STATISTICAL

Decoherence: A Closed-System Approach

**Sebastian Fortin · Olimpia Lombardi ·
Mario Castagnino**

Abstract The aim of this paper is to review a new per-spective about decoherence, according to which formalisms originally devised to deal just with closed or open systems can be subsumed under a closed-system approach that generalizes the traditional account of the phenomenon. This new viewpoint dissolves certain conceptual difficulties of the orthodox open-system approach but, at the same time, shows that the openness of the quantum system is not the essential ingredient for decoherence, as commonly claimed. Moreover, when the behavior of a decoherent system is described from a closed-system perspective, the account of decoherence turns out to be more general than that supplied by the open-system approach, and the quantum-to-classical transition defines unequivocally the realm of classicality by identifying the observables with classical-like behavior.

Keywords Quantum decoherence · Closed system · Relevant observables

1 Introduction

Since more than two decades ago, the environment-induced decoherence (EID) approach is considered a new orthodoxy in the physicists community [1, 2]. It has been fruitfully

S. Fortin (✉)
CONICET, Department of Physics, FCEN (UBA),
Buenos Aires, Argentina
e-mail: sfortin@gmx.net

O. Lombardi
CONICET and FCEN (UBA), Buenos Aires, Argentina

M. Castagnino
CONICET, IAFE (CONICET-UBA), IFIR and FCEN (UBA),
Buenos Aires, Argentina

applied in many areas of physics and supplies the basis of new technological developments. In the foundations of physics community, EID has been viewed as a relevant element for the interpretation of quantum mechanics [3, 4] and for the explanation of the emergence of classicality from the quantum world [5–8].

The great success of EID has given rise to the idea that decoherence necessarily requires the interaction between an open quantum system and an environment of many, potentially infinite, degrees of freedom. However, the historical roots of the decoherence program can be found in certain attempts to explain the emergence of classicality in closed systems. In turn, other approaches have been proposed at present, and the openness of the system is not an essential factor in several of them. These new approaches are usually conceived as rival to EID, or even as dealing with different physical phenomena.

On the basis of our previous works on the subject, the aim of this paper is to supply a comprehensive presentation of a new perspective, according to which formalisms originally devised to deal just with closed or open systems can be subsumed under a closed-system approach that generalizes the traditional account of the phenomenon. With this purpose, we will briefly review the historical development of the decoherence program in Section 2, and we will stress some conceptual difficulties of the EID program in Section 3. In Section 4, we will recall the orthodox approach to decoherence, emphasizing its open-system character. Section 5 will be devoted to introducing a closed-system perspective and to using it to reformulate the orthodox open-system EID approach. On this basis, in Section 6, we will formulate the closed-system approach from a generic viewpoint: in this framework, the account of decoherence turns out to be more general than that supplied by the orthodox open-system approach. In order to

illustrate this theoretical claim, in Section 7, we will consider the description of the well-known spin-bath model in the context of the closed-system framework, as well as the description of a generalization of that model. This task will allow us to argue, in Section 8, that decoherence is relative to the decomposition of the whole closed system into a decoherent system and its environment. On this basis, in Section 9, we will cast a new look at the phenomenon of decoherence by considering how the new theoretical framework dissolves the conceptual difficulties of the EID approach and by discussing the relationship between decoherence and dissipation. Section 10, will be devoted to describe the quantum-to-classical transition from the new closed-system approach, which will allow us to unequivocally define the realm of classicality by identifying the observables with classical-like behavior. In Section 11, we will compare our development with other approaches to decoherence. Finally, in Section 12, we will resume the main results of our paper and draw our conclusions.

2 The Historical Development of the Decoherence Program

From a historical perspective, the decoherence program finds its origin—though, of course, not under this name—in the attempts to explain how a coherent pure state becomes a final decohered mixture with no interference terms. Three general periods can be identified in the development of this program:

- **First period** (closed systems). In the 1950s and the early 1960s, some authors directed their attention to the emergence of classical macroscopic features from quantum microscopic descriptions [9–12]. In this period, the issue was treated in the context of the study of irreversibility and, therefore, closed systems were considered. On this basis, the states indistinguishable from the viewpoint of certain gross observables were described by the same coarse-grained state, whose evolution was proved to reach equilibrium in a certain relaxation time. The main problem of this period was that the relaxation times so obtained turned out to be too long to account for experimental results see [13].
- **Second period** (open systems). In the 1970s, the emergence of classicality begun to be conceived in terms of quantum measurement and, then, was addressed from an open-system perspective [14–16]. On the basis of these precedents, the EID approach was systematized and developed mainly by Zurek and his collaborators in a great number of works [7, 8, 17, 18, 20, 21]. In this context, an open system is considered in interaction with its environment, and the evolution of

its reduced state is studied. EID proves that in many physical models, the interference terms of the reduced state rapidly vanish and the system decoheres in an extremely short decoherence time. This result solves the main problem of the first period; however, the foundations of the EID program are still threatened by certain conceptual problems derived from its open-system perspective (we will return on this point in the next section).

- **Third period** (open and closed systems). Although "EID" is still considered almost as a synonym for "decoherence," other approaches have been proposed to face the conceptual difficulties of EID in the last times [22–28]. Some of these accounts are clearly non-dissipative [29–33], that is, not based on the dissipation of energy from the system to the environment. Among them, the self-induced decoherence approach shows that a closed quantum system with continuous spectrum may decohere by destructive interference [34–43].

In spite of the fact that formalisms for open and closed systems coexist at present, both kinds of approaches are often presented in the literature as alternative scenarios for decoherence, and even as theories dealing with different physical phenomena. In the next sections, we will challenge this common view by showing that the different formalisms can be understood in the context of a general closed-system framework.

3 The Conceptual Difficulties of Environment-Induced Decoherence

In spite of the great success of EID, and of the fact that it is conceived as the orthodox approach for decoherence, it still has to face two conceptual difficulties.

3.1 The Open-System Problem

According to EID, decoherence is a consequence of the interaction between an open system and its environment; this process is what einselects the quantum states that become the candidates to classical states. Therefore, decoherence must always be accompanied by other manifestations of openness, such as the dissipation of energy into the environment. Zurek even considers that the prejudice which seriously delayed the solution of the problem of the emergence of classicality is itself rooted in the fact that the role of the openness of the system was traditionally ignored [7, 8].

If only open systems may decohere, the issue of the emergence of classicality in closed systems from this perspective, in particular, in the Universe as a whole, cannot even be

posed—see [44]. Zurek expresses the criticism to EID in the following terms: *the Universe as a whole is still a single entity with no 'outside' environment, and, therefore, any resolution involving its division into systems is unacceptable* ([45], p.181). This objection has led to the development of the non-dissipative approaches to decoherence which, for this reason, are usually viewed as alternative or rival to the EID approach.

3.2 The Defining Systems Problem

When the EID approach is applied to cosmology, the Universe is split into some degrees of freedom representing the system, and the remaining degrees of freedom that are supposed to be non-accessible and, therefore, play the role of the environment (see, e.g., [46]). The same strategy is followed in the case of internal environments, such as collections of phonons or other internal excitations. The possibility of internal environments shows that EID supplies no general criterion for distinguishing between the system and its environment: the partition of the whole closed system is decided case by case and usually depends on the previous assumption of the observables that will behave classically (see discussion in [37]).

The absence of a general criterion for deciding where to place the cut between system and environment is a serious difficulty for an approach that insists on the essential role played by the openness of the system in the emergence of classicality. Zurek recognizes this problem as a shortcoming of his proposal: *In particular, one issue which has been often taken for granted is looming big as a foundation of the whole decoherence program. It is the question of what are the 'systems' which play such a crucial role in all the discussions of the emergent classicality. This issue was raised earlier, but the progress to date has been slow at best* ([47], p.22).

As we will see, these problems, which seem to be serious conceptual obstacles for the EID approach, lose their original strength when decoherence is understood from a new general perspective. For this purpose, we will begin by comparing the open-system perspective and the closed-system perspective in the next two sections.

4 The Open-System Perspective

As it is well known, given a closed system S_T represented in the Hilbert space $\mathcal{H}_T$, its state $\rho_T(t) \in \mathcal{H}_T \otimes \mathcal{H}_T$ evolves according to the Liouville–von Neumann equation, $i\hbar \, d\rho_T/dt = [H, \rho_T]$, where H is the system Hamiltonian. If the task is to describe the behavior of S_T as a whole, there is no theoretical difference between using its state ρ_T and using the expectation values $\langle O \rangle_{\rho_T}$ of its

observables $O \in \mathcal{H}_T \otimes \mathcal{H}_T$: given the state, we can compute the expectation values of all the observables of the system, and given the expectation values of all the observables (or of the projectors corresponding to a basis of the Hilbert space), we can compute the system state. Therefore, both descriptions are equivalent in this case: although the state-based description is the commonly used one, all the information of physical interest is also given by the expectation values of the system's observables.

If the system is composite, its initial state can be obtained as the tensor product of the states of its subsystems. For example, in the case of two subsystems S_1 and S_2, represented in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_T = \mathcal{H}_1 \otimes \mathcal{H}_2$, and in initial states $\rho_1(0)$ and $\rho_2(0)$, respectively, the initial state of the composite system $S_T = S_1 \cup S_2$ is computed as $\rho_T(0) = \rho_1(0) \otimes \rho_2(0)$ and, conversely, the initial states of the subsystems can be obtained from the initial state of the composite system by means of the operation of partial trace:

$$\rho_1(0) = Tr_2(\rho_T(0)) \in \mathcal{H}_1 \otimes \mathcal{H}_1 \quad \text{and}$$
$$\rho_2(0) = Tr_1(\rho_T(0)) \in \mathcal{H}_2 \otimes \mathcal{H}_2 \tag{1}$$

The question is how to describe the behavior of the subsystems *at any time*.

4.1 States and Observables From the Open-System Perspective

The usual strategy consists in generalizing the partial trace procedure for all times by considering that the reduced states, obtained as

$$\rho_1(t) = Tr_2(\rho_T(t)) \in \mathcal{H}_1 \otimes \mathcal{H}_1 \quad \text{and}$$
$$\rho_2(t) = Tr_1(\rho_T(t)) \in \mathcal{H}_2 \otimes \mathcal{H}_2 \tag{2}$$

are the quantum states of the subsystems S_1 and S_2 in the same sense as $\rho_T(t)$ is the quantum state of the whole closed system S_T (a strategy that is at least controversial: see [48–50]; we will come back to this point below). However, unlike $\rho_T(t)$, the reduced states $\rho_1(t)$ and $\rho_2(t)$ do not always evolve according to the unitary Liouville–von Neumann equation: in the case of interacting subsystems, they evolve according to non-unitary master equations, whose specific forms depend on the particular features of the involved subsystems and, as a consequence, must be constructed case by case.

Nevertheless, if $O_1 \in \mathcal{H}_1 \otimes \mathcal{H}_1$ is any observable of S_1 and $O_2 \in \mathcal{H}_2 \otimes \mathcal{H}_2$ is any observable of S_2, the expectation values of these observables can be computed as

$$\langle O_1 \rangle_{\rho_1(t)} = Tr\left(\rho_1(t)O_1\right) \quad \text{and}$$
$$\langle O_2 \rangle_{\rho_2(t)} = Tr(\rho_2(t)O_2) \tag{3}$$

In fact, the reduced state of a subsystem of a larger system is defined precisely as the density operator by means of which the expectation values of all the observables belonging to that subsystem can be computed. This means that if we focus only on, say, S_1, we can rely on the description given by its reduced state $\rho_1(t)$, since it supplies all the information that we can obtain when we have experimental access only to that subsystem. Therefore, analogously to the closed-system case, in this situation, it is also true that there is no difference between describing the open system S_1 by means of its reduced state $\rho_1(t)$ and describing it by means of the expectation values $\langle O_1 \rangle_{\rho_1(t)}$ of its observables.

4.2 Environment-Induced Decoherence: An Open-System Approach

The orthodox EID approach considers the system S under study—represented in a Hilbert space $\mathcal{H}_S$—in interaction with an environment E—represented in a Hilbert space $\mathcal{H}_E$—that induces decoherence. This approach is based on the analysis of the evolution of the reduced state $\rho_S(t)$ of S represented in a certain pointer basis of $\mathcal{H}_S$. Either by explicitly computing $\rho_S(t)$ or by analyzing the master equation, it can be determined whether, under certain conditions, the reduced state operator becomes diagonal or not in that basis. The diagonalization of its reduced state is viewed as a manifestation of the decoherence of the open system S.

The EID approach proves that in many physical models, the non-diagonal terms of the reduced state rapidly tend to vanish after an extremely short decoherence time t_D:

$$\rho_S(t) \xrightarrow{t \gg t_D} \rho_S^d \tag{4}$$

where ρ_S^d is diagonal in the pointer basis. Thus, it is usually said that the system S decoheres as a consequence of its interaction with the large number of degrees of freedom of the environment E.

We know that the state $\rho_S(t)$ is a Hermitian operator, so it can always be diagonalized in the Smith basis. So, what is special about the state diagonalization in EID? According to EID, the diagonalization of the state must be studied in the preferred basis or pointer basis. Pointer states have some very special proprieties namely [51]:

1. They are singled out by the dynamics, remaining basically unchanged and
2. They result in the smallest entropy increase.

5 The Closed-System Perspective

5.1 States and Observables From the Closed-System Perspective

Although the open-system perspective is what underlies the usual practice in physics, it is not unavoidable. The behavior of the open subsystems can also be described from a closed-system perspective by considering the expectation values of the observables relevant in any case. For example, if we still focus only on the subsystem S_1, the relevant observables $O_R \in \mathcal{H}_T \otimes \mathcal{H}_T$ of the whole closed system $S_T = S_1 \cup S_2$ are those that act only on that subsystem:

$$O_R = O_1 \otimes I_2 \in \mathcal{H}_T \otimes \mathcal{H}_T \tag{5}$$

where the $O_1 \in \mathcal{H}_1 \otimes \mathcal{H}_1$ are the observables of S_1, and $I_2 \in \mathcal{H}_2 \otimes \mathcal{H}_2$ is the identity of the space of observables of S_2. The expectation values of these relevant observables can be computed as

$$\begin{aligned} \langle O_R \rangle_{\rho_T(t)} &= Tr\left(\rho_T(t)O_R\right) = Tr(\rho_T(t)\left(O_1 \otimes I_2\right)) \\ &= Tr(\rho_1(t)O_1) = \langle O_1 \rangle_{\rho_1(t)} \end{aligned} \tag{6}$$

Equation (6) clearly shows that if we want to describe only S_1, its reduced state is not indispensable. The physically relevant information about that subsystem can also be obtained by studying the state $\rho_T(t)$ of the whole closed system S_T and its relevant observables $O_R = O_1 \otimes I_2$. This means that there is no difference between describing the open system S_1 by means of its reduced state $\rho_1(t)$ and describing it from a closed-system perspective by means of the expectation values of the relevant observables $O_R = O_1 \otimes I_2$ of the closed composite system S_T in the state $\rho_T(t)$.

5.2 Environment-Induced Decoherence From a Closed-System Perspective

Although the EID approach relies on the diagonalization of the reduced state, the same phenomenon can also be described from a closed-system perspective (we will consider the discrete case, but analogous arguments can be developed in the continuous case). In fact, analogously to the case of the previous subsection, if we call the whole closed system $U = S \cup E$, represented in the Hilbert space $\mathcal{H}_U$ and whose state is $\rho_U(t)$, in this case, the relevant observables of U are of the form

$$O_R = O_S \otimes I_E \in \mathcal{H}_U \otimes \mathcal{H}_U \tag{7}$$

where the $O_S \in \mathcal{H}_S \otimes \mathcal{H}_S$ are the observables of S, and $I_E \in \mathcal{H}_E \otimes \mathcal{H}_E$ is the identity of the space of observables of

E. The expectation values of these relevant observables can be computed as (see (6))

$$
\begin{aligned}
\langle O_R \rangle_{\rho_U(t)} &= Tr(\rho_U(t) O_R) = Tr(\rho_U(t) (O_S \otimes I_E)) \\
&= Tr(\rho_S(t) O_S) = \langle O_S \rangle_{\rho_S(t)}
\end{aligned}
\tag{8}
$$

It is clear that the evolution of the reduced state $\rho_S(t)$ (see (4)) has its counterpart in the evolution of the expectation values:

$$
\langle O_S \rangle_{\rho_S(t)} \xrightarrow{t \gg t_D} \langle O_S \rangle_{\rho_S^d} = \sum_i \rho_{Sii}^d O_{Sii}
\tag{9}
$$

where the ρ_{Sii}^d and the O_{Sii} are the diagonal components of ρ_S^d and of O_S in the pointer basis, respectively. But since $\langle O_R \rangle_{\rho_U(t)} = \langle O_S \rangle_{\rho_S(t)}$ (see (8)), from the closed-system perspective we can study the time evolution of the expectation value $\langle O_R \rangle_{\rho_U(t)}$: if, for any O_R, such a function tends to settle down, in an extremely short time, in a value $k = \sum_i P_i O_{Sii}$, where $0 \leq P_i \leq 1$ and $\sum_i P_i$, then it can be said that $P_i = \rho_{Sii}^d$ (see (9)). As a consequence,

$$
\begin{aligned}
\langle O_R \rangle_{\rho_U(t)} = \langle O_S \rangle_{\rho_S(t)} &\xrightarrow{t \gg t_D} \sum_i P_i O_{Sii} \\
&= \sum_i \rho_{Sii}^d O_{Sii} = \langle O_S \rangle_{\rho_S^d}
\end{aligned}
\tag{10}
$$

and one can also say that the system has decohered in the pointer basis (see [52]). Moreover, any expectation value $\langle O_S \rangle_{\rho_S^d}$ can also be expressed as the expectation value of the corresponding relevant observable $O_R = O_S \otimes I_E$ in a coarse-grained state $\rho_U^{cg} \in \mathcal{H}_U \otimes \mathcal{H}_U$:

$$
\langle O_S \rangle_{\rho_S^d} = \langle O_R \rangle_{\rho_U^{cg}}
\tag{11}
$$

The state ρ_U^{cg} can be obtained as $\rho_S^d \otimes \widetilde{\delta}_E$, where $\widetilde{\delta}_E \in \mathcal{H}_E \otimes \mathcal{H}_E$ is a normalized identity operator with coefficients $\widetilde{\delta}_{E\alpha\beta} = \delta_{\alpha\beta}/\sum_\gamma \delta_{\gamma\gamma}$. It is quite clear that ρ_U^{cg} is not the quantum state of the closed system U; nevertheless, if we trace off the degrees of freedom of the environment, we recover the diagonalized reduced state of the system:

$$
Tr_E(\rho_U^{cg}) = \rho_S^d
\tag{12}
$$

This means that the coarse-grained state ρ_U^{cg} supplies the same information about the open system S as the reduced state ρ_S^d, but now from the viewpoint of the closed system U. Therefore, from (10) we can conclude that

$$
\langle O_R \rangle_{\rho_U(t)} \xrightarrow{t \gg t_D} \sum_i P_i O_{Sii} = \langle O_R \rangle_{\rho_U^{cg}}
\tag{13}
$$

By following the EID approach, up to this point, we have analyzed the case based on a particular set of relevant observables. In the next section, we will see that the conception of the phenomenon of decoherence can be generalized when different sets of relevant observables are considered.

6 Decoherence: A General Closed-System Approach

As emphasized by Omnés [13], decoherence is just a particular case of the general problem of irreversibility in quantum mechanics. The problem of irreversibility can be roughly expressed in the following terms. Since the quantum state $\rho(t)$ follows an unitary evolution, it cannot reach a final equilibrium state for $t \rightarrow \infty$. Therefore, if the non-unitary evolution towards equilibrium is to be accounted for, a further element has to be added to the unitary evolution. From the most general viewpoint, this element consists in the splitting of the maximal information about the system into a relevant part and an irrelevant part: whereas the irrelevant part is disregarded, the relevant part is retained and its evolution may reach a final equilibrium situation.

This broadly expressed idea can be rephrased in operators language. The maximal information about the system is given by the set of all its potentially possible observables. By selecting a particular subset $\mathcal{O}_R$ of this set, we restrict the maximal information to a relevant part: the expectation values $\langle O_R \rangle_{\rho(t)}$ of the observables $O_R \in \mathcal{O}_R$ express the relevant information about the system. Of course, the decision about which observables are to be considered as relevant depends on the particular purposes in each situation; but without this restriction, irreversible evolutions cannot be described. As we have said above, when EID is described from the closed-system perspective, the phenomenon can be viewed as a particular kind of time evolution of the expectation values of certain relevant observables of the closed system U. In that particular case, the relevant observables are of the form $O_R = O_S \otimes I_E$. The question is whether that account of decoherence can be generalized (see [53–55] for details).

Let us begin by considering the expectation values of certain relevant observables $O_R \in \mathcal{H}_U \otimes \mathcal{H}_U$, which now are not necessarily of the form $O_R = O_S \otimes I_E$. Let us suppose that for any O_R, those expectation values tend to settle down, in an extremely short time, in a value $k = \sum_i P_i O_{Rii}$, where $0 \leq P_i \leq 1$ and $\sum_i P_i$ (see the analogous case in EID in (10)):

$$
\langle O_R \rangle_{\rho_U(t)} \xrightarrow{t \gg t_D} \sum_i P_i O_{Rii}
\tag{14}
$$

Of course, the P_i are not the diagonal elements of a time-independent ρ_U, since the state $\rho_U(t)$ of the closed system evolves unitarily. Nevertheless, the sum of (14) can also be expressed as

$$
\sum_i P_i O_{Rii} = \sum_i \rho_{Uii}^d O_{Rii}
\tag{15}
$$

where the ρ_{Uii}^d can be conceived as the components of a kind of coarse-grained state ρ_U^d, diagonal in a basis $\{|\alpha_i\rangle\}$ of $\mathcal{H}_U$, which plays the role of pointer basis. In other words,

$\langle O_R \rangle_{\rho_U(t)}$ converges, in an extremely short time, to a value that can be computed as if the system were in a state ρ_U^d represented by a diagonal density operator :

$$\langle O_R \rangle_{\rho_U(t)} \xrightarrow{t \gg t_D} \sum_i P_i \, O_{Rii} = \sum_i \rho_{Uii}^d \, O_{Rii} = \langle O_R \rangle_{\rho_U^d}$$

(16)

If (13) and (16) are compared, it is easy to see that in both cases, the expectation values of certain relevant observables of the whole closed system tend very rapidly to certain time-independent values that can be computed as expectation values of those relevant observables in a time-independent diagonal state. Therefore, if (13) describes decoherence, there is no reason to deny that (16) is also a description of the same phenomenon. This means that, although the off-diagonal terms of $\rho_U(t)$ never vanish through the unitary evolution, decoherence obtains because it is a coarse-grained process: the system decoheres from the observational viewpoint given by any observable belonging to the space $\mathcal{O}_R$.

It might be objected that whereas the time-independent diagonal state ρ_U^{cg} is defined in terms of the reduced state ρ_S^d in the case of EID, which is the quantum state of the open system, the time-independent diagonal ρ_U^d is not the quantum state of the closed system in the general approach. However, this objection is based on the assumption that the reduced state of an open system is its quantum state, a position that is seriously challenged by the difference between proper and improper mixtures (see [48, 49]): reduced states are improper mixtures, they are only *a calculational tool* for computing expectation values ([50] , p.48). For this reason, even in the particular context of decoherence, Schlosshauer warns us *against a misinterpretation of reduced density matrices as describing a proper mixture of states* ([50], p.69; see also [56]). Therefore, although it is true that ρ_U^d is not the state of the closed system U but a calculational tool for computing expectation values after the extremely short decoherence time, the same can be said of the reduced state ρ_S^d regarding the open system S.

It is interesting to emphasize in what sense this closed-system perspective for understanding decoherence is more general than the orthodox open-system perspective. The EID approach always selects a set of relevant observables that correspond to a subsystem, with its associated Hilbert space and its reduced state. From the closed-system perspective, by contrast, the set of the relevant observables that decohere is completely generic and may not correspond to a subsystem. For example, one might be interested in the decoherence of a single observable of the closed system U, which certainly does not define a subsystem but, nevertheless, may manifest a classical-like behavior.

Summing up, the closed-system perspective has allowed us to conceptualize decoherence from a viewpoint more general than that supplied by the EID approach: decoherence induced by the environment is a particular case of a phenomenon that involves the evolution of the expectation values of certain relevant observables towards a specific time-independent value. There are, then, certain processes that fall under the concept of decoherence from this general viewpoint, but are not conceptualized as decoherence from the EID approach. It might be thought that this is a merely semantic issue; however, the fact that the difference has physical content will become clear when the problem of the transition from quantum to classical will be considered.

7 The Spin-Bath Model From the Closed-System Approach

In this section, we will analyze the spin-bath model, both in the traditional and in the generalized versions, as presented in previous works, from the open-system approach. We will show how this model leads to conclusions that sound paradoxical when read from an open-system perspective.

7.1 The Traditional Spin-Bath Model

This is a very simple model that has been exactly solved in previous papers ([18, 19]). Here, we will study it from the closed-system perspective presented in the previous section.

Let us consider a closed system $U = P \cup P_1 \cup P_2 \cup ... \cup P_N = P \cup \left(\bigcup_{i=1}^N P_i \right)$, where (1) P is a spin-1/2 particle represented in the Hilbert space $\mathcal{H}_P$, and (2) each P_i is a spin-1/2 particle represented in its Hilbert space $\mathcal{H}_i$. The Hilbert space of the composite system is, then,

$$\mathcal{H} = (\mathcal{H}_P) \otimes \left(\bigotimes_{i=1}^N \mathcal{H}_i \right)$$

(17)

For the particle P, the two eigenstates of the spin operator $S_{S,\vec{v}}$ in direction $\vec{v}$ are $|\Uparrow\rangle$ and $|\Downarrow\rangle$, such that $S_{S,\vec{v}} |\Uparrow\rangle = \frac{1}{2} |\Uparrow\rangle$ and $S_{S,\vec{v}} |\Downarrow\rangle = -\frac{1}{2} |\Downarrow\rangle$. For each particle P_i, the two eigenstates of the corresponding spin operator $S_{i,\vec{v}}$ in direction $\vec{v}$ are $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$, such that $S_{i,\vec{v}} |\uparrow_i\rangle = \frac{1}{2} |\uparrow_i\rangle$ and $S_{i,\vec{v}} |\downarrow_i\rangle = \frac{1}{2} |\downarrow_i\rangle$. Therefore, a pure initial state of U reads

$$|\psi_0\rangle = (a \, |\Uparrow\rangle + b \, |\Downarrow\rangle) \bigotimes_{i=1}^N (\alpha_i | \uparrow_i\rangle + \beta_i | \downarrow_i\rangle)$$

(18)

where the coefficients a, b, α_i, β_i are such that satisfy $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$. If the self-Hamiltonians H_P of P and H_i of P_i are taken to be zero, and there is no interaction among the P_i, then the total Hamiltonian H

of the composite system is given by the interaction between the particle P and each particle P_i. For instance (see [18]),

$$H = \frac{1}{2} \left(|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow| \right) \otimes \sum_{i=1}^{N} g_i \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right) \otimes \left(\bigotimes_{\substack{j=1 \\ j\neq i}}^{N} I_j \right) \tag{19}$$

where $I_j = |\uparrow_j\rangle\langle\uparrow_j| + |\downarrow_j\rangle\langle\downarrow_j|$ is the identity operator on the subspace $\mathcal{H}_j$ and the g_i are the coupling constants.

7.1.1 Decomposition 1

In the typical situation studied by the EID approach, the open system S is the particle P and the remaining particles P_i play the role of the environment E: $S = P$ and $E = \bigcup_{i=1}^{N} P_i$. Then, the decomposition for this case is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_P) \otimes \left(\bigotimes_{i=1}^{N} \mathcal{H}_i \right) \tag{20}$$

and the relevant observables O_R of U are those corresponding to the particle P:

$$O_R = O_S \otimes I_E = \left(s_{\Uparrow\Uparrow} |\Uparrow\rangle\langle\Uparrow| + s_{\Uparrow\Downarrow} |\Uparrow\rangle\langle\Downarrow| + s_{\Downarrow\Uparrow} |\Downarrow\rangle\langle\Uparrow| \right.$$
$$\left. + s_{\Downarrow\Downarrow} |\Downarrow\rangle\langle\Downarrow| \right) \otimes \left(\bigotimes_{i=1}^{N} I_i \right) \tag{21}$$

The expectation value of these observables in the state $|\psi(t)\rangle = |\psi_0\rangle \, e^{-iHt}$ is given by (see [57])

$$\langle O_R \rangle_{\psi(t)} = |a|^2 s_{\Uparrow\Uparrow} + |b|^2 s_{\Downarrow\Downarrow}$$
$$+ 2\,\mathrm{Re}[ab^* s_{\Downarrow\Uparrow} r(t)] = \sum\nolimits^d + \sum\nolimits^{nd}(t) \tag{22}$$

where

$$r(t) = \langle \varepsilon_\Downarrow(t) | \varepsilon_\Uparrow(t) \rangle = \prod_{i=1}^{N} \left[|\alpha_i|^2 e^{ig_i t} + |\beta_i|^2 e^{-ig_i t} \right] \tag{23}$$

By means of numerical simulations, it is shown that, for $N \gg 1$, in general $|r(t)|^2 \to 0$ and, therefore, $\sum^{nd}(t) \to 0$: the particle P decoheres in interaction with a large environment E composed by N particles P_i (see [50]; for larger values of N and realistic values of the g_i in typical models of spin interaction, see [57]).

7.1.2 Decomposition 2

Although in the usual presentations of the model the system of interest is P, there are different ways of splitting the whole closed system U. For instance, we can decide to observe a particular particle P_j of what was previously considered the environment, and to consider the remaining particles as the new environment: $S = P_j$ and $E = P \cup \left(\bigcup_{i=1, i\neq j}^{N} P_i \right)$. The total Hilbert space of the closed composite system U is still given by (17), but now the decomposition is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_j) \otimes \left(\mathcal{H}_P \otimes \left(\bigotimes_{\substack{i=1 \\ i\neq j}}^{N} \mathcal{H}_i \right) \right) \tag{24}$$

and the relevant observables O_R of U are those corresponding to the particle P_j:

$$O_R = O_S \otimes I_E = \left(\xi_{\uparrow\uparrow}^j |\uparrow_j\rangle\langle\uparrow_j| + \xi_{\uparrow\downarrow}^j |\uparrow_j\rangle\langle\downarrow_j| \right.$$
$$\left. + \xi_{\downarrow\uparrow}^j |\downarrow_j\rangle\langle\uparrow_j| + \xi_{\downarrow\downarrow}^j |\downarrow_j\rangle\langle\downarrow_j| \right)$$
$$\otimes \left(I_P \otimes \left(\bigotimes_{\substack{i=1 \\ i\neq j}}^{N} I_i \right) \right) \tag{25}$$

The expectation value of these observables in the state $|\psi(t)\rangle$ is given by ([57])

$$\langle O_R \rangle_{\psi(t)} = |\alpha_j|^2 \xi_{\uparrow\uparrow}^j + |\beta_j|^2 \xi_{\downarrow\downarrow}^j$$
$$+ 2\,\mathrm{Re}[\alpha_j \beta_j^* \xi_{\downarrow\uparrow}^j e^{ig_j t}] = \sum\nolimits^d + \sum\nolimits^{nd}(t) \tag{26}$$

In this case, numerical simulations are not necessary to see that the time-dependent term of (26) is an oscillating function which, therefore, has no limit for $t \to \infty$. This result is not surprising, but completely reasonable from a physical point of view. In fact, with the exception of the particle P, the remaining particles of the environment E are uncoupled to each other: each P_i evolves as a free system and, as a consequence, E is unable to reach a final stable state.

7.2 A Generalized Spin-Bath Model

Let us now consider a closed system $U = A \cup B$ where

(1) The subsystem A is composed of M spin-1/2 particles A_i, with $i = 1, 2, ..., M$, each one represented in its Hilbert space $\mathcal{H}_{A_i}$: in each A_i, the two eigenstates of the spin operator $S_{A_i, \overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\Uparrow_i\rangle$ and $|\Downarrow_i\rangle$.

(2) The subsystem B is composed of N spin-1/2 particles B_k, with $k = 1, 2, ..., N$, each one represented in its Hilbert space $\mathcal{H}_{B_k}$: in each B_k, the two eigenstates of the spin operator $S_{B_k, \overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\uparrow_k\rangle$ and $|\downarrow_k\rangle$.

The Hilbert space of the composite system $U = A \cup B$ is then

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \left(\bigotimes_{i=1}^{M} \mathcal{H}_{A_i} \right) \otimes \left(\bigotimes_{k=1}^{N} \mathcal{H}_{B_k} \right) \tag{27}$$

and a pure initial state of U reads

$$|\psi_0\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \left(\bigotimes_{i=1}^{M}(a_i\,|\Uparrow_i\rangle + b_i\,|\Downarrow_i\rangle)\right)$$
$$\otimes \left(\bigotimes_{k=1}^{N}(\alpha_k|\uparrow_k\rangle + \beta_k|\downarrow_k\rangle)\right) \quad (28)$$

with $|a_i|^2 + |b_i|^2 = 1$ and $|\alpha_k|^2 + |\beta_k|^2 = 1$. As in the original spin-bath model, the self-Hamiltonians H_{A_i} and H_{B_k} are taken to be zero, and there is no interaction among the particles A_i nor among the particles B_k. As a consequence, the total Hamiltonian of the composite system is given by

$$H = \frac{1}{2}\left(|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|\right)$$
$$\times \left(\sum_{i=1}^{M}\frac{1}{2}\left(|\Uparrow_i\rangle\langle\Uparrow_i| - |\Downarrow_i\rangle\langle\Downarrow_i|\right) \otimes \left(\bigotimes_{\substack{j=1\\j\neq i}}^{M}I_{A_j}\right)\right)$$
$$\otimes \left(\sum_{k=1}^{N}g_k\left(|\uparrow_k\rangle\langle\uparrow_k| - |\downarrow_k\rangle\langle\downarrow_k|\right) \otimes \left(\bigotimes_{\substack{l=1\\l\neq k}}^{N}I_l\right)\right) \quad (29)$$

where $I_{A_j} = |\Uparrow_j\rangle\langle\Uparrow_j| + |\Downarrow_j\rangle\langle\Downarrow_j|$ and $I_{B_k} = |\uparrow_k\rangle\langle\uparrow_k| + |\downarrow_k\rangle\langle\downarrow_k|$ are the identity operators on the subspaces $\mathcal{H}_{A_j}$ and $\mathcal{H}_{B_k}$, respectively. Let us notice that the (19) of the original model is the particular case of (29) for $M = 1$.

7.2.1 Decomposition 1

We can consider the decomposition where A is the open system S and B is the environment E. This is a generalization of Decomposition 1 in the traditional spin-bath model: the only difference is that here S is composed of $M \geq 1$ particles instead of only one. Then, the decomposition is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left(\bigotimes_{i=1}^{M}\mathcal{H}_{A_i}\right) \otimes \left(\bigotimes_{k=1}^{N}\mathcal{H}_{B_k}\right) \quad (30)$$

and the relevant observables O_R are those corresponding to A:

$$O_R = O_S \otimes I_E = O_A \otimes \left(\bigotimes_{i=1}^{N}I_i\right) \quad (31)$$

When the expectation value $\langle O_R\rangle_{\psi(t)} = \sum^d + \sum^{nd}(t)$ of the observables O_R in the state $|\psi(t)\rangle$ is computed, two cases can be distinguished:

- **Case (a)**: $M \ll N$
  Numerical simulations show that $\sum^{nd}(t) \to 0$ very fast for increasing time (see Fig. 2 of [57]). This means

that, as expected, a small open system $S = A$ of M particles decoheres in interaction with a large environment $E = B$ of $N \gg M$ particles.

- **Case (b)**: $M \gg N$ or $M \simeq N$
  Numerical simulations show that $\sum^{nd}(t)$ exhibits an oscillating behavior and, then, it does not approach zero for increasing time (see Figs. 3 and 4 of [57]). This means that when the environment $E = B$ of N particles is not large enough when compared with the open system $S = A$ of M particles, S does not decohere.

7.2.2 Decomposition 2

In this case, we decide to observe only one particle of A. This amounts to splitting the closed system U into two new subsystems: the open system S is, say, the particle A_M, and the environment E is $\left(\bigcup_{i=1}^{M-1}A_i\right) \cup \left(\bigcup_{k=1}^{N}B_k\right)$. Let us notice that the Decomposition 2 of the traditional spin-bath model is a particular case of this one, for $N = 1$ (where N plays the role of the M of this case). The decomposition here is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left(\mathcal{H}_{A_M}\right) \otimes \left(\left(\bigotimes_{i=1}^{M-1}\mathcal{H}_{A_i}\right) \otimes \left(\bigotimes_{k=1}^{N}\mathcal{H}_{B_k}\right)\right) \quad (32)$$

and the relevant observables O_R of U are those corresponding to A_M:

$$O_R = O_S \otimes I_E = O_{A_M} \otimes \left(\left(\bigotimes_{i=1}^{M-1}I_i\right) \otimes \left(\bigotimes_{k=1}^{N}I_k\right)\right) \quad (33)$$

When the expectation value $\langle O_R\rangle_{\psi(t)} = \sum^d + \sum^{nd}(t)$ is computed, numerical simulations show that if $N \gg 1$, $\sum^{nd}(t) \to 0$ very fast for increasing time (see Figs. 5, 6, and 7 of [57]). This means that the particle A_M decoheres when $N \gg 1$, independently of the value of M. But since the particle A_M was arbitrarily selected, the same argument holds for any particle A_i of A. Then, when $N \gg 1$ and independently of the value of M, any particle A_i decoheres in interaction with its environment E of $N + M - 1$ particles. On the other hand, the symmetry of the whole system U allows us to draw analogous conclusions when the system S is one of the particles of B: when $M \gg 1$ and independently of the value of N, any particle B_k decoheres in interaction with its environment E of $N + M - 1$ particles.

7.3 Analyzing Results

Let us consider the generalized spin-bath model when $M = N \gg 1$. In this case, the subsystem $A = \bigcup_{i=1}^{M}A_i$ does not decohere (Decomposition 1), but the particles A_i, considered independently, do decohere (Decomposition 2). In

other words, in spite of the fact that certain particles decohere and may behave classically, the subsystem composed by all of them retains its quantum nature. We have also seen that since $M \gg 1$, all the particles B_k, considered independently, decohere. Then, in this case, not only all the A_i but also all the B_k decohere. This means that all the particles of the closed system $U = \left(\bigcup_{i=1}^{M} A_i \right) \cup \left(\bigcup_{k=1}^{N} B_k \right)$ may become classical when considered independently, although the whole system U certainly does not decohere and, therefore, retains its quantum character.

The fact that certain particles may be classical or quantum depending on how they are considered sounds paradoxical in the context of an approach that explains decoherence as the result of an interaction between open systems. This difficulty can also be seen as a manifestation of the looming big problem of defining the open systems involved in decoherence. The irony of this story is that such a problem is the consequence of what has been considered to be the main advantage of the EID program, its open-system perspective, according to which particles interacting with other particles are well-defined open systems, and the collections of those particles are open systems too. So, the problem is to decide which one of all these open systems is the system S that decoheres or, in other words, where to place the cut between the system S and its environment E.

The open-system approach not only leads to the looming big problem but also, in a certain sense, disregards the well-known holism of quantum mechanics: a quantum system is not the mere collection of its parts and the interactions among them (Fig. 1) . In order to retain its holistic nature, a quantum system has to be considered as a whole: the open subsystems are only partial descriptions of the whole closed system. It is on the basis of this closed-system perspective that a different conceptual viewpoint for understanding decoherence can be developed.

8 The Relative Nature of Decoherence

From the closed-system perspective introduced in the previous sections, the discrimination between system and environment turns out to be the selection of the relevant

observables. By following Harshman and Wickramasekara [58], we will use the expression tensor product structure (TPS) to call any factorization $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of a Hilbert space $\mathcal{H}$, defined by the set of observables $\{O_A \otimes I_B, I_A \otimes O_B\}$, such that the eigenbases of the sets $\{O_A\}$ and $\{O_B\}$ are bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. If $\mathcal{H}$ corresponds to a closed system U, the TPS $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ represents the decomposition of U into two open systems S_A and S_B, corresponding to the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. In turn, given the space $\mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ of the observables of U, such a decomposition identifies the spaces $\mathcal{O}_A = \mathcal{H}_A \otimes \mathcal{H}_A$ and $\mathcal{O}_B = \mathcal{H}_B \otimes \mathcal{H}_B$ of the observables of the open systems S_A and S_B, such that $\mathcal{O}_A \otimes I_B \subset \mathcal{O}$ and $I_A \otimes \mathcal{O}_A \subset \mathcal{O}$. In particular, the total Hamiltonian of U, $H \in \mathcal{O}$, can be expressed as $H = H_A \otimes I_B + I_B \otimes H_A + H_{AB}$, where $H_A \in \mathcal{O}_A$ is the Hamiltonian of S_A, $H_B \in \mathcal{O}_B$ is the Hamiltonian of S_B, and $H_{AB} \in \mathcal{O}$ is the interaction Hamiltonian, representing the interaction between the open systems S_A and S_B.

In general, a quantum system U admits a variety of TPSs, that is, of decompositions into S_A and S_B. Among all these possible decompositions, there may be a particular TPS that remains *dynamically invariant* (see [58]). This is the case when there is no interaction between S_A and S_B, $H_{AB} = 0$, and, then,

$$[H_A \otimes I_B, I_B \otimes H_A] = 0 \quad \Rightarrow \quad \exp(-iHt)$$
$$= \exp(-iH_At) \exp(-iH_Bt) \quad (34)$$

Therefore,

$$\rho_A(t) = Tr_B \rho(t) = Tr_B \left(e^{-iHt} \rho_0 e^{iHt} \right)$$
$$= e^{-iH_At}(Tr_B \rho_0)e^{iH_At} = e^{-iH_At} \rho_{A0} e^{iH_At} \quad (35)$$

$$\rho_B(t) = Tr_A \rho(t) = Tr_A \left(e^{-iHt} \rho_0 e^{iHt} \right)$$
$$= e^{-iH_Bt}(Tr_A \rho_0)e^{iH_Bt} = e^{-iH_Bt} \rho_{B0} e^{iH_Bt} \quad (36)$$

This means that even if the initial state ρ_0 of U is an entangled state with respect to the TPS $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, S_A and S_B are *dynamically independent*: each one evolves unitarily

Fig. 1 Schema of the interactions among the particles: **a** generalized spin-bath model ($M > 1$), and **b** original spin-bath model ($M = 1$)
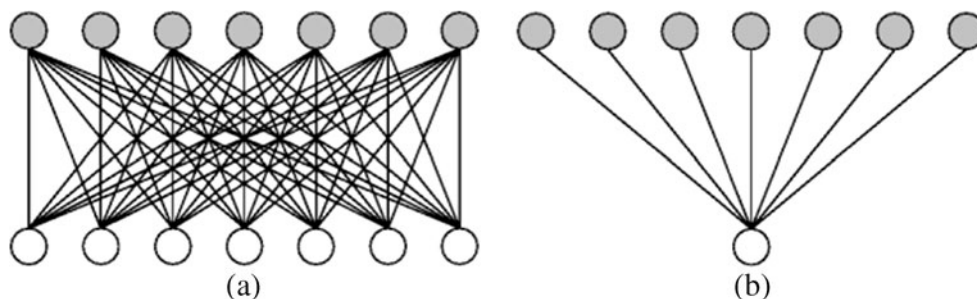
under the action of its own Hamiltonian. As a consequence, the subsystems S_A and S_B resulting from this particular TPS do not decohere.

Once we have excluded the dynamically invariant TPS, all the remaining TPSs of U define subsystems S_A and S_B such that $H_{AB} \neq 0$. As a result of the interaction, S_A and S_B evolve non-unitarily; then, depending on the particular H_{AB}, they may decohere. But the point to stress here is that there is no privileged non-dynamically invariant decomposition of U: each partition of the closed system into S_A and S_B is just a way of selecting the spaces of observables $\mathcal{O}_A$ and $\mathcal{O}_B$.

Once these concepts are considered, the selection of the space $\mathcal{O}_R$ of relevant observables in the EID approach amounts to the selection of a particular TPS, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, such that $\mathcal{O}_R = \mathcal{O}_S \otimes I_E \subset \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$. From this closed-system perspective, it turns out to be clear that there is no essential criterion for identifying the open system and its environment. Given the closed system U, that identification requires two steps: (1) to select a TPS $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ such that $U = S_A \cup S_B$ and (2) to decide that one of the systems resulting from the decomposition, say S_A, is the open system S, and the other, S_B, is the environment E. Since the TPS is defined by the spaces of observables $\mathcal{O}_A$ and $\mathcal{O}_B$, the decomposition of U is just the adoption of a descriptive perspective: the identification of S and E amounts to the selection of the relevant observables in each situation. But since the split can be performed in many ways, with no privileged decomposition, there is no need of an unequivocal criterion for deciding where to place the cut between the system and the environment. Decoherence is not a yes-or-no process, but a phenomenon *relative* to the chosen decomposition of the whole closed quantum system.

From this perspective, the perplexities derived from the generalized spin-bath model vanish. In fact, when we consider the whole closed system U, there is no difficulty in saying that from the viewpoint of the space of observables, say $\mathcal{O}_{A_1}$, (corresponding to the particle A_1) there is decoherence, but from the viewpoint of the space of observables $\mathcal{O}_A$ (corresponding to the open subsystem $A = \bigcup_{i=1}^{M} A_i$) there is no decoherence. Moreover, even if there is decoherence from the viewpoint of all the $\mathcal{O}_{A_i}$, this does not imply decoherence from the viewpoint of $\mathcal{O}_A$ since, as it is well known, $\mathcal{O}_A$ is not the mere union of the $\mathcal{O}_{A_i} \otimes \left(\bigotimes_{j=1, j \neq i}^{M} I_j \right)$. In other words, in agreement with quantum holism, the open subsystem A is not the mere collection of the particles A_i; then, it is reasonable to expect that the behavior of A cannot be inferred from the behavior of all the A_i. In the same sense, it is not surprising that there is no decoherence from the viewpoint of the total space of observables $\mathcal{O}$ of U, in spite of the fact that there is decoherence from the viewpoint of anyone of the $\mathcal{O}_{A_i}$ and $\mathcal{O}_{B_k}$, corresponding to the particles A_i and B_k, respectively. And since the privileged

viewpoint does not exist, the conclusions about decoherence have to be relativized to the particular observational perspective selected in each case.

9 A New Look at Environment-Induced Decoherence

9.1 Dissolving the Conceptual Problems

When decoherence is understood in the new general framework, the conceptual difficulties of the EID program turn out to be not as serious as originally supposed. In fact:

1. With respect to the open-system problem, it does not make sense to say that closed quantum systems may not decohere: a closed quantum system may decohere from certain observational perspectives, that is, from the viewpoint of certain spaces of observables. Furthermore, in spite of the fact that EID focuses on open systems, it can also be formulated from the perspective of the composite closed system and, in this case, meaningful relationships between the behavior of the whole system and the behavior of its subsystems can be explained (see [52, 53]).

2. The defining systems problem is simply dissolved by the fact that the splitting of the closed system into an open subsystem and an environment is just a way of selecting the relevant observables of the closed system. Since there are many different sets of relevant observables depending on the observational viewpoint adopted, the same closed system can be decomposed in many different ways: each decomposition represents a decision about which degrees of freedom are relevant and which can be disregarded in any case. Since there is no privileged or essential decomposition, there is no need of an unequivocal criterion for deciding where to place the cut between the system and the environment. As said above, decoherence is not a yes-or-no process, but a phenomenon relative to the chosen decomposition. Therefore, Zurek's looming big problem is not a real threat to the EID approach: the supposed challenge dissolves once the relative nature of decoherence is taken into account (see [59, 60]).

Although the new framework neutralizes the conceptual difficulties of the EID approach, it also points to some warnings about the way in which the proposal is usually presented. From the new perspective, the insistence on the essential role played by the openness of a system and its interaction with the environment in the phenomenon of decoherence sounds rather misleading. The essential physical fact is that among all the observational viewpoints that

may be adopted to study a closed system, some of them determine a subset of relevant observables for which the system decoheres.

9.2 Decoherence and Dissipation

As pointed out in Section 2, certain presentations of the EID approach suggest the existence of a certain relationship between decoherence and dissipation, as if decoherence were a physical consequence of or, at least, were strongly linked to energy dissipation. Some particular models studied in the literature on the subject tend to reinforce this idea by describing the behavior of a small open system—typically, a particle—immersed in a large environmental bath. On this basis, the EID approach has been considered a dissipative approach, by contrast to non-dissipative accounts of decoherence that constitute the heterodoxy in the field [29–33].

The fact that energy dissipation is not a condition for decoherence has been clearly emphasized by Schlosshauer, who claims that *decoherence may, but does not have to, be accompanied by dissipation, whereas the presence of dissipation also implies the occurrence of decoherence* ([50], p.93). This fact is explained by stressing that the loss of energy from the system is a classical effect, leading to thermal equilibrium in the relaxation time, whereas decoherence is a pure quantum effect that takes place in the decoherence time, many orders of magnitude shorter than the relaxation time: *If dissipation and decoherence are both present, they are usually quite easily distinguished because of their very different timescales* ([50], p.93). According to the author, it is this crucial difference between relaxation and decoherence timescales that explains why we observe macroscopic objects to follow Newtonian trajectories — effectively created through the action of decoherence—with no manifestation of energy dissipation, such as a slowing-down of the object. Schlosshauer recalls an example used by Joos et al. [61]: the planet Jupiter has been revolving around the sun on a Newtonian trajectory for billions of years, while its motional state has remained virtually unaffected by any dissipative loss.

This explanation, although correctly stressing the difference between decoherence and dissipation, seems to present both phenomena on the same footing: an open system would first become classical through decoherence, and would then relax due to energy dissipation. Following this view, dissipation involves the loss of energy from the system to the environment, while decoherence amounts to a sort of dissipation of coherence that leads the open system, in a very short time, to the classical regime: the environment plays the role of a sink that carries away the information about the system ([50], p.85). The results obtained in the generalized spin-bath model show that the coherence–dissipation

or information–dissipation picture has to be considered with great caution, as a mere metaphor. In fact, to the extent that decoherence is a relative phenomenon, no flow of a non-relative quantity from the open system to the environment can account for decoherence. In particular, although energy dissipation and decoherence are in general easily distinguished because of their different timescales, the very reason for their difference is that energy dissipation is not a relative phenomenon but results from the effective flow of a physical entity, whereas decoherence is relative to the observational partition of the whole closed system selected in each situation. On the other hand, decoherence can be explained in terms of the flow of information from the open system to the environment if information is also conceived as a relative magnitude ([62, 63]).

10 The Quantum-To-Classical Transition

As it is well known, one of the main aims of the decoherence program is to explain *the transition from quantum to classical* ([20], p.36). For this purpose, it is expected that the phenomenon of decoherence cancels the quantum features that preclude classicality. In this section, we will see how this may happen.

Let us begin by recalling that in the quantum domain, the expectation values are the experimentally accessible magnitudes: we do not measure states but expectation values. Therefore, the purpose must be to obtain quantum expectation values with a classical-like form. The expectation value of a classical observable O, whose possible values are o_i, is computed as

$$\langle O \rangle = \sum_i P_i \, o_i \qquad (37)$$

where P_i is the probability corresponding to the value o_i. In quantum mechanics, by contrast, the expectation value of an observable O in the state ρ is computed as

$$\langle O \rangle_\rho = Tr(\rho O) = \sum_{ij} \rho_{ji} \, O_{ij} = \sum_i \rho_{ii} \, O_{ii}$$
$$+ \sum_{i \neq j} \rho_{ji} \, O_{ij} = \sum{}^D + \sum{}^{ND} \qquad (38)$$

where the O_{ij} and the ρ_{ji} are the components of O and ρ in a given basis. It is important to note that in this approach, the terms of the mean values must be written in the same preferred basis of EID (however, it is also possible to define the baseline from the mean values [64, 65]). Therefore, if the purpose is that quantum expectation values behave as in the classical case, we need to identify the situation in which they acquire a form that resembles the classical one, that is, a situation in which the non-diagonal sum $\sum{}^{ND}$ vanishes and only the diagonal sum $\sum{}^D$ remains.

10.1 The Quantum-to-Classical Transition in Environment-Induced Decoherence

Since the non-diagonal terms of a quantum state do not have a classical analogue, in the EID approach, the diagonalization of the reduced state of the system is viewed as a manifestation of the quantum-to-classical transition: the states of the pointer basis, einselected by decoherence, are the candidates to classical states (strictly speaking, in order to obtain a classical-like description, it is necessary to apply the Wigner transformation and the limit $\hbar \longrightarrow 0$, see [7]). In most models, the system is a particle and, as a consequence, it is usually said that the particle becomes—or, at least, begins to behave as if it were— classical.

In fact, the diagonalization of the reduced state $\rho_S(t)$ of S in the pointer basis leads to a situation in which certain expectation values acquire a classical-like form. Since ρ_S^d is diagonal, it can be conceived as representing a classical-like probability that makes the interference terms represented by $\sum^{ND}$ to be zero (see (9)):

$$\langle O_S \rangle_{\rho_S^d} = \sum_i \rho_{Sii}^d \, O_{Sii} \tag{39}$$

As a consequence, the expectation value of any observable O_S^c of S that is diagonal in the pointer basis (the same of EID) will have the classical form of (37):

$$\langle O_S^c \rangle_{\rho_S^d} = \sum_i \rho_{Sii}^d \, o_{Si}^c \tag{40}$$

where the diagonal components ρ_{Sii}^d of ρ_S^d play the role of the classical-like probabilities corresponding to the possible values —eigenvalues— o_{Si}^c of the observable O_S^c.

In summary, the decoherence of the observables corresponding to the system S selects the pointer basis in which the reduced state $\rho_S(t)$ becomes diagonal after the decoherence time and, in turn, that pointer basis identifies the observables of S whose expectation values acquire a classical form. As a consequence, instead of saying that the system S behaves as if it were classical or that the states of the pointer basis become classical, it would be more accurate to say that *the observables O_S^c of S behave as if they were classical observables*. Moreover, on the basis of the arguments of the previous sections, from the closed-system perspective one could also say that the observables $O_R^c = O_S^c \otimes I_E$ of the whole closed system $U = S \cup E$ behave as if they were classical observables.

10.2 The Quantum-to-Classical Transition: A More General Approach

As we have said in Section 6, from our viewpoint, decoherence is a process that leads $\langle O_R \rangle_{\rho_U(t)}$ to converge, in an extremely short time, to a value that can be computed as if the system were in a state represented by a diagonal density operator (see (16)):

$$\langle O_R \rangle_{\rho_U^d} = \sum_i \rho_{Uii}^d \, O_{Rii} \tag{41}$$

Now—analogously to the EID case—the expectation value of any observable O_R^c of U that is diagonal in the pointer basis will have the classical form of (37):

$$\langle O_R^c \rangle_{\rho_U^d} = \sum_i \rho_{Uii}^d \, o_{Ri}^c \tag{42}$$

where the diagonal components ρ_{Uii}^d of ρ_U^d play the role of the classical-like probabilities corresponding to the possible values—eigenvalues—o_{Ri}^c of the observable O_R^c.

In complete analogy with the quantum-to-classical transition in EID, here, the decoherence of the relevant observables of U selects the pointer basis in which the state ρ_U^d is diagonal and, in turn, this pointer basis identifies, among all the relevant observables, those whose expectation values acquire a classical form. But in this case, the relevant observables do not need to correspond to a subsystem of U: it is possible to focus attention on any group of observables of the closed system, or even on a single observable, in order to know if they behave in a classical-like manner.

Since according to the EID approach, only open systems may decohere, the issue of the emergence of classicality in closed systems from this perspective, in particular, in the Universe as a whole, cannot even be posed since *the state of a perfectly isolated fragment of the Universe —of, for that matter, of the quantum universe as a whole— would evolve forever deterministically* ([45], p.181). Nevertheless, this difficulty can be overcome from the closed-system perspective. In fact, once the idea of the decoherence and the classicality of systems or states is replaced by the view that certain observables decohere and may acquire a classical-like behavior, the restriction to open systems loses its strength.

Moreover, the perplexities resulting from the generalized spin-bath model can also be solved. In fact, they derive from supposing that the open subsystems become classical when interacting with their environments. From the open-system perspective, on the contrary, one can univocally identify, among all the observables of the closed system, which acquire a classical-like behavior: the classical world is unambiguously defined by the set observables whose expectation values acquire a classical form.

11 Decoherence Without Decoherence

Current works indicate that the most suitable scenario for early cosmology is the one of inflation [66–71]. This is because the whole structure of the universe can be traced

to the primordial fluctuations during an accelerated phase of the early universe. Several works show that EID can solve the quantum-classical transition of the universe, by separating the different degrees of freedom in system and environment. Since there is no a standard way to divide the universe, some authors take the inflation as a system and the gravitons as the environment [72], while other authors divide the universe between long and short frequencies [73].

However, there is an alternative approach proposed by Kiefer, Polarski, and Starobinsky [74–76], it studies the problem of quantum-classical transition without the need to attend to decoherence. The predictions obtained through mean values are indistinguishable from stochastic averaging.

According to paper [76], inflation can appear in energy scales where spacetime can be described as a classic curved spacetime where fluctuations are defined. Inflation fluctuations $\delta\phi(x, t)$ can be studied as scalar fields without mass. It is convenient to consider the rescaled quantity $a\delta\phi(x, t) \equiv y(x, t)$ and the conformal time $\eta = \int \frac{dt}{a(t)}$. The symbol $'$ will be used to denote a derivative with respect to η. The quantization of the perturbation $y(x, \eta)$ is the usual quantization, and p is the momentum conjugate to y,

$$p = \frac{\partial \mathcal{L}(y, y')}{\partial y'} \tag{43}$$

where $\mathcal{L}$ is the Lagrangian that corresponds to the Hamiltonian:

$$H = \frac{1}{2}\int d^3k \left[k(a(k)a^\dagger(k) + a^\dagger(-k)a(-k)) \right. $$
$$\left. +i\frac{a'}{a}(a^\dagger(k)a^\dagger(-k) - a(k)a(-k)) \right] \tag{44}$$

$a(k)$ is the time-dependent annihilation operator

$$a(k) = \frac{1}{\sqrt{2}}\left(\sqrt{k}y(k) + \frac{i}{\sqrt{k}}p(k)\right) \tag{45}$$

$y(k, \eta)$ and $p(k', \eta)$ satisfy the canonical commutation relation
$$[y(k, \eta), p(k', \eta)] = i\delta^{(3)}(k - k') \tag{46}$$

This equation expresses the quantum character of the perturbation $y(k, \eta)$.

If we compute the time evolution in the Heisenberg representation, we have that the following:

$$y(k, \eta) = f_k(\eta)a_k + f_k^*(\eta)a_{-k}^\dagger \tag{47}$$

where $a_k = a(k, \eta_0)$ and the modes f_k satisfy the equation

$$f_k'' + \left(k^2 - \frac{a''}{a}\right)f_k = 0 \tag{48}$$

We can write the (47) in a more suggestive way

$$y(k, \eta) = \sqrt{2k}f_{k1}(\eta)y_k - \sqrt{\frac{2}{k}}f_{k2}(\eta)p_k \tag{49}$$

where $y_k = y(k, \eta_0)$ and $p_k = p(k, \eta_0)$, $f_{k1} = \text{Re}(f_k)$, $f_{k2} = \text{Im}(f_k)$. Analogously

$$p(k, \eta) = \sqrt{\frac{2}{k}}g_{k1}(\eta)p_k + \sqrt{2k}g_{k2}(\eta)y_k \tag{50}$$

11.1 Quantum-Classical Transition: Pragmatic Version

This formulation is relevant to understand the quantum-classical transition. For example, Kiefer and Polarski [77] assume that there is a limit in which f_{k2} and g_{k1} (or f_{k1} and g_{k2}, respectively) behave as

$$f_{k2} \longrightarrow 0 \quad \text{and} \quad g_{k1} \longrightarrow 0$$
$$\text{or}$$
$$f_{k1} \longrightarrow 0 \quad \text{and} \quad g_{k2} \longrightarrow 0$$

Then from (49) and (50), we can see that the non-commutativity of operators y_k and p_k is no longer relevant

$$\left[y(k, \eta), p^\dagger(k', \eta)\right] = 2\left(f_{k1}(\eta)g_{k1}(\eta)\right.$$
$$\left. + f_{k2}(\eta)g_{k2}(\eta)\right)[y_k, p_k] \longrightarrow 0 \tag{51}$$

To understand the physical meaning of this limit, we can consider a classical stochastic system where the dynamics is still described by equations of the form (49), but with $y(k, \eta_0)$ and $p(k, \eta_0)$ representing random initial values. If f_{k2} and g_{k1} vanish, we get

$$p(k, \eta) = p_{cl}(y(k, \eta)) = \frac{g_{k2}}{f_{k1}}y(k, \eta) \tag{52}$$

This equation is valid when y and p are operators and also if they are classical stochastic variables (y_{cl} and p_{cl}). In this case, for a given realization there is a pair (y_{cl}, p_{cl}). For this reason Kiefer and Polarski claim.

Then the quantum system is effectively equivalent to the classical random system, which is an ensemble of classical trajectories with a certain probability associated to each of them ([76] pp. 4).

A concrete example in which f_{k2} and g_{k1} tend to zero is the perturbation on de Sitter space [77]. When the decaying mode becomes vanishingly small and this is neglected, we are in the limit of a random stochastic process.

11.2 Pragmatic View, EID and the Closed-System Perspective

Kiefer and Polarski present their explanation of the quantum-classical transition of the universe as a pragmatic view. This is because although we can obtain classical statistics from quantum statistics, there is not an associated interpretation. The main characteristic of the pragmatic view is that it focuses on the mean values and it does not use any reduced state.

Most interpretations of quantum mechanics emphasize the quantum state, assigning to it some kind of ontology. In fact, in EID, the focus is in the reduced state. For this reason, the authors believe that their point of view, that does not consider the states, is simply a pragmatic view. And to understand the phenomenon, it is necessary to introduce EID.

However, we believe that this formalism can be given in a proper interpretive framework that does not consider to the state, then EID is not necessary.

The pragmatic view basically consists of studying the mean values directly, without taking into account the state, and it determines when these values do take the form of classical statistics and when they do not. This way of studying decoherence is similar to the approach proposed by the closed-system perspective.

11.2.1 The Hidden Decoherence of the Pragmatic View

As it is known, a quantum system evolves under the Schrödinger equation. As the evolution is unitary, the system cannot reach equilibrium or decoherence.

As explained in previous sections, if we choose appropriate relevant observables, we can describe the phenomenon of decoherence in the mean values of the closed system. Therefore, the quantum-classical transition of the pragmatic view has to ignore some kind of decoherence. This is difficult to determine if the decoherence is understood from the point of view of EID, since it divides the whole system into the system and the environment, computes the reduced state of the system and considers its diagonalization. However, according to the closed-system perspective, decoherence is understood as the vanishing of the interference terms in the mean values, then the problem is simplified. In fact, the pragmatic view selects the observable and ignores evanescent modes.

The expression (51) is misleading because from (46), (49), and (50), we know that

$$f_{k1}(\eta)g_{k1}(\eta) + f_{k2}(\eta)g_{k2}(\eta) = \frac{1}{2} \qquad (53)$$

then

$$[y(k, \eta), p^{\dagger}(k', \eta)] \nrightarrow 0 \qquad (54)$$

Then Kiefer and Polarski say

Clearly, when $f_{k2}(\eta)$, $g_{k1}(\eta)$ are unobservable, this coherence becomes unobservable as well. This is the case when the decaying mode is so small that we have no access to it in observations ([76] pp. 5).

This means that coherence becomes unobservable from the observational point of view, as in the closed-system perspective. Since f_{k2} and g_{k1} are undetectable, it requires a gross observable that is not capable to detect the presence of f_{k2} and g_{k1}. That is, we need to select a relevant observable O_R. If, for example, we have only access to the observable

$$O_R = (|f_{k2}(\eta)| + |g_{k1}(\eta)|)\, I, \qquad (55)$$

then

$$\langle O_R \rangle \longrightarrow 0 \qquad (56)$$

In this case we can say that the system decoheres from the observational point of view. Then the pragmatic view of Kiefer, Polarski and Starobinsky is fully compatible with the closed-system perspective.

12 Conclusions

The aim of this paper has been to argue that decoherence can be viewed from a closed-system perspective, which improves the understanding of the phenomenon. After recalling the historical development of the decoherence program and stressing some conceptual difficulties of the EID program, we have introduced the closed-system approach from which the orthodox EID approach can be reformulated. In this theoretical framework, we have analyzed the spin-bath model, both in its traditional and in a generalized version: by considering different partitions of the whole closed system in both cases, we have shown that decoherence depends on the way in which the relevant observables are selected. On this basis, the following conclusions can be drawn:

(1.) Decoherence is a phenomenon relative to which degrees of freedom of the whole closed system are

considered relevant and which are disregarded in each situation.

(2.) Since there is no privileged or essential decomposition of the closed system, there is no need of an unequivocal criterion for identifying the systems involved in decoherence. Therefore, the looming big problem—which, according to Zurek, poses a serious threat to the whole decoherence program—dissolves in the light of the relativity of decoherence.

(3.) Due to its relative nature, decoherence cannot be accounted for in terms of dissipation of energy or of any other non-relative magnitude.

(4.) The open-system perspective leads to the unambiguous identification, among all the observables of the closed system, of those that will have a classical-like behavior because their expectation values acquire a classical form.

Once the phenomenon of decoherence is de-substantialized in this way, one might ask in what sense it can be still understood as the result of the action of an environment that destroys the coherence between the states of an open quantum system.

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