exist several methods based on different ways of generating random variables. We could mention, for example, the generator of Janicki and Weron [13] or the method developed by Chambers et al. [14]. According to some authors [15], the lack of uniformity in the parameterization methods can sometimes lead people to avoid this kind of distribution. Indeed, several techniques can be found in the literature: methods based on statistical orders [16, 17] estimators or those that use the characteristic function [18–21]. Today, the debate of parameter estimation for stable laws remains open. For further information about these methods, see [5, 15] or [22].

2.2 Stable Processes in Physics

From the 1980s onward, stable Lévy processes became increasingly used in physics,[10] particularly in statistical physics. The first studies on the subject were those of Kolmogorov [24, 25] on the scale invariance of turbulence in the 1940s. This theme was subsequently addressed by many physicists and mathematicians, particularly by Mandelbrot in the 1960s when he defined his fractal mathematics[11] and applied it not only to the phenomenon of turbulence but also to the behaviour of financial markets as I will explain it.

According to Hughes [26], power laws and scaling properties appeared in physics at the same period, when Kolgomorov's research (1941) about turbulence has progressively been widespread in the discipline. Progressively scaling laws have been studied by physicists such as Kadanoff [27], Domb and Hunter [28] or Fisher [29]. However, as Stanley [30] explained it, there were no physically based justifications, at that time, for the existence of scaling laws. More precisely, Stanley [30] (p. 18) wrote, "the scaling hypothesis is at best unproved and indeed, to some workers represents an ad hoc assumption entirely devoid of physical content".[12] Hughes [26] explained that throughout the 1970s and the 1980s, physicists got theoretical and experimental evidence supporting the scaling laws and the existence of power laws in a lot of physical phenomena. These evidences have been increasingly observed and published starting from the 1990s [31]: chaotic dynamics of complex systems [32, 33], front dynamics in reaction–diffusion systems [34], fractional diffusion [35], thermodynamics of anomalous diffusion [36], dynamical foundation on non-canonical equilibrium [37], quantum fractional kinetics [38], diffusion by flows in porous media [39], noise amplifications [40], and kinetic Ising and spherical models [41].

This high degree of interest for stable Lévy processes went beyond the frontiers of physics since we observed an increasing number of papers proposing stable Lévy processes as the most appropriate statistical description for a great number of distinct phenomena such as seismic series and earthquakes [42], time series statistical analysis of DNA [43], primary sequences of protein like copolymers [44], spreading of epidemic processes [45], flights of an albatross [46] and human memory retrieval [47].[13] As Dubkov et al. [31] (p. 2650) mentioned,

[7] See Nolan [5] or Samorodnitsky and Taqqu [11] for different equivalent definitions of stable Lévy distributions.

[8] See Weron [12] regarding the fact that $\alpha>2$ does not necessarily exclude the Lévy-stable regime.

[9] The parameter α is called the 'characteristic exponent' and shows the index of stability of the distribution. The value of this exponent refers to the shape of the distribution: the lower α is, the more often extreme events are observed. In financial terms, this parameter is then an indicator of risk since it describes how often important variations can occur. The parameter γ is the scale indicator that can be any positive number. It refers to the 'random size', i.e. the importance of the variance, whose regularity is given by the exponent α. In a sense, γ can be seen as an indicator of statistical dispersion. α and γ allow the risk of a process to be broken down into two types—a 'risk of shape' (α) and a 'size risk' (γ)—which is well known in neoclassical finance since it is supposed to be reduced thanks to diversification. However, financial economists neglect the 'risk of shape', which is often reduced to a Gaussian distribution. The parameter β is the skewness and gives information about the symmetry of the distribution. Finally, the parameter δ is a localization factor that shifts the distribution right if $\delta>0$ and left if $\delta<0$.

[10] See Frisch et al. [23] for an analysis of the influence of Lévy processes in physics.

[11] Although modern probability theory was properly created in the 1930s, in particular through the works of Kolmogorov, it was not until the 1950s that Kolmogorov's axioms became the dominant paradigm in this discipline.

[12] See Stanley [30] for a review of theoretical literature related to the scaling laws in the 1960s.

[13] See Dubkov et al. [31] for an amazing list of stable Lévy-type phenomena in physical, biological, chemical and social systems.

> it is astonishing how the same diffusion equation can describe the behavior of neutrons in a nuclear reactor, the light in the atmosphere, the stock markets values rate on financial exchange, particles of flower dust suspended in a fluid and so on. The fact that completely different by nature phenomena are described by identical equations is a direct indication that the matter concerns not the concrete mechanism of the phenomenon but rather the same common quality of whole class of similar phenomena.

However, although stable Lévy processes can be observed in a large variety of phenomena, the physicality of this statistical pattern generated a lot of debates, as explained in the following section.

3 'Not Physically Relevant!'

Coupled with the growing literature on stable Lévy processes, the old issue of the physically plausible dimension of stable processes reemerged. Gupta et al. [48, 49] rejected the applicability in physics of these processes, claiming that they have "mathematical properties that discourage a physical approach because they have infinite variance" [48] (p. 32). This opposition between theoretical properties and empirical applicability of these processes is clarified by Mantegna and Stanley [50] (p. 4) as follows:

> Stochastic processes with infinite variance, although well-defined mathematically, are extremely difficult to use and, moreover, raise fundamental questions when applied to real systems. For example, in physical systems, the second moment is often related to the system temperature, so infinite variance implies an infinite temperature.

Consequently, in the 1990s, physicists seemed to be facing the same problematic contradictions as financial economists in the 1960s: on the one hand, the increasing number of empirical evidences supporting power laws and, on the other hand, the problematic theoretical feature of these processes (the infinite character of variance).[14]

The second reason for rejecting stable Lévy processes in physics is that these processes are also based on an asymptotic argument referring to the generalized version of the central limit theorem proposed by Gnedenko and Kolmogorov [52], according to which, if many independent random variables are added whose probability distributions have power law tails such as $p_i(x_i)\sim|x_i|^{-(1+\alpha)}$ with an index $0<\alpha<2$, their sum will be distributed according to a stable Lévy distribution. In the Gaussian case ($\alpha=2$), no single term contributes to a finite fraction of the whole sum (i.e. there is no significant deviation but a lot of small deviations). Conversely, if we consider the case $\alpha<2$, the whole sum and the largest term have the same order of magnitude. The largest term in the sum contributes to a finite fraction of the sum and then there may be significant deviations even if $N\rightarrow\infty$. Moreover, an asymptotic stable Lévy process leads to an infinite variance, which is not physically plausible.

All physical systems refer to real phenomena that have neither infinite parameters nor an asymptotic behaviour. By choosing to characterize financial distributions using stable Lévy processes, physicists have to solve two theoretical problems in order to fit the statistical tools to the reality they are studying: on the one hand, they look for a finite variance and, on the other hand, they want to work in a (locally)[15] non-asymptotic framework. In this perspective, a number of authors have developed new statistical methods to 'standardize' the 'α-stable' distributions so that variance is no longer infinite. Of course, this idea of truncation is nothing new in physics, where a lot of works have considered it empirical evidence because all physical systems are finite. However, all truncations resulting from the finite dimension of physical systems necessarily imply a gap between asymptotical results and empirical data. In this perspective, truncations are seen as the 'least bad' solution to the applicability of asymptotic mathematics in physics. By developing theoretical solutions for truncated distributions, physicists applied asymptotical results without misrepresenting them since they provided a specific formulation of the gap between these results and empirical ones. They provided a theoretical formulation of these boundary conditions by making truncation physically plausible and theoretically justified. In a sense, these truncations developed by econophysicists are in line with the works dedicated to the physical interpretation of asymptote in physics [53]. Before reviewing the truncation techniques, I will introduce, in the next section, the area of knowledge they have been developed.

4 Econophysics and the Emergence of Analytical Solutions

During the last two decades, a considerable number of physicists have begun using physics concepts to understand economic phenomena. Econophysics, as a specific label and conceptual practice, was first coined in 1996 by physicists in

[14] Note that a number of studies have been carried out on a new data dependency structure to replace the concept of variance with the notion of 'covariation' [51]. However, these studies have not been unanimously accepted by physicists.

[15] I will explain and illustrate what I mean by 'locally' in the following sections.

Stanley et al. [54] in a paper published in *Physica A*. As the name suggests, econophysics presents itself as a hybrid discipline that can be defined in methodological terms as "a quantitative approach using ideas, models, conceptual and computational methods of statistical physics" applied to economic and financial phenomena [55].

The influence of physics on economics is nothing new. A number of writers have studied the 'physical attraction' exerted by economists on physics [56, 57]. But as McCauley [58] points out, in spite of these theoretical and historical links between physics and economics, econophysics represents a fundamentally new approach that differs from preceding influences [59]. Its practitioners are not economists taking their inspiration from the work of physicists to develop their discipline, as has been seen repeatedly in the history of economics. This time, it is physicists that are going beyond the boundaries of their discipline, studying various problems raised by social sciences in light of their methods. Econophysicists are not attempting to integrate physics concepts into economics as it exists today, but rather are seeking to ignore, even to deny, this discipline in an endeavour to replace the theoretical framework that currently dominates it with a new framework derived directly from statistical physics[16] [60, 61].

Econophysicists present their field of research as a new way of thinking about the economic and financial systems through the 'lenses' of physics. Since econophysics is a very new field, it mainly focuses on financial economics, even if some works exist on macroeconomics.[17] Just as classical economics imported models from classical physics as formulated by Lagrange [55] and financial economics built on the model of Brownian motion also imported from physics, so too does econophysics try to model economic phenomena using analogies taken from modern condensed matter physics and its associated mathematical tools and concepts.

Econophysics is directly in line with the development of the so-called complexity science in the 1990s [55], for which economic systems are obvious candidates for a treatment in terms of 'complexity' because they are composed of multiple components (agents) interacting in such a way as to generate the macro-properties of economic systems and subsystems [64]. These macro-properties can be characterized in terms of statistical regularities, i.e. the fact that statistical properties appear and reappear in many diverse phenomena [58]. This statistical regularity is often characterized by econophysicists through power laws (and more generally stable Lévy processes), which are at the heart of econophysics.[18]

Econophysicists want to describe the world like it is and not like it would be according to an idealized theoretical framework. For them, real data must be described statistically through a physically plausible process. In this perspective, the Gaussian perspective and its idea of convergence is called into question by econophysicists on one hand because the Gaussian convergence only happens at infinity and, on the other hand, because the distribution becomes Gaussian only at its centre and not in its tails, such that this asymptotic regime cannot characterize the empirical distributions.[19]

In this perspective, the asymptotic convergence does not describe financial distributions which follow a power law [71–73]. In their works, econophysicists could have used the generalized version of CLT, according to which the sum of many independent and identically distributed random variables within a power law distribution converges towards a stable Lévy law. This asymptotic result could then describe the form of the empirical distributions, but it generates an infinite variance, which is empirically inconsistent with the studies of physical systems. In other words, if econophysicists wanted to describe data with non-Gaussian but stable distributions, they had to develop statistical solutions to avoid the asymptotical (theoretical) convergence towards central limit theorem (CLT) and generalised CLT as well. During the past two decades, econophysicists have developed sophisticated statistical tools (truncated stable processes) to escape this 'asymptotic trap'. However, although these new processes better fit the financial

[16] During the past decades, a lot of physics models have been used in economics, but these were mainly used for their mathematical description of physical phenomena. Over time, these imported models have become mainstream (see the Black and Scholes model, for example). This trend is not observed in econophysics. In this perspective, econophysicists do not try to connect their works with the preexisting economic theory. For an epistemological analysis of this attitude, see Gingras and Schinckus [60].

[17] Some econophysicists work on the emergence of money [62] or on global demand [63].

[18] As Farmer and Geanakoplos [65] emphasized, power laws could be "consistent with the economic equilibrium but still need to be explained by other means". For further information, see Calvet and Fisher [66].

[19] Of course, financial economists know that financial distributions are not empirically Gaussian. Therefore, they developed a large variety of statistical processes to better understand the dynamics of financial markets, but all these statistical solutions imply either a Gaussian world or a non-Gaussian framework (jump processes). The first category of works refers to what we call the autoregressive conditional heteroskedasticity-type models which remained implicitly based on a Gaussian framework since they consider that the unconditional distributions are governed by Gaussian processes whose tails can be fatter and explained in terms of conditional distribution (for which a stable Lévy processes can be applied). In opposition, econophysicists developed a data-driven approach by focusing on 'raw data' without Gaussian standardization, which is the reason why they work directly on unconditional distribution (i.e. observed). The second category of papers refers to jump processes which are based on discontinuous processes such as the variance gamma process [67], the generalized hyperbolic process [68] and the CGMY process [69]. However, because financial economists knew that stable Lévy processes generated infinite variance, they never used these processes and rather developed a great number of statistical solutions whose proliferation does not favour the crystallization of a unified financial theory. For further information about this distinction, see Rachev et al. [70].

data, some of their statistical properties still raise problems with regard to their empirical applications. This is why econophysicists have developed specific truncation methods.

5 Econophysics and Stable Lévy Processes

5.1 Toward a Finite Variance: Truncated Lévy Processes

The third section explained the importance of stable Lévy processes in the statistical characterization of physical systems. However, despite this importance, Vinogradov [1] (p. 5795) explained that few works exist on the deformation shape of these processes.

> Notwithstanding the fact that truncated Lévy flights have received wide acceptance for the description of stochastic processes of a different nature, the special research on the influence of the deformation shape on the stochastic process characteristics has been absent up until now.

Whilst this paper reminds the necessity to develop a physically plausible deformation shape, this section presents, more specifically, the main steps related to the progressive integration of truncation techniques into physical sciences.

The idea of truncation providing finite statistical moments is nothing new in physics since it refers to the application of asymptotic mathematics to physical systems. This technique can already be found in the 1700s, with the famous St. Petersburg paradox [74]. The most widespread solution to solve the infinite variance problem is to truncate Lévy distributions. The truncation of a Lévy distribution consists in normalizing it using a particular function so that its variance becomes finite.[20] This kind of technique, introduced by Mantegna [76], uses stable Lévy distributions with the specific condition that there is a cutoff length for the price variations, above which the distribution function is set to zero in the simplest case [77] or decreases exponentially [48, 49]. These functions are chosen in order to obtain the best fit with the empirical data [77] or are derived from models such as percolation theory [49], or the generalized Fokker–Plank equation [79].

The idea is to combine a statistical distribution factor and a gradual cutoff after a certain step size, which may be due to the limited sample under study. Gupta and Campanha [49] generalized this approach with a generic probability distribution where l is the cutoff at which the distribution begins to deviate from Lévy distribution. This deviation is associated with a function depending on the time and the cutoff parameter l. For a small value of x, $P(x)$ takes a value very close to the one expected for a stable Lévy process. But for large values of n, $P(x)$ will tend to the value predicted for a Gaussian process. In this framework, the probability of taking a step size (x) at any time is defined by

$$P(x) = L(x)g(x) \tag{2}$$

where the $L(x)$ is a stable distribution and $g(x)$ a truncation function. In the simplest case (abrupt truncation), the truncation function $g(x)$ is equal to a constant k and the abrupt truncation process can be characterized by

$$f(x) = \begin{cases} k\,L(x) & |x| \le l \\ 0 & |x| > l \end{cases} \tag{3}$$

In other words, if x is not too large, a truncated Lévy distribution behaves very much like a stable Lévy distribution since most of the value expected for x fall in the Lévy-like region. When x is beyond the crossover value, we are in the Gaussian regime and the classical central limit theorem can be applied.

This truncation results from the central limit theorem, according to which there is a competition between normal distribution and power law distribution for very large samples. In an abrupt truncation, the central limit theorem can be applied for the asymptotic case, but a power law distribution is kept for finite samples (before the cross-value where the distribution switches to a Gaussian convergence), where power laws never really disappear because even on a very large sample extreme events can appear (maybe more rarely, but with a higher amplitude).

The idea is to reduce the fat tails of the stable Lévy distribution without deforming the central part of the distribution in order to decompose it into a stable part for the short and mid-term and a Gaussian part for the long term,[21] as shown on the following plot (Fig. 1).

Thanks to the truncated Lévy, physicists can have a finite variance and then a more realistic analysis. However, these truncated processes are still based on an asymptotic argument coming from the convergence towards a Gaussian distribution, even if this process of convergence is very slow. The asymptotic traps have been partially solved by physicists since they are able to escape locally (only for $<N^*$) to this asymptotic behaviour.

The more important drawback of truncated stable Lévy distributions is that they are not stable[22] and not infinitely divisible because the truncation of the distribution is abrupt. This implies that its shape is changing at different time horizons and that distributions at different time horizons do not obey scaling relations.[23] More precisely, scaling turns

[20] The class of truncated Lévy processes is referred to as KoBol by mathematicians. See Schoutens [75].

[21] See Figueiredo et al. [80] for a statistical explanation of this slow convergence towards a Gaussian framework.

[22] Only non-truncated Lévy processes are stable [81].

[23] Indeed, the variable x progressively converges towards a Lévy distribution for $x<N^*$ whilst it converges towards a normal distribution when x is beyond the crossover value N^*.

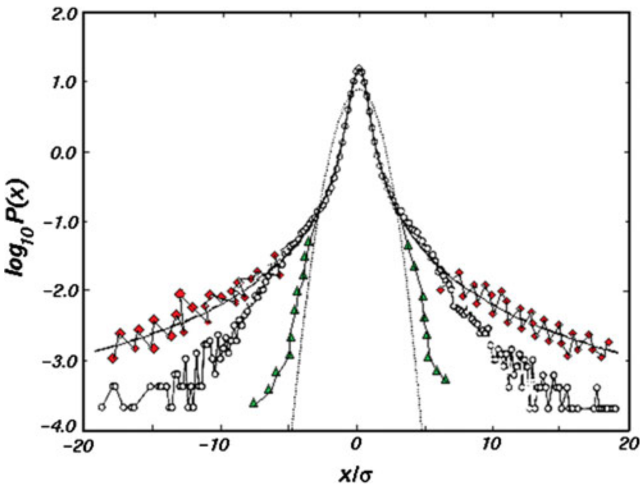

Fig. 1 Empirical probability density function for high-frequency price differences of the Xerox stock traded in the New York Stock Exchange during the 2-year period, 1994 and 1995. This is a semi-logarithmic scaled plot of the probability distributions of the progressive truncated Lévy process characterized by a α=1.5. We can observe that for low values of n (*red diamonds*), the central part of the distributions is better described by the Lévy-stable regime in *solid line*. The larger the values of n are, the more Gaussian the process becomes (see intermediated values of n, *circles*). For very large values of n (*triangles*), the truncated process has already switched into a Gaussian regime (*dotted line*; adapted from [50])

out to be approximate and valid for a finite time interval only. In other words, we observe the scaling property only for the Lévy regime. For longer time intervals, scaling must break down. However, scaling properties observed in the Lévy regime can be very useful in the analysis of finance data since this concept allows us to describe the statistical features of financial distributions independently from time horizons. When this abrupt truncation emerged in the 1990s, some physicists have claimed that this kind of truncation is only useful for very specific cases but that this methodology would not be physically plausible enough because a physical system rarely changes abruptly.

Physicists therefore had good reason to go beyond abrupt truncated stable Lévy processes, which is a very special case and which does not have any physical basis [50]. I will deal with this problem in the next section.

5.2 Towards Physically Plausible Processes: Gradual and Exponential Truncation

In real systems, the variance of a stationary process is necessarily finite. In this perspective, when we try to describe a system with stable Lévy processes, an unavoidable cutoff is needed. Mantegna and Stanley [77] introduced truncated Lévy processes to solve this problem. As presented in the previous section, these truncated stable distributions provide a finite variance, but the probability of taking a step is abruptly cut to zero at a certain critical step. However, Gupta and Campanha [48, 49] or Matsushita et al. [82] called the physical meaning of abrupt truncated Lévy processes into question. More precisely, the first wrote:

> The dynamical properties of a complex physical or social system depend on the dynamical evolution of a large number of non-linear coupled subsystems. Thus, it is expected that in general, the probability of taking a step [a variation] should decrease gradually and not abruptly, in a complex way with step size due to limited physical capacity. [49] (p. 232)

In this perspective, new truncated stable distributions have been introduced by Koponen [78] who considered the truncation with an exponential cutoff. Koponen [78] considered a truncation in which the cutoff is a decreasing exponential function. The idea was to combine a statistical distribution factor and a gradual cutoff after a certain step size, which may be due to the limited physical capacity of the systems under study. By using Eq. 2, an exponentially truncation function can be characterized by

$$g(x) = \begin{cases} 1 & |x| \leq l \\ \exp\left\{-\left(\frac{|x|-l}{k}\right)^{\beta}\right\} & |x| > l \end{cases} \tag{4}$$

where l is the cutoff at which the distribution begins to deviate from the Lévy distribution and $\exp\left\{-\left(\frac{|x|-l}{k}\right)^{\beta}\right\}$ is a decreasing function depending on the time and the cutoff parameter (l). k and β are constants related to truncation. Using this truncation function, Gupta and Campanha [49] defined a gradual truncation in which the probability of taking a step of size (x) at any time is given by

$$P(x) = \begin{cases} k\,L(x) & |x| \leq l \\ k\,L(x)\exp\left\{-\left(\frac{|x|-l}{k}\right)^{\beta}\right\} & |x| > l \end{cases} \tag{5}$$

In line with abruptly truncated Lévy distributions, exponentially and gradually truncated Lévy distributions have a finite variance; they make it possible to avoid the asymptotic behaviour locally and have scaling properties (for the part of the sample before the truncation) [83]. However, this kind of distribution has an advantage from an empiricist perspective: it is more adapted to describing reality[24] than abruptly truncated Lévy distributions because it is more physically justified since real systems rarely evolve abruptly [48]. However, despite all these statistical features, gradually truncated Lévy distributions still have an empirical

[24] This kind of distribution has been applied to currency data. See Figueiredo et al. [80] or Gupta and Campanha [49] for empirical tests of this kind of distribution.

drawback: the gradual truncation underestimates the fat tails [82] whilst the cutoff of an exponential truncation results from an asymptotic approximation of a stable distribution valid for large values. In order to improve the gradually truncated Lévy distributions by keeping their statistical feature, physicists developed a more sophisticated process that better fits the data. I will introduce this process in the next section.

5.3 Towards Data-Driven Processes: Exponentially Damped Truncation

Gupta and Campanha [48, 49] pointed that the gradually truncated Lévy distributions break down in the presence of positive variations. This also means that tails are fatter than those estimated by gradually truncated Lévy distributions. This drawback is significant since in real systems we also observe increasing deviations for which a more realistic process must necessarily be developed. Matsushita et al. [82] introduced what they called ‘exponentially damped truncated distributions’, which encompass the previous cases. These new truncated Lévy distributions are more justified from a physical point of view and fit the data better. By using Eq. 1, Matsushita et al. [82] (see also [1]) defined the exponentially damped truncation function as

$$f(X_{\Delta t}) = \begin{cases} 1 & |x| \leq l \\ \left(\Delta t^{-1/\alpha}|x_{\Delta t}| + \theta\right)^{\beta_1} \exp\{H(x_{\Delta t})\} & l \leq |x| \leq l_{\max} \\ 0 & |x| > l_{\max} \end{cases} \quad (6)$$

where $H(x_{\Delta t}) = \lambda_1 + \lambda_2[1-|x_{\Delta t}|/l_{\max}]^{\beta_2} + \lambda_3(|x_{\Delta t}|-l)$ and θ, λ_1, $\lambda_2 \leq 0$, $\lambda_3 \leq 0$, β_1, β_2 and β_3 are parameters describing the deviations from the Lévy.[25] As mentioned before, this exponentially damped truncation is a generalization of the other case since the exponentially truncated case given by Nakao [83] can be found by setting $\theta=0$, $\beta_1=-1-\alpha$ and $\beta_3=1$, whilst it generates a similar function to the one given by the gradual case which can be found if $\theta=\lambda_1=\lambda_2=\beta_1$; finally, abrupt truncation can also be found by choosing the appropriate parameters such that $H(x_{\Delta t}) \to -\infty$. The truncation parameters (l and $l_{\max}$) are estimated to provide an optimized fit for the tails. l is the step size at which variations begin to deviate from the stable law following an exponentially decreasing process; $l_{\max}$ is the step size at which variations follow a process in line with an abruptly truncated distribution which can then be characterized with a Gaussian law. In other words, this is a truncation in two steps that preserves all statistical features observed for previous cases.

[25] Matsushita et al. [82] used the maximum likelihood approach for estimating α and γ and nonlinear least squares for the other parameters.

The development of more data-driven truncations is very important for econophysics since it integrates all previous theoretical problems in order to better describe the empirical leptokurticity of financial distributions. In this time of financial crisis, this contribution is valuable and could play an important role in risk and portfolio management in the future because it offers new statistical measures for the estimation of financial risks (see [84] for further details on the potential contributions of econophysics on financial economics).

6 A Very Old Issue in Financial Economics

6.1 Stable Lévy Processes in Financial Economics

As shown by Jovanovic and Schinckus [85], the emergence of econophysics is directly embedded in the history of financial economics. Mandelbrot was the first to attempt to describe financial distributions using stable Lévy processes by showing that statistical evolution of financial returns can be studied as a turbulence phenomenon using a Paretian process. In 1963, Mandelbrot [86] demonstrated how what Lévy referred to as ‘α-stable’ (α=1.7) processes were entirely suitable for studying the discontinuity of price changes. To characterize this variability with respect to abrupt or discontinuous variations, Mandelbrot and Wallis [87] talked of a ‘Noah effect’.[26]

Fama [10] dealt with the statistical properties of stable processes, which he presented in an economic perspective. Two years later, Fama [89] gave a mathematical reinterpretation of modern portfolio theory developed by Markowitz (together with Sharpe’s diagonal model) in a Paretian statistical framework, but he was unable to provide a theoretical interpretation of his work because the parameter of risk was infinite [10]. In this regard, Fama observed that a general increasing of diversification has a direct impact on the evolution of the scale factor (γ) of the stable distribution. More precisely, an increasing diversification reduces the scale of the distribution of the return on a portfolio (only when the characteristic exponent $\alpha>1$).[27] Fama therefore proposed to substitute the variance by the scale factor (γ) of the stable distribution in order to approximate the dispersion of financial distributions. In the same vein, Samuelson [90] provided an “efficient portfolio selection for Pareto–Lévy investments” in which the scale parameter was used as an

[26] Mandelbrot and Wallis [87] referred indirectly to the biblical tale of Noah. When a ‘deluge’ (stock market crash) is observed on financial markets, “even a big bank or brokerage house may resemble a small boat in a huge storm” [88] (p. 222).

[27] When $\alpha=1$, there is no impact of an increasing diversification on the scale factor; when $\alpha<1$, this scale factor increases in case of increasing diversification [89] (p. 412).

approximation of variance (because the scale parameter is proportional to the mean absolute deviation). Samuelson presented the computation of the efficient frontier as a problem of nonlinear programming solvable by the Kuhn–Tucker techniques. However, even though he demonstrated the theoretical possibility of finding an optimal solution for a stable Lévy distribution, Samuelson did not give an example of his technique. In their attempt[28] to provide a new way of thinking about the notion of uncertainty within a Lévy stable framework, these authors were unable to provide a theoretical interpretation of their work because the parameter of risk (variance) was infinite [5].

Many researchers considered the infinite-variance hypothesis unacceptable because it is meaningless in the financial economics framework. Variance and the expected mean are the two main variables for theoretical interpretations. In the 1960s, the period in which financial economics was constituted as a scientific discipline, the relationship between risk and return was taken from Markowitz' work [92, 93]. Markowitz associated risk with variance and return with the mean. In this perspective, if variance were infinite (as it is in a Lévy process), it became impossible to understand the notion of risk as Markowitz had defined it.

In addition to these difficulties, the authors had to face the indeterminacy of variance on the one hand and, on the other, the fact that no computational definition yet existed for evaluating the parameters of stable Lévy processes. Fama [89] himself regretted this point. He explained that the next step in the acceptability of Lévy processes in financial economics would be "to develop more adequate statistical tools for dealing with stable Paretian distributions" [89] (p. 429). A reminder of this statistical problem is found in papers dedicated to the estimation of parameters of stable distributions [94]. In addition, some authors expressed their skepticism about the opportunity to use Lévy processes. Officer [95] (p. 811) explained that financial data "have some but not all properties of a stable process" and that several "inconsistencies with the stable hypothesis were also observed". He concluded that the evolution of financial markets could not be described through a Lévy process. The use of Lévy in finance has then been progressively abandoned, and this point has not been really discussed in the literature since it implied a new measure of risk [89]. Ten years after his 1965 article, Fama [94] himself preferred to use normal distribution to describe monthly variations, thereby abandoning α-stable distributions. In his extension of the modern portfolio theory to the Paretian framework, Fama [89] (p. 416) deplored the fact that no computational definition yet existed for evaluating this parameter. This led him to conclude:

> Although the model discussed in the previous sections provides a complete theoretical structure for a portfolio model in a stable Paretian market, there are several difficulties involved in applying the model in practical situations". [89] (p. 414)

However, as I explained in the first part of this paper, a statistical response (truncation of stable Lévy processes) to this indeterminate nature of variance was developed by econophysicists in their attempt to describe the financial market through stable Lévy processes [76, 77]. This analytical solution also contributed to the emergence and the progressive institutionalization of econophysics (see [60] on this point).

6.2 Why These Solutions Did Not Emerge in Financial Economics

Why have truncation techniques been developed by physicists although financial economists faced this problem of infiniteness of variance in the 1960s? This question directly refers to the way of doing science.

Rickles [55] (p. 6) explains that "the real empirical data are certainly at the core of this whole enterprise [econophysics] and the models are built around it, rather than some non-existent, ideal market [as in economics]".[29] Stanley et al. [54] (p. 157) stress that, "in contrast to standard economics, econophysicists begin empirically with real data that one can analyze in some detail but without prior models". By starting with real data, econophysicists set the empirical dimension at the heart of their studies [54, 104, 105].

In opposition to the a priori assumptions (i.e. perfect rationality, Gaussian world) developed by the economic mainstream [55], there are a lot of methodological differences between econophysics and economics. By using the opposition 'principle theories versus constructive theories' proposed by Einstein in 1905 concerning his formulation of special relativity in comparison with Lorentz's approach, Rickles [64] explained that econophysics is a constructive field founded on a synthetic and empirical approach whilst economics is rather a principles-based field focusing on an more analytical and a priori reasoning. For further details about this opposition, see Rickles [64] or Schinckus [106].

We can illustrate this difference between economists and econophysicists by the way they work on stylized facts such as fat tails or financial crashes. Economists consider that

[28] In this non-Gaussian research, we can also mention Praetz [91] who dealt with scaled t distributions whose empirical results were, according to him, better than stable distribution.

[29] For debates about these oppositions between economics and econophysics, see [101–103].

prices changes obey a lognormal probability distribution with a kurtosis around zero (a mesokurtic distribution).

This a priori perspective implies that massive fluctuations are very unlikely. However, real data have a positive kurtosis and, thus, a leptokurtic distribution in which extreme events have a greater probability. Mandelbrot and Hudson [88] (p. 4) argued that economists' a-priori-ism leads them to underestimate the likelihood of a financial crash: "The standard theory, as taught in business schools around the world, would estimate the odds of that final, August 31[1998] collapse at one in 20 million". However, as Kahana [107] points out, there were several financial crises during the twentieth century. Economic theory seems to be unable to describe this kind of extreme event.

By beginning with economic and financial data, econophysicists develop models in which some extreme events (such as the financial crises of the 1980s and 1990s) can occur: a financial crash can be studied, for example, as a phase transition and particularly as a specific heat jump [108]. This is not the case for neoclassical economics in which a financial crisis has a very small probability of occurrence.[30] Empiricism leads to a specific perspective in which physics appears to be the main discipline appropriated to help our understanding of economic phenomena. In this framework, only a physicalist language with a physical methodology should be used to describe economic complex systems, as McCauley [103] (p. 7) explains.

> Mathematicians do work in economics but they tend to be postulatory and to ignore data […] Chemists and biologists are trained to concentrate on details. Physicists are trained to see the connections between seemingly different phenomena, to try to get a glimpse of the big picture and to present the simplest possible mathematical description of a phenomenon that includes as many links as are necessary, but not more.

This stance is also adopted by McCauley [58] or Bouchaud [104] (p. 238) who presents it almost as a necessity.

> finance is becoming an empirical (rather than axiomatic) science […] This means that any statistical model, or theoretical idea, can and must be tested against available data, as physicists are (probably better than other communities) trained to do.

This kind of physicalism suggests implicitly that economic phenomenon can be reduced to a language coming from physics. That does not mean that every social theorists must use a physicalist language but rather that all terms used in other disciplines can be translated in terms of physics (i.e. all theoretical terms must be empirically founded). This kind of reductionism refers to the terms of science and not to scientific laws. The fact that all scientific terms can be translated into empirical stance does not imply that laws developed in other disciplines must be reduced to physicalist laws [109]. Econophysicists do not want to replace all socioeconomic models by theirs—they just claim that these socioeconomic models should have empirical dimension.

Econophysicists describe economic phenomena through models explaining how emergence appears at the macro-level of economic systems. Epistemologically, econophysics is founded on the observation of statistical regularities, i.e. the fact that statistical properties appear and reappear in many diverse phenomena [58]. As mentioned before, this statistical regularity can be characterized by the power laws that are at the heart of econophysics. As Stanley et al. [54] (p. 288) express it:

> It is becoming clear that almost any system comprised of a large number of interacting units has the potential of displaying power-law behavior. Since economic systems are, in fact, comprised of a large number of interacting units has the potential of displaying power-law behavior, it is perhaps not unreasonable to examine economic phenomena within the conceptual framework of scaling.

In this perspective, the emergence of econophysics and the analytical solutions it provided for the problem of infinite variance refer to the more positivist[31] attitude. Although stable Lévy processes seemed to be physically irrelevant, they fitted the empirical data. Because they mainly focused on the data, econophysicists tried therefore to develop analytical solutions in order to make these processes physically plausible, whilst a priorist economists' attitude preferred to work with different assumptions (leading them to abandon stable Lévy processes in the 1970s).[32]

[30] Moreover, in neoclassical theory, the very rare financial crises are caused by exogenous factors. For further information about positivism in economics, see [96-100]

[31] From a positivist perspective, power laws can be seen as either 'structural laws' [110] (p. 6) or theoretical laws (in a Carnapian sense) describing the structure of the observed phenomena. They are theoretical laws whose unobservable terms (complexity and emergence) create sufficient empirical evidence to give them a cognitive meaning. Like the existence of a magnetic field, complexity and emergence are not directly observed, but the empirical observations allow their existence to be inferred. Complexity and emergence are not logically deduced from power laws because the explanation provided by econophysics is not strictly deductive but rather 'deductive-statistical' [110], (p. 5). See Schinckus [106] for further details about the neo-positivism in econophysics.

[32] See [84] for further information about this historical comparison.

7 Conclusion

This paper presented an overview of the statistical solutions (truncations) developed by econophysicists to make stable Lévy processes physically plausible. Of course, the idea of truncation is nothing new in physics since it is often considered empirical evidence given that all physical systems have boundary conditions. However, when truncations result from the finite dimension of physical systems, they necessarily appear as the 'least bad solution' in the applicability of asymptotic mathematics to physics. By developing theoretical solutions for truncated distributions, physicists have applied asymptotical results without misrepresenting asymptotic properties since they give them a specific formulation by making truncation physically plausible and theoretically justified. That is why these analytical solutions developed by econophysicists can also be extended to other fields of physics.

This paper also showed how the emergence of these truncated stable distributions is directly associated with econophysics which provides solutions to the theoretical problems encountered by financial economists in the 1960s.

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