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PARTICLES AND FIELDS

Hidden Symmetries in the Two-Dimensional Isotropic Antiferromagnet

**S. A. Leonel · A. C. R. Mendes · W. Oliveira ·
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Abstract We discuss the two-dimensional isotropic anti-ferromagnet in the framework of gauge invariance. Gauge invariance is one of the most subtle useful concepts in theoretical physics, since it allows one to describe the time evolution of complex physical system in arbitrary sequences of reference frames. All theories of the fundamental interactions rely on gauge invariance. In Dirac's approach, the two-dimensional isotropic antiferromagnet is subject to second-class constraints, which are independent of the Hamiltonian symmetries and can be used to eliminate certain canonical variables from the theory. We have used the symplectic embedding formalism developed by a few of us to make the system under study gauge invariant. After carrying out the embedding and Dirac analysis, we systematically show how second-class constraints can generate hidden symmetries. We obtain the invariant second-order Lagrangian and the gauge-invariant model Hamiltonian. Finally, for a particular choice of factor ordering, we derive the functional Schröodinger equations for the original Hamiltonian and for the first-class Hamiltonian and show them to be identical, which justifies our choice of factor ordering.

1 Introduction

Since the discovery of high-T_c superconductivity [1], interest in the two-dimensional Heisenberg magnets has been intensively revived. The study of magnetism in two dimensions (2D) has motivated much theoretical and experimental work [2, 3, 5] and led to substantial progress in the understanding of 2D magnetism [7–9]. This includes the $O(3)$ nonlinear sigma model, which describes the continuum classical limit of the 2D Heisenberg antiferromagnet.

A consistent, systematic study of constrained systems was first established by Dirac [10]. The main goal of the so-called Dirac formalism was to obtain the Dirac brackets, the bridge to the commutators in quantum theory. With its categorization of constraints as of first or second class, primary or secondary, etc., this formalism has become one of the standards for the analysis of constrained theories. Faddeev and Jackiw [18] proposed a first-order Lagrangian geometric method for the symplectic quantization of constrained systems, which is different from the traditional Hamiltonian Dirac approach. In the Faddeev–Jackiw method, we need not introduce primary constraints as in the Dirac formalism, which stems from the definition of the canonical momenta. Nor is it necessary to classify constraints as of first or second class, primary or secondary. All the constraints are held to the same standard. Barcelos-Neto and Wotzasek proposed the symplectic formalism [19], an improved version

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of the Faddeev–Jackiw method for the case in which the constraints are only partially eliminated.

The central goal of the symplectic formalism is to turn the system into a first-order Lagrangian with certain auxiliary fields, the definition of that Lagrangian being independent of how the auxiliary fields are introduced. The first-order Lagrangian, which consists of a few symplectic variables and their generalized canonical momenta, gives the symplectic two-form matrix f_{ij}. In the symplectic formalism, the system can be classified as constrained or unconstrained, depending on the singular behavior of the symplectic two-form matrix. As a result, the algorithm runs into one of three alternatives. In the first alternative, if the symplectic two-form matrix is nonsingular, it can be inverted to yield generalized brackets, which correspond to Dirac brackets.

In the second alternative, if the symplectic two-form matrix is singular, there may be some nontrivial zero mode, which generates the constraints in the context of the symplectic quantization method. The constraints can then be transported to the canonical sector by means of appropriate Lagrangian multipliers and are regarded as conjugate canonical one-form components, whereas the Lagrange multipliers are treated as symplectic variables. With this new first-order Lagrangian, a finite number of iterations usually suffices to make the symplectic two-form matrix nonsingular and to yield the generalized brackets of the symplectic variables, which coincide with those in the Dirac formalism.

Finally, even with a singular two-form matrix, it may be that the original zero modes impose no constraints on the dynamical variables at the first stage of iteration. In the absence of additional constraints, the original canonical sector in the first-order Lagrangian is unchanged, and we can say that the system has a gauge symmetry with the field-transformation rules supported by the zero modes.

Recently, the functional Schrödinger representation has been systematically used to quantize different field theories. Different theories have derived divers predictions, part of which are physically appealing, from the wave-functionals obtained so far. The so-called *vacuum angle* is one of the important theoretical features of gauge theories obtained from the functional Schrödinger representation with no instanton approximation [21].

We will make use of a general canonical formalism of embedding developed by a few of us on the basis of the symplectic formalism [22], which embeds a second-class system into one with gauge invariance. The first-class Hamiltonian leads to the same classical theory as the original one. We shall then derive the functional Schrödinger equation using the Dirac first-class quantization formalism [10].

This paper is organized as follows. In Section 2, we obtain the Dirac brackets of the model, via two distinct methods: the usual Dirac algorithm and the symplectic

formalism. In Section 3, we embed the isotropic antiferromagnet into a gauge-symmetric system and discuss the interesting physical situation in which a second-class constraint acts as symmetry generator of the hidden symmetries of the model. In Section 4, we derive the functional Schrödinger equations of the original and first-class models and show that they are all identical. In Section 5, we present our concluding remarks and future perspectives. Finally, in Appendices A and B, we briefly review the symplectic formalism and present the general theory of symplectic embedding, respectively. For briefness, throughout the paper, we write "Lagrangian" and "Hamiltonian" for the Lagrangian and Hamiltonian densities, respectively.

2 Dirac Brackets

2.1 Dirac Brackets via the Dirac Formalism

One of the simplest theoretical descriptions of a 2D isotropic antiferromagnet at low temperatures is given by the Lagrangian

$$\mathcal{L}' = \frac{J}{2}\, \partial_\mu S^i \partial^\mu S_i \qquad (\mu = 0, 1, 2), \tag{1}$$

where $\vec{S} = (S_1, S_2, S_3)$ is a three-dimensional vector subjected to the constraint $S_i^2 = 1$ and satisfies the usual spin Poisson bracket. The metric has signature $(+, -, -)$ and we are using the convention of sum over repeated indices. After a Legendre transformation, the canonical Hamiltonian is given by the expression

$$\mathcal{H}' = \frac{\pi_i^2}{2J} + \frac{J}{2}\, (\partial_k S_i)^2, \tag{2}$$

where $k = 1, 2$ and π_i is the momenta canonically conjugate to the coordinates S_i.

We are particularly interested in the fields satisfying a condition that follows from the fact that the energy is finite (static solution). In this case, the Hamiltonian that follows from the Lagrangian (1) is $\mathcal{H}'' = \frac{J}{2}\, (\partial_k S_i)^2$, which is the classical continuous version, valid at low temperatures, of the 2D isotropic antiferromagnetic Heisenberg model [25], described by the quantum discrete Hamiltonian

$$H = -J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j, \qquad (J < 0), \tag{3}$$

where i, j represent lattice cites and the bracket in the sum indicates nearest neighbors, and the fields satisfy the constraint $S_i^2 = 1$. Note that the first term on the right-hand side of (2) can be dropped since Lorentz invariance allows the moving finite-π solution to be obtained by a boost of the static $\pi = 0$ solution. For details, the interested reader can consult [26]. $\mathcal{H}''$ is also the Hamiltonian for the $O(3)$ nonlinear sigma model.

Instead of (1), consider the Lagrangian

$$\mathcal{L} = \frac{J}{2} \partial_\mu S^i \partial^\mu S_i - \frac{1}{2} \lambda \left(S^i S_i - 1 \right), \qquad (4)$$

where we have introduced a Lagrange multiplier λ to account for the constraint $S_i^2 = 1$.

From (4) we obtain the Hamiltonian

$$\mathcal{H} = \frac{\pi_i^2}{2J} + \frac{J}{2} (\partial_k S_i)^2 + \frac{1}{2} \lambda \left(S_i^2 - 1 \right). \qquad (5)$$

From the persistence in time of the constraint $\Omega_1 = S_i^2 - 1$, we obtain the constraint $\Omega_2 = S_i \pi^i$. In the Dirac-Hamiltonian approach, these second-class constraints are the only constraints on the system. The nonsingular constraint matrix C is given by the equality

$$C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} S_i^2 \delta(x - y). \qquad (6)$$

The inverse of C yields the usual Dirac brackets of the theory, namely

$$\{S_i(x), S_j(y)\}^* = 0,$$

$$\{S_i(x), \pi_j(y)\}^* = \left(\delta_{ij} - \frac{S_i S_j}{S_i^2} \right) \delta(x - y),$$

$$\{\pi_i(x), \pi_j(y)\}^* = \frac{(S_i \pi_j - S_j \pi_i)}{S_i^2} \delta(x - y). \qquad (7)$$

2.2 Dirac Brackets via the Symplectic Formalism

To implement the symplectic method, let us introduce the auxiliary variables π^i, so that the original second-order Lagrangian in the velocity, (4), can be written as the first-order Lagrangian

$$\mathcal{L}^{(0)} = \pi^i \dot{S}_i - V^{(0)}, \qquad (8)$$

with

$$V^{(0)} = \frac{1}{2J} \pi_i^2 + \frac{1}{2} \lambda \left(S_i^2 - 1 \right) - \frac{J}{2} (\partial_k S_i)^2. \qquad (9)$$

The symplectic coordinates are $\xi_\alpha^{(0)} = (S_i, \pi_i, \lambda)$ and the superscript (0) indicates that we are at iteration zero. The symplectic tensor given by (90) is computed in this case as

$$f^{(0)} = \begin{pmatrix} 0 & -\delta^{ij} & 0 \\ \delta^{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(x - y). \qquad (10)$$

This matrix being singular, we consider the following zero mode,

$$\nu^{(0)} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix}. \qquad (11)$$

Contraction of this zero mode with the gradient of the symplectic potential $V^{(0)}$ in (9) yields the following constraint

$$\Omega_1 = \frac{1}{2} (S_i^2 - 1). \qquad (12)$$

To follow the symplectic formalism, we must introduce this constraint into the canonical sector of the first-order Lagrangian (8) by means of a Lagrange multiplier ρ, which yields the first-iteration Lagrangian

$$\mathcal{L}^{(1)} = \pi^i \dot{S}_i + \Omega_1 \dot{\rho} - V^{(1)} \qquad (13)$$

with

$$V^{(1)} = V^0 |_{\Omega_1 = 0} = \frac{1}{2J} \pi_i^2 + \frac{J}{2} (\partial_k S_i)^2. \qquad (14)$$

The new symplectic coordinates are $\xi_\alpha^{(1)} = (S_i, \pi_i, \rho)$, with the following one-form canonical momenta,

$$A_{S_i}^{(1)} = \pi^i, \quad A_{\pi_i}^{(1)} = 0, \quad A_\rho^{(1)} = \frac{1}{2} \left(S_i^2 - 1 \right). \qquad (15)$$

The corresponding symplectic tensor $f^{(1)}$ is given by the matrix

$$f^{(1)} = \begin{pmatrix} 0 & -\delta^{ij} & S^i \\ \delta^{ji} & 0 & 0 \\ -S^j & 0 & 0 \end{pmatrix} \delta(x - y), \qquad (16)$$

which is singular and has a zero mode that generates a new constraint,

$$\Omega_2 = \frac{1}{J} S_k \pi^k. \qquad (17)$$

We next introduce the constraint Ω_2 into the first-iteration Lagrangian (13) with a Lagrange multiplier ζ, to obtain the second-iteration Lagrangian

$$\mathcal{L}^{(2)} = \pi^i \dot{S}_i + \frac{1}{2} \left(S_i^2 - 1 \right) \dot{\rho} + \frac{1}{J} \left(S_i \pi^i \right) \dot{\zeta} - V^{(2)}, \qquad (18)$$

with $V^{(2)} = V^{(1)} |_{\Omega_1 = 0}$. The symplectic coordinates are now $\xi_\alpha^{(2)} = (S_i, \pi_i, \rho, \zeta)$, and the new one-form canonical momenta are

$$A_{S_i}^{(2)} = \pi_i, \qquad\qquad A_{\pi_i}^{(2)} = 0,$$

$$A_\rho^{(2)} = \frac{1}{2} S_i^2 - 1, \qquad A_\zeta^{(2)} = \frac{1}{J} S_i \pi^i. \qquad (19)$$

The corresponding matrix $f^{(2)}$ is

$$f^{(2)} = \begin{pmatrix} 0 & -\delta^{ij} & S^i & \frac{1}{J}\pi^i \\ \delta^{ji} & 0 & 0 & \frac{1}{J}S^i \\ -S^j & 0 & 0 & 0 \\ -\frac{1}{J}\pi^j & -\frac{1}{J}S^j & 0 & 0 \end{pmatrix} \delta(x - y), \qquad (20)$$

which is nonsingular. We immediately identify the Dirac brackets, given by the equalities

$$\{S_i(x), S_j(y)\}^* = \{\rho(x), \rho(y)\}^* = \{\zeta(x), \zeta(y)\}^* = 0,$$

$$\{S_i(x), \pi_j(y)\}^* = \left(\delta_{ij} - \frac{S_i S_j}{S_i^2}\right)\delta(x-y),$$

$$\{\pi_i(x), \pi_j(y)\}^* = \frac{(S_i \pi_j - S_j \pi_i)}{S_i^2}\delta(x-y).$$

$$\{S_i(x), \rho(y)\}^* = -\frac{S_i}{S_i^2}\delta(x-y),$$

$$\{\pi_i(x), \rho(y)\}^* = \frac{\pi_i}{S_i^2}\delta(x-y),$$

$$\{\pi_i(x), \zeta(y)\}^* = -\frac{J S_i}{S_i^2}\delta(x-y), \tag{21}$$

which coincide with the results in Ref. [7] for the variables S_i and π_i. In principle, this indicates that the model lacks gauge symmetry.

3 Hidden Symmetries in the 2D Isotropic Antiferromagnet

To disclose the hidden symmetry in the $2D$ isotropic antiferromagnet, again following the symplectic embedding formalism, we now extend the original phase space with a Wess-Zumino (WZ) field. To this end, we introduce two arbitrary functions $\Psi(S_i, \pi_i, \theta)$ and $G(S_i, \pi_i, \theta)$ into the first-order Lagrangian as follows:

$$\tilde{\mathcal{L}}^{(0)} = \pi^i \dot{S}_i + \Psi \dot{\theta} - \tilde{V}^{(0)}, \tag{22}$$

where the symplectic potential is

$$\tilde{V}^{(0)} = \frac{1}{2J}\pi_i^2 + \frac{1}{2}\lambda\left(S_i^2 - 1\right) + \frac{J}{2}(\partial_k S_i)^2 + G(S_i, \pi_i, \theta), \tag{23}$$

with $G(S_i, \pi_i, \theta)$ satisfying (98) and (99), in Appendix B.

The symplectic coordinates are $\tilde{\xi}_\alpha^{(0)} = (S_i, \pi_i, \lambda, \theta)$, with the following one-form canonical momenta:

$$\tilde{A}_{S_i}^{(0)} = \pi_i, \quad \tilde{A}_{\pi_i}^{(0)} = 0,$$
$$\tilde{A}_\lambda^{(0)} = 0, \quad \tilde{A}_\theta^{(0)} = \Psi. \tag{24}$$

As dictated by the symplectic embedding formalism, the corresponding matrix $\tilde{f}^{(0)}$, given by the equality

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -\delta^{ij} & 0 & \frac{\partial \Psi_y}{\partial S_i^x} \\ \delta_{ji} & 0 & 0 & \frac{\partial \Psi_y}{\partial \pi_i^x} \\ 0 & 0 & 0 & \frac{\partial \Psi_y}{\partial \lambda^x} \\ -\frac{\partial \Psi_x}{\partial S_j^y} & -\frac{\partial \Psi_x}{\partial \pi_j^y} & -\frac{\partial \Psi_x}{\partial \lambda^y} & 0 \end{pmatrix}\delta(x-y), \tag{25}$$

must be singular, which leads to $\partial \Psi_y / \partial \lambda_i^x = 0$, i.e., $\Psi \equiv \Psi(S_i, \pi_i, \theta)$. This matrix has a zero mode, which we identify with the gauge-symmetry generator. To pull out the hidden symmetry, we force the zero mode to satisfy the relation

$$\int d^3 y \; v_\alpha^{(0)}(x) \; f_{\alpha\beta}(x, y) = 0, \tag{26}$$

which allows us to compute Ψ.

Let us start with the symmetry generated by the following zero mode,

$$v^{(0)} = \begin{pmatrix} 0 \\ S_i \\ 0 \\ 1 \end{pmatrix}. \tag{27}$$

Since this zero mode and the symplectic matrix (25) satisfy (26), we find that

$$\Psi = \frac{1}{2}S_i^2. \tag{28}$$

To start the second step, we require that the contraction of the zero mode with the potential gradient generates no additional constraints. The correction terms can be explicitly computed as functions of θ. Integration yields the correction to first-order in θ:

$$\mathcal{G}^{(1)}(S_i, \pi_i, \theta) = -\frac{1}{J}S_i \pi^i \theta. \tag{29}$$

Substitution in the last term on the right-hand side of (23) yields the new Lagrangian

$$\tilde{\mathcal{L}}^{(0)} = \pi^i \dot{S}_i + \Psi \dot{\theta} - \frac{J}{2}(\partial_k S_i)^2 - \frac{1}{2}\lambda\left(S_i^2 - 1\right)$$
$$- \frac{1}{2J}\pi_i^2 + \frac{1}{2J}S_i \pi^i \theta - \sum_{n=2}^{\infty}\mathcal{G}^{(n)}. \tag{30}$$

At this point, the model is not yet gauge invariant because the contraction of the zero mode $v^{(0)}$ with the gradient of the potential $V^{(0)}$ is nonzero. This calls for computation of the remaining correction terms $\mathcal{G}^{(n)}(n = 2, 3, \ldots)$ as functions of θ. In practice, to carry out the sum in the last term on right-hand side of (30), we only have to demand that the zero mode generate no new constraint. The correction term $\mathcal{G}^{(2)}$ is therefore given by the equality

$$\mathcal{G}^{(2)} = \frac{1}{2J}S_i^2 \theta^2, \tag{31}$$

and the correction terms $\mathcal{G}^{(n)}(n \geq 3)$ are null. Substitution of these results into the first-order Lagrangian (30) yields the result

$$\tilde{\mathcal{L}}^{(0)} = \pi^i \dot{S}_i + \frac{1}{2}S_i^2 \dot{\theta} - \frac{J}{2}(\partial_k S_i)^2 - \frac{1}{2}\lambda\left(S_i^2 - 1\right)$$
$$- \frac{1}{2J}\pi_i^2 + \frac{1}{J}S_i \pi^i \theta - \frac{1}{2J}S_i^2 \theta^2. \tag{32}$$

The zero mode $\nu^{(0)}$ no longer producing new constraints, the model is symmetric and, in compliance with the symplectic formalism, the symmetry generator is the zero mode.

We now want to recover the invariant second-order Lagrangian from the first-order form in (32). To this end, the canonical momenta must be eliminated from the Lagrangian (32). The canonical momenta computed from the equation of motion for the π_i are

$$\pi_i = J\dot{S}_i + S_i\theta. \tag{33}$$

When (33) is inserted in the first-order Lagrangian (32), the following expression for the second-order Lagrangian is obtained:

$$\tilde{\mathcal{L}} = \frac{J}{2}\partial_\mu S^i \partial^\mu S_i - (\dot{S}_i S^i)\theta - \frac{1}{2}\left(S_i^2 - 1\right)\lambda, \tag{34}$$

with the following gauge-invariant Hamiltonian,

$$\tilde{\mathcal{H}} = \frac{1}{2J}\pi_i^2 + \frac{J}{2}(\partial_k S_i)^2 + \frac{1}{J}\pi^i S_i\theta + \frac{1}{2J}S_i^2\theta^2 + \frac{\lambda}{2}\left(S_i^2 - 1\right). \tag{35}$$

The symplectic formalism identifies the zero mode with the generator of the infinitesimal gauge transformations $\delta\tilde{\xi}_\alpha^{(0)} = \varepsilon\nu^{(0)}$, namely,

$$\delta S_i = 0,$$
$$\delta\pi_i = \varepsilon S_i,$$
$$\delta\lambda = 0,$$
$$\delta\theta = \varepsilon. \tag{36}$$

These transformations introduce Hamiltonian changes of the form

$$\delta\tilde{\mathcal{H}} = 0. \tag{37}$$

Henceforth, we are interested in disclosing the hidden symmetry of the isotropic antiferromagnet in the original phase space (S_i, π_i). To this end, we will apply the Dirac method to obtain the set of constraints on the gauge-invariant isotropic antiferromagnet described by the Lagrangian (34) and Hamiltonian (35). We therefore have that

$$\phi_1 = \pi_\lambda,$$
$$\phi_2 = -\frac{1}{2}\left(S_i^2 - 1\right), \tag{38}$$

and

$$\varphi_1 = \pi_\theta,$$
$$\varphi_2 = \frac{1}{J}S_i\pi_i - \frac{1}{J}S_i^2\theta, \tag{39}$$

where π_λ and π_θ are the canonical momenta conjugated to λ and θ, respectively. The corresponding Dirac matrix is singular. However, a few constraints have nonvanishing Poisson brackets, which point to both second- and first-class constraints. To solve this problem, we separate the former

from the latter via constraint analysis. The set of first-class constraints is

$$\chi_1 = \pi_\lambda,$$
$$\chi_2 = -\frac{1}{2}\left(S_i^2 - 1\right) + \pi_\theta, \tag{40}$$

while the set of second-class constraints is

$$\varsigma_1 = \pi_\theta,$$
$$\varsigma_2 = \frac{1}{J}S_i\pi_i - \frac{1}{J}S_i^2\theta. \tag{41}$$

Since the second-class constraints are assumed to be equal to zero in a strong way, the Dirac brackets are

$$\{S_i(x), S_j(y)\}^* = 0,$$
$$\{S_i(x), \pi_j(y)\}^* = \delta_{ij}\delta(x-y),$$
$$\{\pi_i(x), \pi_j(y)\}^* = 0. \tag{42}$$

Hence, the gauge-invariant Hamiltonian can be rewritten in the form

$$\tilde{\mathcal{H}} = \frac{1}{2J}\pi_i^2 + \frac{J}{2}(\partial_k S_i)^2 - \frac{1}{2}\frac{(S_i\pi^i)^2}{S_i S^i} + \frac{\lambda}{2}\left(S_i^2 - 1\right)$$
$$= \frac{1}{2J}\pi_i M^{ij}\pi_j + \frac{J}{2}(\partial_k S_i)^2 + \frac{\lambda}{2}\left(S_i^2 - 1\right), \tag{43}$$

where the phase space metric M^{ij} is given by the equality

$$M^{ij} = \delta^{ij} - \frac{S^i S^j}{S_k^2}, \tag{44}$$

which is a singular matrix.

The set of first-class constraints becomes

$$\chi_1 = \pi_\lambda,$$
$$\chi_2 = -\frac{1}{2}\left(S_i^2 - 1\right). \tag{45}$$

The constraint χ_2, originally a second-class constraint, becomes the generator of gauge symmetries and satisfies the first-class property

$$\{\chi_2, \tilde{H}\} = 0. \tag{46}$$

In view of this result, the infinitesimal gauge transformations are computed as

$$\delta S_i = \varepsilon\{S_i, \chi_2\} = 0,$$
$$\delta\pi_i = \varepsilon\{\pi_i, \chi_2\} = \varepsilon S_i,$$
$$\delta\lambda = 0, \tag{47}$$

where ε is an infinitesimal.

It is easy to verify that the Hamiltonian (43) is invariant under these transformations because the S_i are eigenvectors of the phase space metric M_{ij} with null eigenvalues.

4 The Functional Schrödinger Equation

The $2D$ isotropic antiferromagnet is described by the $O(3)$ nonlinear sigma model given by the Lagrangian

$$
\begin{aligned}
L &= \frac{J}{2} \int d^2x \, \partial_\mu S^a \partial^\mu S^a \\
&= \frac{J}{2} \int d^2x \left[(\partial_0 S^a)^2 - (\partial_i S^a)^2 \right] (\mu = 0, 1, 2; i = 1, 2),
\end{aligned}
\tag{48}
$$

with the constraint

$$
\Omega_1 = S^a S^a - 1.
\tag{49}
$$

Here, a is an index related to the $O(3)$ symmetry group, and the metric has signature $(+, -, -)$.

To start our search for the functional Schrödinger, we strongly impose the constraint Ω_1 in (49). This lets us write one of the fields, say S^3, in terms of the fields S^1 and S^2:

$$
S^3 = \sqrt{1 - S^i S^i} \qquad (i = 1, 2).
\tag{50}
$$

From (50), we obtain the result

$$
\partial_\mu S^3 = -\frac{S^i \partial_\mu S^i}{\sqrt{1 - S^i S^i}}.
\tag{51}
$$

Introducing (50) and (51) in (48), we express the model in the fields S^1 and S^2:

$$
L = \frac{J}{2} \int d^2x \, g_{ij} \partial_\mu S^i \partial^\mu S^j,
\tag{52}
$$

where

$$
g_{ij} = \delta_{ij} + \frac{S_i S_j}{1 - S^i S^i}.
\tag{53}
$$

Let us now construct the model Hamiltonian, for subsequent quantization. We compute the momenta

$$
\pi_i = \frac{\partial L}{\partial (\dot{S}^i)}.
\tag{54}
$$

Then, we have that

$$
\pi_i = J g_{ij} \dot{S}^j.
\tag{55}
$$

In order to write the model in its Hamiltonian form, we must invert (55), so that we can express the *velocities* in terms of the momenta. The computation of the inverse of g_{ij} gives us

$$
\tilde{g}^{ij} = \delta^{ij} - S^i S^j,
\tag{56}
$$

so that

$$
\dot{S}^i = \frac{1}{J} \tilde{g}^{ij} \pi_j.
\tag{57}
$$

The model Hamiltonian, the general expression for which is

$$
H = \int d^2x (\pi_i \dot{S}^i - L),
\tag{58}
$$

takes the particular form,

$$
H = \int d^2x \left(\frac{1}{2J} \tilde{g}^{ij} \pi_i \pi_j + \frac{J}{2} g_{ij} \partial_k S^i \partial_k S^j \right),
\tag{59}
$$

where ∂_k denotes the partial derivatives with respect to the spatial coordinates. By definition, the (S^i, π_j) form canonically conjugated pairs, with the usual Poisson brackets,

$$
\{S_i(x), \pi_j(y)\} = \delta_{ij} \delta^2(x - y).
\tag{60}
$$

To derive functional Schrödinger equation for the $2D$ isotropic antiferromagnet, we introduce the wave-functional $\Psi[S^i, t]$ and treat S^i and π_i as quantum operators. In other words, in the field representation, the momenta are replaced by the following functional derivatives:

$$
\pi_i(x) \longrightarrow -i \frac{\delta}{\delta S^i(x)},
\tag{61}
$$

where we have set $\hbar = 1$.

The wave-functional Ψ satisfies the functional Schrödinger equation

$$
i \frac{\partial}{\partial t} \Psi[S^i, t] = \hat{H}[S^i, t] \Psi[S^i, t],
\tag{62}
$$

where $\hat{H}$ is the operator version of the Hamiltonian in (59).

Since $\tilde{g}^{ij}$ depends on the fields, the kinetic term in the on the right-hand side of (59) will give rise to factor-ordering ambiguities upon quantization. To resolve these ambiguities, we choose a particular factor ordering: we write all field functions to the left of the momenta operators. To justify this choice, we note that in the study of the first-class constrained version of the model, the ordering is consistent with the operator version of the classical-constraint algebra. Moreover, the first-class Hamiltonian that will be derived below leads to the same functional Schrödinger equation.

Given (59) and the aforementioned particular choice of factor ordering, we obtain the following functional Schrödinger equation for the isotropic antiferromagnet:

$$
\int d^2x \left(\frac{1}{2J} \tilde{g}^{ij} \frac{\delta^2 \Psi}{\delta S^i \delta S^j} + \frac{J}{2} g_{ij} \partial_k S^i \partial_k S^j \Psi \right) = i \frac{\partial}{\partial t} \Psi.
\tag{63}
$$

Since the Hamiltonian (59) does not explicitly depend on time, we may factor out the time dependence of the wave-functional and write the equality

$$
\Psi[S^i, t] = e^{-Et} \Psi[S^i].
\tag{64}
$$

In view of (63), we then see that $\Psi[S^i]$ must satisfy the time-independent functional Schrödinger equation

$$
\int d^2x \left(\frac{1}{2J} \tilde{g}^{ij} \frac{\delta^2 \Psi}{\delta S^i \delta S^j} + \frac{J}{2} g_{ij} \partial_k S^i \partial_k S^j \Psi \right) = E \Psi.
\tag{65}
$$

It is clear that the solution of (65) will yield the energies E for the studied model.

Let us now consider the $2D$ isotropic antiferromagnet as a first-class constrained field theory. We also want to write the functional Schrödinger equation for our model. To this end, we use Dirac's prescription to canonically quantize first-class constrained systems [10]. As we shall see below, the functional Schrödinger equation will be the same as the one in (63). It proves convenient to write the first-class Hamiltonian of the model in the following way:

$$H = \int d^2x \left[\frac{1}{2J} \bar{g}^{ab} \pi_a \pi_b + \frac{J}{2} \partial_i S^a \partial_i S^a - \lambda \left(S_a^2 - 1 \right) + v_\lambda \pi_\lambda \right],$$
(66)

where

$$\bar{g}^{ab} = \delta^{ab} - \frac{S^a S^b}{S^a S^a}.$$
(67)

We note that $\pi_\lambda = 0$ is a first-class constraint and v_λ is a Lagrangian multiplier. The formulation via (66) is classically equivalent to the initial one, via (59). In the appropriate gauge, the equations of motion for the physical fields in (66) and in (59) are the same [27].

Only one modification of the functional Schrödinger method we have described is necessary to comply with Dirac's prescription for first-class constrained systems. The wave-functional must be annihilated by the operator version of the constraints, besides satisfying the functional Schrödinger equation [28, 32]. In our case, the requirement that the operator version of the constraint χ_2, (45), annihilate the wave-functional imposes no condition upon Ψ. We therefore make the canonical transformation [27],

$$S^3 \longrightarrow -\pi_3, \quad \pi_3 \longrightarrow S^3,$$
(68)

which changes χ_2 and H to

$$\tilde{\chi}_2 = \pi_3 \pi_3 + S^i S^i - 1 = 0,$$
$$\tilde{H} = \int d^2x \left\{ \frac{1}{2J} \left[\pi_i \pi_i - \left(\frac{S^i S^j}{\pi_3 \pi_3 + S^k S^k} \right) \pi_i \pi_j \right] \right.$$
$$+ \frac{J}{2} \left[\partial_x S^i \partial_x S^i + \partial_x \pi_3 \partial_x \pi_3 \right]$$
$$+ \frac{1}{2J} \left[S^3 S^3 + 2 \left(\frac{S^i S^3}{\pi_3 \pi_3 + S^k S^k} \right) \pi_i \pi_3 \right.$$
$$\left. - \left(\frac{S^3 S^3}{\pi_3 \pi_3 + S^k S^k} \right) \pi_3 \pi_3 \right] - \lambda(\pi_3 \pi_3 + S^i S^i - 1)$$
$$+ v_\lambda \pi_\lambda \right\}.$$
(69)

We are now ready to write the equations for the wave-functional $\Psi[S^3, S^i, \lambda]$. The first two will be obtained from the operator version of the constraints $\pi_\lambda = 0$, (45), and $\tilde{\chi}_2$, (69), which annihilate Ψ. The last one is the functional Schrödinger equation, which will be derived from the

operational version of the Hamiltonian $(\hat{\tilde{H}})$, (69). Thus, in the field representations, we have that

$$\frac{\delta \Psi}{\delta \lambda} = 0,$$
(70)

$$-\frac{\delta^2 \Psi}{\delta (S^3)^2} + (S^i S^i - 1)\Psi = 0,$$
(71)

$$i \frac{\partial \Psi}{\partial t} = \int d^2x \left[\frac{1}{2J} \left(-\frac{\delta^2 \Psi}{\delta (S^i)^2} + (S^i S^j) \frac{\delta^2 \Psi}{\delta S^i \delta S^j} \right) \right.$$
$$+ \frac{J}{2} \left(\partial_x S^i \partial_x S^i \Psi - \partial_x \frac{\delta}{\delta S^3} \partial_x \frac{\delta}{\delta S^3} \Psi \right)$$
$$+ \frac{1}{2J} \left(S^3 S^3 \Psi - 2(S^i S^3) \frac{\delta^2 \Psi}{\delta S^i \delta S^3} \right.$$
$$\left. + (S^3 S^3) \frac{\delta^2 \Psi}{\delta (S^3)^2} \right],$$
(72)

where we have explicitly used (71) to deal with the operators in the denominators of the fractions in $\hat{\tilde{H}}$. The particular choice of factor ordering in (70)–(72) preserves the classical-constraint algebra and the involution of the constraints with the Hamiltonian.

From (70) and (71), we have that

$$\Psi[S^i, S^3, \lambda, t] = \Psi[S^i, S^3, t],$$
(73)

$$\Psi[S^i, S^3, t] = \exp\left[\int d^2y\, S^3 \sqrt{S^i S^i - 1} \right] \Psi_{phys}[S^i, t].$$
(74)

Substitution of the right-hand side of (74) for Ψ in the functional Schrödinger equation (72) yields the equality

$$\int d^2x \left(\frac{1}{2J} \tilde{g}^{ij} \frac{\delta^2 \Psi_{phys}}{\delta S^i \delta S^j} + \frac{J}{2} g_{ij} \partial_k S^i \partial_k S^j \Psi_{phys} \right) = E \Psi_{phys},$$
(75)

where g_{ij} and $\tilde{g}^{ij}$ are given by (53) and (56), respectively, and the time dependence of Ψ_{phys} is given by (64). A few terms proportional to the Dirac delta function at the point zero, i.e., $\delta(0)$, appear in the derivation of the (75). These terms contribute energy infinities, which can be removed by the usual regularization techniques.

Comparison of (75) with (65) shows that they are identical.

5 Conclusions

In this work, we have obtained the Dirac brackets of the two-dimensional isotropic antiferromagnet by two different ways: via the usual Dirac formalism and the symplectic formalism. We have used the symplectic embedding formalism

to unveil hidden symmetries in the 2D isotropic antiferromagnet. We have obtained the gauge-invariant Lagrangian and Hamiltonian for the system.

In this context, the equivalence between the $O(3)$ nonlinear sigma model and the CP^1 model deserves mention [34]. In fact, the dynamical variables of the CP^1 model are the pair of complex fields $Z(x) = (Z_1(x), Z_2(x))$ which are constrained to lie on the unit three-sphere $S^3 : Z^*Z = |Z|^2 = |Z_1|^2 + |Z_2|^2 = 1$. This model is described by the nonlinear Lagrangian $\mathcal{L} = \partial_\mu Z^* \partial^\mu Z - (Z^* \partial_\mu Z)(Z \partial^\mu Z^*)$ and it is invariant under a local $U(1)$ gauge symmetry. Using the Hopf bundle [36], $S^a = Z^* \sigma^a Z$, which characterizes maps from S^3 to S^2, where the σ^a are the Pauli matrices, we can see that the Lagrangian (4) is equivalent to the Lagrangian of the CP^1 model. In view of this, the existence of a hidden gauge symmetry in the model could have been expected.

For a particular choice of factor ordering, we wrote the functional Schrödinger equation for the first-class Hamiltonian and for the original Hamiltonian and showed that they are all identical, which justifies our factor-ordering choice. In a future paper, we intend to present the gauge-invariant version for the $2D$ anisotropic antiferromagnet, as well as its functional Schrödinger equation, and discuss the spectra of both the isotropic and anisotropic models.

Appendix A: Symplectic Formalism

In this appendix, we will present a brief review of the symplectic formalism. Let us consider a general first-order Lagrangian described by the symplectic variables ξ^i and their generalized canonical momenta a_i,

$$\mathcal{L} = a_i(\xi)\dot{\xi} - \mathcal{H}(\xi), \tag{76}$$

which is obtained from a conventional second-order Lagrangian by introducing certain auxiliary fields and performing the Legendre transformation. The symplectic potential $V(\xi)$ is none other than the Hamiltonian $\mathcal{H}(\xi)$, as we can see by Legendre transforming the first-order Lagrangian in (76). Thus, the Lagrangian in (76) may be rewritten in the form

$$\mathcal{L}dt = a_i(\xi)d\xi - \mathcal{H}(\xi)dt, \tag{77}$$

and the first term on the right side defines the canonical one-form $a_i d\xi^i \equiv a(\xi)$. Using the variational principle, we obtain the dynamical equations of motion

$$f_{ij}\dot{\xi}^j = \frac{\partial}{\partial \xi^i}\mathcal{H}(\xi), \tag{78}$$

where

$$f_{ij} = \frac{\partial a_j}{\partial \xi^i} - \frac{\partial a_i}{\partial \xi^j}, \tag{79}$$

which is called the symplectic two-form, $\frac{1}{2}f_{ij}d\xi^i d\xi^j \equiv f(\xi)$.

Usually, the geometric structure of the theory is fully determined by the generalized canonical momenta $a_i(\xi)$ and is insensitive to the functional form of $V(\xi)$. The symplectic matrix f_{ij} gives the geometric structure in phase space. Theories are classified as unconstrained or constrained, depending on whether f_{ij} has an inverse or not, respectively.

In the unconstrained case, we can directly obtain equations of motion such that

$$\dot{\xi}^i = f^{ij}\frac{\partial \mathcal{H}(\xi)}{\partial \xi^j}. \tag{80}$$

In this case, we can obtain the generalized symplectic brackets as

$$\dot{\xi}^i = \{\xi^i, \mathcal{H}(\xi)\} = \{\xi^i, \xi^j\}\frac{\partial \mathcal{H}(\xi)}{\partial \xi^j}. \tag{81}$$

Comparing (80) with (81), we obtain the relations between the symplectic two-form matrix and the generalized symplectic bracket

$$f^{ij} = \{\xi^i, \xi^j\}, \tag{82}$$

which correspond to the Dirac brackets.

In the more interesting case, when f_{ij} is singular and constraints arise such that

$$\Omega^{(a)} = \tilde{v}^{(a)i}\frac{\partial}{\partial \xi^i}\mathcal{H}(\xi) = 0, \tag{83}$$

where $\tilde{v}^{(a)i}$ are called zero modes, the superscript i corresponds to the symplectic variables ξ^i, and (a) denotes the number of constraints. Furthermore, from the persistence in time of the constraints in (83), we have that

$$\dot{\Omega}^{(a)} = \frac{\partial \Omega^{(a)}}{\partial \xi^i}\dot{\xi}^i = 0. \tag{84}$$

When this happens, we can modify part of the canonical sector to make the symplectic two-form matrix invertible. To this end, we introduce appropriate Lagrange multipliers and incorporate the requirement in (84), that constraints in the first-order Lagrangian must be stable under time evolution. Thus, the modified first-order Lagrangian is described by the expression

$$\mathcal{L}^{(k)} = a_i^{(k)}(\xi)\dot{\xi}^{(k)i} + \Omega_i^{(k)}\dot{\alpha}^{(k)i} - \mathcal{H}^{(k)}(\xi), \tag{85}$$

where the integer k in the superscript denotes the iteration number to generate the modified nonsingular symplectic matrix, and the kth iteration Hamiltonian,

$$\mathcal{H}^{(k)}(\xi) = \mathcal{H}^{(k-1)}(\xi)\Big|_{\Omega_i^{(k-1)}=0}, \tag{86}$$

corresponds to the reduced Hamiltonian of the theory.

If we come to a nonsingular f_{ij} after a finite number of iterations, we interrupt the iterative sequence and obtain the Dirac brackets from the inverse of the matrix f_{ij}. Otherwise, the sequence grows to infinity, in which case the zero mode plays an important role, generating a gauge symmetry, and the transformation rules are given by the zero mode, such that

$$\delta\xi^{(k)i} = \varepsilon\tilde{v}^{(k)i}, \tag{87}$$

where ε is a function of time.

In this step of the method, we then need some gauge-fixing conditions, which are a kind of constraint. These constraints must be introduced in the canonical sector of the first-order Lagrangian in (85). Following the prescription of the symplectic formalism as already described, we obtain the Dirac brackets, which are the bridge to the quantum commutators.

Appendix B: Brief Review of the General Theory of Symplectic Embedding

This appendix closely follows the ideas in Ref. [22].

Consider a general noninvariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i, \dot{a}_i, t)$, $(i = 1, 2, \ldots, N)$, where a_i and $\dot{a}_i$ are the space and velocities variables, respectively. Notice that this model results in no loss of generality or physical content. In the symplectic method, the zeroth-iterative first-order Lagrangian one-form is written as

$$\mathcal{L}^{(0)}dt = A_\theta^{(0)}d\xi^{(0)\theta} - V^{(0)}(\xi)dt, \tag{88}$$
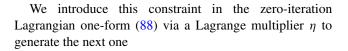
and the symplectic variables are

$$\xi^{(0)\theta} = \begin{cases} a_i, & \text{with } \theta = 1, 2, \ldots, N \\ p_i, & \text{with } \theta = N+1, N+2, \ldots, 2N, \end{cases} \tag{89}$$

where $A_\theta^{(0)}$ are the canonical momenta and $V^{(0)}$ is the symplectic potential. From the Euler–Lagrange equations of motion, we obtain the symplectic tensor

$$f_{\theta\beta}^{(0)} = \frac{\partial A_\beta^{(0)}}{\partial \xi^{(0)\theta}} - \frac{\partial A_\theta^{(0)}}{\partial \xi^{(0)\beta}}. \tag{90}$$

If the two-form $f \equiv \frac{1}{2}f_{\theta\beta}d\xi^\theta \wedge d\xi^\beta$ is singular, the symplectic matrix (90) has a zero mode $v^{(0)}$ that generates a new constraint when it is contracted with the gradient of the symplectic potential,

$$\Omega^{(0)} = v^{(0)\theta}\frac{\partial V^{(0)}}{\partial \xi^{(0)\theta}}. \tag{91}$$

We introduce this constraint in the zero-iteration Lagrangian one-form (88) via a Lagrange multiplier η to generate the next one

$$\begin{aligned}\mathcal{L}^{(1)}dt &= A_\theta^{(0)}d\xi^{(0)\theta} + d\eta\Omega^{(0)} - V^{(0)}(\xi)dt, \\ &= A_\gamma^{(1)}d\xi^{(1)\gamma} - V^{(1)}(\xi)dt \quad (\gamma = 1, 2, \ldots, 2N+1),\end{aligned} \tag{92}$$

and

$$\begin{aligned}V^{(1)} &= V^{(0)}|_{\Omega^{(0)}} = 0, \\ \xi^{(1)\gamma} &= (\xi^{(0)\theta}, \eta), \\ A_\gamma^{(1)} &= (A_\theta^{(0)}, \Omega^{(0)}).\end{aligned} \tag{93}$$

The first-iteration symplectic tensor is then

$$f_{\gamma\beta}^{(1)} = \frac{\partial A_\beta^{(1)}}{\partial \xi^{(1)\gamma}} - \frac{\partial A_\gamma^{(1)}}{\partial \xi^{(1)\beta}}. \tag{94}$$

If this tensor is nonsingular, we stop the iterative process and obtain the Dirac brackets among the phase space variables from the inverse matrix $(f_{\gamma\beta}^{(1)})^{-1}$. Consequently, the Hamilton equation of motion can be computed and solved, as discussed in Ref. [33]. It is well known that a physical system can be described at least classically in terms of a symplectic manifold M. From the physical point of view, M is the phase space of the system, while a nondegenerate closed 2-form f can be identified with the Poisson bracket. To fix the dynamics of the system, we only have to specify a real-valued function (Hamiltonian) H in phase space. In other words, one of these real-valued function solves the Hamilton equation, namely,

$$\iota(X)f = dH, \tag{95}$$

and determines the classical dynamical trajectories of the system in phase space.

If f is nondegenerate, (95) has a unique solution. The nondegeneracy of f means that the linear map $\flat : TM \to T^*M$, defined by $\flat(X) := \flat(X)f$, is an isomorphism. Equation (95) therefore has a unique solution for any Hamiltonian $[X = \flat^{-1}(dH)]$. If, by contrast, the tensor has a zero mode, a new constraint arises, and the iterative process goes on until the symplectic matrix becomes nonsingular or singular. If this matrix is nonsingular, we will be able to determine the Dirac brackets. Reference [33] considers the case of degenerate f in detail.

The central idea in this embedding formalism is to introduce extra fields into the model in order to obstruct the solutions of the Hamiltonian equations of motion. We introduce two arbitrary functions that depend on the original phase space and on WZ variables, namely, $\Psi(a_i, p_i)$ and $G(a_i, p_i, \eta)$, in the first-order Lagrangian one-form as follows:

$$\tilde{\mathcal{L}}^{(0)}dt = A_\theta^{(0)}d\xi^{(0)\theta} + \Psi d\eta - \tilde{V}^{(0)}(\xi)dt, \tag{96}$$

with

$$\tilde{V}^{(0)} = V^{(0)} + G(a_i, p_i, \eta), \tag{97}$$

where the arbitrary function $G(a_i, p_i, \eta)$ is expanded in the WZ field, given by

$$G(a_i, p_i, \eta) = \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta), \qquad \mathcal{G}^{(n)}(a_i, p_i, \eta) \sim \eta^n, \tag{98}$$

and satisfies the following boundary condition:

$$G(a_i, p_i, \eta = 0) = 0. \tag{99}$$

We extend the symplectic variables to include the WZ variable $\tilde{\xi}^{(0)\tilde{\theta}} = (\xi^{(0)\theta}, \eta)(\tilde{\theta} = 1, 2, \ldots, 2N + 1)$ and the first-iterative symplectic potential becomes

$$\tilde{V}^{(0)}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta). \tag{100}$$

In this context, the new canonical momenta are

$$\tilde{A}_{\tilde{\theta}}^{(0)} = \begin{cases} A_{\theta}^{(0)}, & (\tilde{\theta} = 1, 2, \ldots, 2N) \\ \Psi, & (\tilde{\theta} = 2N + 1) \end{cases} \tag{101}$$

and the new symplectic tensor is

$$\tilde{f}_{\tilde{\theta}\tilde{\beta}}^{(0)} = \frac{\partial \tilde{A}_{\tilde{\beta}}^{(0)}}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} - \frac{\partial \tilde{A}_{\tilde{\theta}}^{(0)}}{\partial \tilde{\xi}^{(0)\tilde{\beta}}}, \tag{102}$$

that is,

$$\tilde{f}_{\tilde{\theta}\tilde{\beta}}^{(0)} = \begin{pmatrix} f_{\theta\beta}^{(0)} & f_{\theta\eta}^{(0)} \\ f_{\eta\beta}^{(0)} & 0 \end{pmatrix}. \tag{103}$$

To sum up, we have two steps: in the first, we compute $\Psi(a_i, p_i)$, while in the second, we calculate $G(a_i, p_i, \eta)$. At the beginning, the first step we impose that this new symplectic tensor ($\tilde{f}^{(0)}$) have a zero mode $\tilde{\nu}$, which leads to the following condition:

$$\tilde{\nu}^{(0)\tilde{\theta}} \tilde{f}_{\tilde{\theta}\tilde{\beta}}^{(0)} = 0. \tag{104}$$

At this point, f becomes degenerate. Consequently, we introduce an obstruction to solve, in a unique way, the Hamilton equation of motion in (95). Assuming that the zero mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is

$$\tilde{\nu}^{(0)} = \begin{pmatrix} \mu^{\theta} & 1 \end{pmatrix}, \tag{105}$$

and using the relation in (104) together with (103), we find a set of equations, namely,

$$\mu^{\theta} f_{\theta\beta}^{(0)} + f_{\eta\beta}^{(0)} = 0, \tag{106}$$

where

$$f_{\eta\beta}^{(0)} = \frac{\partial A_{\beta}^{(0)}}{\partial \eta} - \frac{\partial \Psi}{\partial \xi^{(0)\beta}}. \tag{107}$$

The matrix elements μ^{θ} are chosen to disclose the desired gauge symmetry. In this formalism, the zero mode

$\tilde{\nu}^{(0)\tilde{\theta}}$ is the gauge-symmetry generator. This characteristic is important because it opens the possibility of disclosing the desired hidden gauge symmetry from the noninvariant model. It enables the symplectic embedding formalism to deal with noninvariant systems. From (104), we obtain differential equations involving $\Psi(a_i, p_i)$, (106), and after straightforward computation, determine $\Psi(a_i, p_i)$.

To compute $G(a_i, p_i, \eta)$ in the second step, we impose that no additional constraints arise from the contraction of the zero mode $(\tilde{\nu}^{(0)\tilde{\theta}})$ with the gradient of the potential $\tilde{V}^{(0)}(a_i, p_i, \eta)$. This condition generates a general differential equation, which reads

$$\tilde{\nu}^{(0)\tilde{\theta}} \frac{\partial \tilde{V}^{(0)}(a_i, p_i, \eta)}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} = 0, \tag{108}$$

$$\mu^{\theta} \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \mu^{\theta} \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \mu^{\theta} \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \cdots$$
$$+ \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} + \cdots = 0. \tag{109}$$

Equations (108) and (110) allow us to compute all correction terms $\mathcal{G}^{(n)}(a_i, p_i, \eta)$ in order of η. Since this polynomial expansion in η is equal to zero, the coefficient of each power of η must vanish identically. This determines each correction term of order η^n. For the linear correction term, we have that

$$\mu^{\theta} \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{110}$$

For the quadratic term, we find that

$$\mu^{\theta} \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{111}$$

More generally, the following recursive equation for $n \geq 2$ results:

$$\mu^{\theta} \frac{\partial \mathcal{G}^{(n-1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \eta)}{\partial \eta} = 0, \tag{112}$$

which allows us to compute the remaining correction terms in order of η.

This process is iterated until (108) vanish identically so that the extra term $G(a_i, p_i, \eta)$ can be explicitly obtained. The gauge-invariant Hamiltonian, identified with the symplectic potential, is obtained in the form

$$\tilde{\mathcal{H}}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + G(a_i, p_i, \eta), \tag{113}$$

and the zero mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is identified with the generator of an infinitesimal gauge transformation, given by

$$\delta \tilde{\xi}^{\tilde{\theta}} = \varepsilon \tilde{\nu}^{(0)\tilde{\theta}}, \tag{114}$$

where ε is an infinitesimal.

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