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Distribution of Mutual Information in Multipartite States

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Abstract Using the relative entropy of total correlation, we derive an expression relating the mutual information of n -partite pure states to the sum of the mutual informations and entropies of its marginals and analyze some of its implications. Besides, by utilizing the extended strong subadditivity of von Neumann entropy, we obtain generalized monogamy relations for the total correlation in three-partite mixed states. These inequalities lead to a tight lower bound for this correlation in terms of the sum of the bipartite mutual informations. We use this bound to propose a measure for residual three-partite total correlation and discuss the non-applicability of this kind of quantifier to measure genuine multipartite correlations.

Keywords Distribution of multipartite correlations · Relative entropy of total correlation · Generalized monogamy relations · Residual correlations

1 Introduction

The correlations among the constituents parts of a system have been at the central stage of discussions regarding fundamental concepts of quantum physics since nearly a decade after its formulation [1–3]. In quantum information science, the quantum part of correlations is believed to be one of the

main factors responsible by the so-called quantum advantage [4–9]. However, classical correlation has also been proven worthy of investigation. For example, the derivative of bipartite classical correlation can be used to indicate critical points of quantum phase transition [10, 11]. Besides, the sudden change phenomenon of the classical correlation between two qubits during its decoherent dynamics [12, 13] was shown to characterize the emergence of the pointer bases in a quantum measurement process [14].

With respect to multipartite systems, researches concerning its correlations are important both from the practical point of view (e.g., because of large-scale implementations of protocols in quantum information science) and also for the foundations of physics (e.g., understanding the rising of collective behavior is essential in investigations of quantum and classical phase transitions). The structure of the correlations presented in general multipartite states has been investigated using different techniques in Refs. [15–21]. In this article, considering finite dimensional systems and using relative entropy-based measures of correlations, we address some instances of the problem of distribution of total correlation—which encompasses both the classical and quantum ones—in multi-particle systems.

In what concerns the quantification of total correlation, for bipartite states, a well-justified (both physically [22] and operationally [23]) measure for total correlation is obtained via a direct generalization of Shannon's classical mutual information [24]. This quantifier is dubbed quantum mutual information and is defined as:

$$I(\rho_{12}) = S(\rho_1) + S(\rho_2) - S(\rho_{12}), \quad (1)$$

with ρ_s being the density operator (i.e., $\rho_s \geq 0$ and $\text{tr}(\rho_s) = 1$) on the Hilbert's space \mathcal{H}_s of system s ($\rho_s \in \mathcal{D}(\mathcal{H}_s)$) and $S(\rho_s) = -\text{tr}(\rho_s \log_2 \rho_s)$ being its von Neumann's entropy. Above and hereafter, $\rho_s = \text{tr}_{s'}(\rho_{ss'})$ is the reduced state of

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subsystem s , obtained by tracing out the other parties s' of the whole system.

On the other hand, for multipartite states, the situation is less understood. Venn's diagram-based approaches may lead to negative measures of correlation [25, 26]. One quantifier free from this problem was introduced in Ref. [27] as follows:

$$I(\rho_{1\dots n}) = \sum_{s=1}^n S(\rho_s) - S(\rho_{1\dots n}). \quad (2)$$

In Ref. [28], this issue was addressed by quantifying the total correlation in a multipartite state by how distinguishable it is from uncorrelated (product) states. Using the quantum relative entropy (QRE) [26, 29, 30],

$$S(\rho||\sigma) = \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \sigma), \quad (3)$$

to measure distinguishability between any pair of quantum states ρ and σ , Modi et al. showed that the closest (less distinguishable)¹ product state from any density operator is given by the states of its marginals in the product form. The multipartite mutual information defined in this way is called *relative entropy of total correlation* (RETC) and is given as in (2).

Here, we give a simple, alternative, proof for the result obtained by Modi et al. in Ref. [28].

Proposition 1 *The closest product state of any multipartite state $\rho_{1\dots n}$ is obtained from its marginals in the product form.*

Proof Let $\bigotimes_{s=1}^n \sigma_s$ be any n -partite product state. Using the definition of QRE in (3), one can write (we postpone the proof of this equality to Appendix):

$$S\left(\rho_{1\dots n}||\bigotimes_{s=1}^n \sigma_s\right) = S\left(\rho_{1\dots n}||\bigotimes_{s=1}^n \rho_s\right) + \sum_{s=1}^n S(\rho_s||\sigma_s). \quad (4)$$

As $S(\rho||\sigma) \geq 0$ with equality if and only if $\rho = \sigma$, we see that

$$S\left(\rho_{1\dots n}||\bigotimes_{s=1}^n \sigma_s\right) \geq S\left(\rho_{1\dots n}||\bigotimes_{s=1}^n \rho_s\right), \quad (5)$$

¹It is worthwhile mentioning that because, in general, $S(\rho||\sigma) \neq S(\sigma||\rho)$ and because quantum relative entropy (QRE) lacks a triangle inequality, it is not a well-defined metric. Nevertheless, as the QRE is positive semidefinite, i.e., $S(\rho||\sigma) \geq 0$ for any pair of quantum states, with equality if and only if $\rho = \sigma$, and because the QRE is related to the probability of confusing (distinguishability of) two quantum states, $S(\rho||\sigma)$ can be seen as a quantifier for how close (or how distant) ρ and σ are from each other in the state space.

with equality obtained only if $\sum_{s=1}^n S(\rho_s||\sigma_s) = 0$, i.e., if $\rho_s = \sigma_s \forall s$. Therefore,

$$\begin{aligned} I(\rho_{1\dots n}) &= \min_{\bigotimes_{s=1}^n \sigma_s} S(\rho_{1\dots n}||\bigotimes_{s=1}^n \sigma_s) \\ &= S(\rho_{1\dots n}||\bigotimes_{s=1}^n \rho_s), \end{aligned} \quad (6)$$

concluding thus the proof of the proposition. \square

In the subsequent sections, we shall regard the distribution of the RETC in n -partite pure (Section 2) and three-partite mixed (Section 3) states.

2 Distribution of Mutual Information in n -Partite Pure States

Let us consider the case of a system with n parties in a pure state $|\psi_{1\dots n}\rangle \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$, with \mathcal{H}_s being the state space for the subsystem s ($s = 1, \dots, n$). Using the definition presented in Section 1 and noting that the uncertainty associated with the preparation of a pure state is null ($S(|\psi_{1\dots n}\rangle) = 0$), one sees that the total mutual information of n subsystems in a pure state $|\psi_{1\dots n}\rangle$ is given by:

$$I(|\psi_{1\dots n}\rangle) = \sum_{s=1}^n S(\rho_s). \quad (7)$$

Below this correlation is related to the mutual informations and entropies of the marginals of $|\psi_{1\dots n}\rangle$. First, we shall define some quantities to be used in the sequence of the article.

Definition 1 The sum of the mutual informations of the $(n-k)$ -partite reductions of $\rho_{1\dots n}$ is defined as $\mathcal{I}_{n-k}(\rho_{1\dots n})$.

Definition 2 The sum of the entropies of the k -partite reductions of $\rho_{1\dots n}$ is defined as $\mathcal{S}_k(\rho_{1\dots n})$.

We observe that Definition 1 only makes sense if $n-k \geq 2$, i.e., if $k = 1, 2, \dots, n-2$. In what follows, we will use these definitions in the proof for the following proposition.

Proposition 2 *The total amount of information shared among n parties in a pure state $|\psi_{1\dots n}\rangle$ can be written as*

$$I(|\psi_{1\dots n}\rangle) = \frac{k! (\mathcal{I}_{n-k}(|\psi_{1\dots n}\rangle) + \mathcal{S}_k(|\psi_{1\dots n}\rangle))}{\prod_{i=1}^k (n-i)}. \quad (8)$$

Proof For the purpose of proving this proposition, it will be helpful first to split the system $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ in two components $\mathcal{H}_x \otimes \mathcal{H}_{\bar{x}}$, with x denoting one or more parties and \bar{x} being the rest of the system. It follows then, from Schmidt's decomposition (see, e.g., Ref. [31]), that $S(\rho_x) = S(\rho_{\bar{x}})$, with $\rho_{x(\bar{x})} = \text{tr}_{\bar{x}(\bar{x})}(|\psi_{1\dots n}\rangle\langle\psi_{1\dots n}|)$.

Now, let us begin by computing the sum of the mutual

informations of the $(n - 1)$ -partite reductions of $|\psi_{1\dots n}\rangle$. In this case, $k = 1$ and

$$\begin{aligned} \mathcal{I}_{n-1}(|\psi_{1\dots n}\rangle) &= \sum_{s=1}^n I(\rho_{\overline{s}}) \\ &= \sum_{s=1}^n \sum_{\substack{s'=1 \\ s' \neq s}}^n S(\rho_{s'}) - \sum_{s=1}^n S(\rho_{\overline{s}}). \end{aligned} \quad (9)$$

There are $n(n - 1)$ one-party entropies in the first term on the right hand side of the last equality. As the n subsystems appear with the same frequency in this term, and $S(\rho_{\overline{s}}) = S(\rho_s)$, we get

$$\mathcal{I}_{n-1} = \frac{(n-1)}{1!} \mathcal{S}_1(|\psi_{1\dots n}\rangle) - \mathcal{S}_1(|\psi_{1\dots n}\rangle). \quad (10)$$

For $k = 2$, if we take the sum of the total correlation of the $(n - 2)$ -partite marginals of $|\psi_{1\dots n}\rangle$, we obtain

$$\begin{aligned} \mathcal{I}_{n-2} &= \sum_{s=1}^n \sum_{s'=s+1}^n I(\rho_{\overline{ss'}}) \\ &= \sum_{s=1}^n \sum_{s'=s+1}^n \sum_{\substack{s''=1 \\ s'' \neq s, s'}}^n S(\rho_{s''}) - \sum_{s=1}^n \sum_{s'=s+1}^n S(\rho_{\overline{ss'}}) \\ &= \frac{(n-1)(n-2)}{2!} \mathcal{S}_1(|\psi_{1\dots n}\rangle) - \mathcal{S}_2(|\psi_{1\dots n}\rangle). \end{aligned} \quad (11)$$

In order to obtain the last equality, and below, we note that there are $n!/(k!(n-k)!)$ different reductions of $|\psi_{1\dots n}\rangle$ comprising k parties.

For $k = 3$, the sum of $(n - 3)$ -partite mutual informations is

$$\begin{aligned} \mathcal{I}_{n-3} &= \sum_{s=1}^n \sum_{s'=s+1}^n \sum_{s''=s'+1}^n I(\rho_{\overline{ss's''}}) \\ &= \sum_{s=1}^n \sum_{s'=s+1}^n \sum_{s''=s'+1}^n \sum_{\substack{s'''=1 \\ s''' \neq s, s', s''}}^n S(\rho_{s'''}) - \mathcal{S}_3 \\ &= \frac{(n-1)(n-2)(n-3)}{3!} \mathcal{S}_1(|\psi_{1\dots n}\rangle) - \mathcal{S}_3(|\psi_{1\dots n}\rangle). \end{aligned} \quad (12)$$

So, by inductive reasoning, one sees that for any value of k in the set $\{1, 2, \dots, n - 2\}$, the following equality holds

$$\mathcal{I}_{n-k} = \frac{\prod_{i=1}^k (n-i)}{k!} \mathcal{S}_1(|\psi_{1\dots n}\rangle) - \mathcal{S}_k(|\psi_{1\dots n}\rangle). \quad (13)$$

Once $I(|\psi_{1\dots n}\rangle) = \mathcal{S}_1(|\psi_{1\dots n}\rangle)$, this equation is seen to be equivalent to (8), concluding thus the proof of the proposition. \square

2.1 Some Particular Cases

Now, we regard the particular case in which $k = 1$ (and therefore $n \geq 3$). It follows from (8) that

$$I(|\psi_{1\dots n}\rangle) = \frac{\mathcal{I}_{n-1}(|\psi_{1\dots n}\rangle)}{(n-2)}. \quad (14)$$

Thus, for three-particle pure states ($n = 3$), the following equality is obtained:

$$I(|\psi_{123}\rangle) = \mathcal{I}_2(|\psi_{123}\rangle). \quad (15)$$

So, the total correlation in $|\psi_{123}\rangle$ is shown to be equal to the sum of the mutual informations of its bipartite marginals. As an example, let us consider the three-qubit Greenberger-Horne-Zeilinger state [32]: $|\text{GHZ}_3\rangle = 2^{-1/2}(|0_1 0_2 0_3\rangle + |1_1 1_2 1_3\rangle)$, where $\{|0_s\rangle, |1_s\rangle\}$ is the computational basis for the qubit s . In the last equation and throughout the article, we shall use the notation: $|\psi_s\rangle \otimes |\phi_{s'}\rangle = |\psi_s \phi_{s'}\rangle = |\psi \phi\rangle$. The reduced states of $|\text{GHZ}_3\rangle$ are $\rho_{ss'} = 2^{-1}(|0_s 0_{s'}\rangle\langle 0_s 0_{s'}| + |1_s 1_{s'}\rangle\langle 1_s 1_{s'}|)$, with $ss' = 12, 13, 23$, and $\rho_r = 2^{-1}(|0_r\rangle\langle 0_r| + |1_r\rangle\langle 1_r|)$, with $r = 1, 2, 3$. Thus $S(\rho_{ss'}) = S(\rho_r) = 1$, which leads to $I(|\text{GHZ}_3\rangle) = 3$ and $I(\rho_{ss'}) = 1$, and consequently to the equalities in (14) and (15).

On the other side, one sees that for $n \geq 4$ the sum of the $(n - 1)$ -partite reductions' total correlation overestimate the total information shared among the n subsystems, i.e.,

$$I(|\psi_{1\dots n}\rangle) < \mathcal{I}_{n-1}(|\psi_{1\dots n}\rangle) \quad (16)$$

for $n \geq 4$. For the sake of exemplifying the applicability of this inequality, we regard again the example of Greenberger-Horne-Zeilinger states, but for four qubits: $|\text{GHZ}_4\rangle = 2^{-1/2}(|0_1 0_2 0_3 0_4\rangle + |1_1 1_2 1_3 1_4\rangle)$. The reduced states of $|\text{GHZ}_4\rangle$ we need here are the three-qubit density operators: $\rho_{ss's''} = 2^{-1}(|000\rangle\langle 000| + |111\rangle\langle 111|)$, with $ss's'' = 123, 124, 133, 234$, and the one-qubit density matrices: $\rho_r = 2^{-1}(|0_r\rangle\langle 0_r| + |1_r\rangle\langle 1_r|)$, with $r = 1, 2, 3, 4$. Hence, $S(\rho_{ss's''}) = S(\rho_r) = 1$. Therefore, it follows that $I(|\text{GHZ}_4\rangle) = 4$ and $I(\rho_{ss's''}) = 2$. Using these values, we obtain the relations:

$$\mathcal{I}_3(|\text{GHZ}_4\rangle) = 8 = (4 - 2)I(|\text{GHZ}_4\rangle) > I(|\text{GHZ}_4\rangle), \quad (17)$$

which satisfy the equality in (14) and the inequality in (16).

3 Distribution of Mutual Information in Three-Partite States

3.1 Generalized Monogamy Relations for Total Correlation

One of the most important inequalities in quantum information theory is the strong subadditivity property of von

Neumann entropy (SSA), by which [33]:

$$S(\rho_{ss'}) + S(\rho_{ss''}) - S(\rho_s) - S(\rho_{123}) \geq 0, \quad (18)$$

where $ss's'' = 123, 231, 321$.

Recently, Carlen and Lieb proved an extended version for the SSA (ESSA) [34]:

$$S(\rho_{ss'}) + S(\rho_{ss''}) - S(\rho_s) - S(\rho_{123}) \geq 2 \max\{S(\rho_{s'}) - S(\rho_{s's''}), S(\rho_{s''}) - S(\rho_{s's'}), 0\} \quad (19)$$

We shall use this inequality to prove the following proposition.

Proposition 3 *The total mutual information of three-partite states imposes the following constraint for the correlations of its bipartite marginals:*

$$I(\rho_{123}) \geq I(\rho_{ss'}) + I(\rho_{ss''}) + 2 \max\{I(\rho_{s's''}) - S(\rho_{s'}), I(\rho_{s's'}) - S(\rho_{s''}), 0\} \quad (20)$$

with $ss's'' = 123, 231, 321$.

Proof Let us begin the proof by rewriting the SSA in (18) as

$$\sum_{i=s,s',s''} S(\rho_i) - S(\rho_{123}) \geq \sum_{i=s,s'} S(\rho_i) - S(\rho_{ss'}) + \sum_{i=s,s''} S(\rho_i) - S(\rho_{ss''}). \quad (21)$$

Or, equivalently,

$$S(\rho_{123} || \rho_s \otimes \rho_{s'} \otimes \rho_{s''}) \geq S(\rho_{ss'} || \rho_s \otimes \rho_{s'}) + S(\rho_{ss''} || \rho_s \otimes \rho_{s''}). \quad (22)$$

Using $S(\rho_{s'}) - S(\rho_{s's''}) = I(\rho_{s's''}) - S(\rho_{s''})$, $S(\rho_{s''}) - S(\rho_{s's'}) = I(\rho_{s's'}) - S(\rho_{s'})$, and the definition of multipartite mutual information presented in Section 1, we see that the last equation, together with the ESSA implies the inequality in (20), concluding thus the proof of the proposition. \square

For three-partite systems whose reduced states are such that $S(\rho_{s'}) \leq S(\rho_{s's''})$ and $S(\rho_{s''}) \leq S(\rho_{s's'})$, we have a weaker version of the inequality in (20):

$$I(\rho_{123}) \geq I(\rho_{ss'}) + I(\rho_{ss''}). \quad (23)$$

This *generalized monogamy relation* entail that the total correlation in a three-particle mixed state restrict the information which a subsystem can share individually with the other two parties of the system. Similar constraints were obtained recently for bipartite quantum correlations using the global quantum discord (see Ref. [35] and references therein). For reviews about the classical and quantum aspects of correlations, see Refs. [36–38].

We observe that (20) is, in general, a stronger version of the monogamy inequality in (23). Although the first is cumbersome, it shows that the three-partite mutual information of any state ρ_{123} limits the total amount of correlation that its bipartite reductions can possess.

3.2 An Inequality for Three-Partite Mutual Information

Proposition 4 *The total mutual information of three-partite mixed states is lower bounded by the sum of the mutual informations of its bipartite marginals as follows:*

$$I(\rho_{123}) \geq \frac{2}{3} \mathcal{I}_2(\rho_{123}). \quad (24)$$

Proof It is straightforward to prove this proposition by combining (23) for $ss's'' = 123, 231, 312$. \square

In words, the lower bound in (24) means that the distinguishability between ρ_{123} and $\rho_1 \otimes \rho_2 \otimes \rho_3$ is greater or equal than two thirds of the sum of the distinguishabilities between its bipartite marginals and their one-particle reductions in the product form.

Now, we show that a subset of the classically correlated states (see for instance, reference [39]) saturates the inequality above. These states can be written as $\chi_{123} = \sum_{i_1=i_2=i_3} p_{i_1 i_2 i_3} |\psi_{i_1} \psi_{i_2} \psi_{i_3}\rangle \langle \psi_{i_1} \psi_{i_2} \psi_{i_3}|$, where $\{p_{i_1 i_2 i_3}\}$ is a probability distribution (that is to say, $p_{i_1 i_2 i_3} \geq 0$ and $\sum_{i_1=i_2=i_3} p_{i_1 i_2 i_3} = 1$) and $\{|\psi_{i_s}\rangle\}$ are local orthonormal basis. Noting that the entropies are given by

$$S(\chi_{123}) = S(\chi_{ss'}) = S(\chi_{ss''}) = \sum_{i_1=i_2=i_3} p_{i_1 i_2 i_3} \log_2 p_{i_1 i_2 i_3}^{-1}, \quad (25)$$

with $ss' = 12, 13, 23$ and $s'' = 1, 2, 3$, one obtain $I(\chi_{ss'}) = S(\chi_{ss''})$ and thus $2\mathcal{I}_2(\chi_{123})/3 = 2S(\chi_{ss'}) = I(\chi_{123})$, which shows that the class of states χ_{123} indeed saturates the inequality in (24), and consequently lead to equality in the SSA [40] and in the ESSA. So, the lower bound in (24) is as tight as it could be.

On the other hand, one can show that for states with no genuine tripartite correlation [41–43], viz., states of the form $\rho_s \otimes \rho_{s's''}$, the equality holds: $I(\rho_s \otimes \rho_{s's''}) = I(\rho_{s's''}) = \mathcal{I}_2(\rho_s \otimes \rho_{s's''})$ (see Section 3.3). In Section 2.1, we proved that a similar equality is obtained for three-partite pure states, namely, $I(|\psi_{123}\rangle) = \mathcal{I}_2(|\psi_{123}\rangle)$. In addition, Streltsov et al. proved that any positive measure of correlation that is non-increasing under quantum operations on at least one of its subsystems is maximal for some pure state [44]. As the quantum relative entropy fulfills this condition [26], we have that for any density operator ρ_{123} there exists a state vector $|\psi_{123}\rangle$ such that $I(|\psi_{123}\rangle) \geq I(\rho_{123})$. As $I(|\psi_{123}\rangle) = \mathcal{I}_2(|\psi_{123}\rangle)$ for all three-partite pure states, one

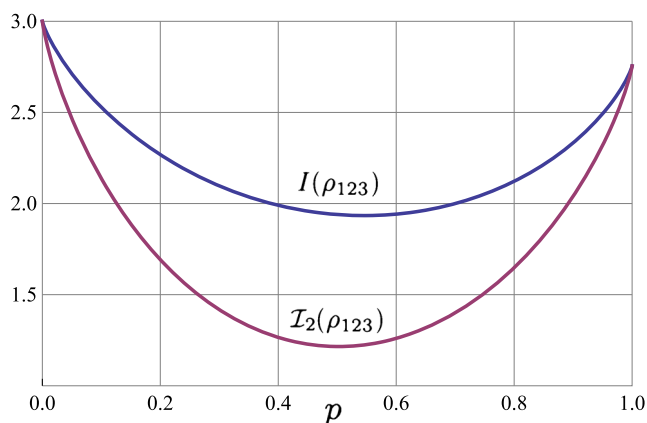


Fig. 1 (Color online). Total mutual information $I(\rho_{123})$ and the sum of bipartite total correlations $\mathcal{I}_2(\rho_{123})$ for the mixture of W and GHZ states in (26) (see text for details)

may ask if the total mutual information of general, mixed, three-partite states is limited by the correlations of its bipartite marginals. Below, we give an example showing that this is not generally the case. Let us regard a mixture of W [45] and GHZ [32] three-qubit states:²

$$\rho_{123} = p|W_3\rangle\langle W_3| + (1-p)|GHZ_3\rangle\langle GHZ_3|, \quad (26)$$

with $|W_3\rangle = 3^{-1/2}(|0_1 0_2 1_3\rangle + |0_1 1_2 0_3\rangle + |1_1 0_2 0_3\rangle)$ and $0 \leq p \leq 1$. The total correlation of ρ_{123} and the sum of the mutual informations of its bipartite marginals are shown in Fig. 1.

We see that $I(\rho_{123}) > \mathcal{I}_2(\rho_{123})$ for all values of p with exception of $p = 0$ and $p = 1$, where the states are pure and we have equality of the correlations ($I(\rho_{123}) = \mathcal{I}_2(\rho_{123})$).

If there are two states $|\psi_{123}\rangle$ and ρ_{123} (i) having the same bipartite reductions and (ii) connected by local quantum operations, i.e., $\rho_{123} = \Lambda_I[|\psi_{123}\rangle]$, with $\Lambda_I : \mathcal{D}(\mathcal{H}_s) \rightarrow \mathcal{D}(\mathcal{H}_s)$, then

$$\begin{aligned} I(\rho_{123}) &= I(\Lambda_I[|\psi_{123}\rangle]) \leq I(|\psi_{123}\rangle) \\ &= \mathcal{I}_2(|\psi_{123}\rangle) = \mathcal{I}_2(\rho_{123}). \end{aligned} \quad (27)$$

So, the negative result above (Fig. 1) rules out the fulfillment of both conditions (i) and (ii) for general pairs of states, though for particular cases, these conditions can be satisfied.

3.3 Residual Versus Genuine Three-Partite Correlations

Monogamy inequalities are frequently used as a starting point to defining measures for residual multipartite entanglement [46] and quantum discord [35]. The inequality

²The states in (26) are typical system-environment states generated for a two-qubit system when one of the qubits suffers decoherence due to the action of composite amplitude and phase channels [43, 47]

above (23) can be used to define a positive quantifier for the residual three-partite total correlation in ρ_{123} as follows:

$$I_r(\rho_{123}) := I(\rho_{123}) - \frac{2}{3}\mathcal{I}_2(\rho_{123}). \quad (28)$$

Genuine n -partite correlations (GnC) are those correlations that cannot be accounted for by looking at $n - 1$ or less subsystems. It is natural to question if I_r does quantify GnC in three-partite states [41–43]. For the sake of answering this question, let us assume that a state ρ_{123} does not presents genuine three-partite total correlation, e.g., $\rho_{123} = \rho_1 \otimes \rho_{23}$. Hence, $\rho_{12} = \rho_1 \otimes \rho_2$ and $\rho_{13} = \rho_1 \otimes \rho_3$. In this case, $I(\rho_{123}) = S(\rho_1 \otimes \rho_{23} || \rho_1 \otimes \rho_2 \otimes \rho_3) = S(\rho_{23} || \rho_2 \otimes \rho_3) = I(\rho_{23})$. As $I(\rho_{12}) = I(\rho_{13}) = 0$, the residual tripartite correlation is given by $I_r(\rho_1 \otimes \rho_{23}) = 3^{-1}I(\rho_{23})$. So, one sees that although the residual total correlation somehow quantifies the correlations in a multipartite state, I_r is nonzero for states which do not possess genuine three-partite total correlation. So, I_r cannot be used for the purpose of quantifying or identifying genuine multipartite correlations.

4 Concluding Remarks

In this article, we addressed the problem of distribution of mutual information in multipartite systems, focusing on n -partite state vectors and three-partite density operators. We obtained a general relation for the relative entropy of total correlation of n -partite pure states in terms of the mutual informations and entropies of its marginals. The total correlation of three-partite pure states was shown to be completely accounted for by the correlations of its bipartite reductions. However, for systems in a pure state and with $n > 3$ subsystems, the sum of the mutual informations of the $(n - 1)$ -partite reductions of $|\psi_{1\dots n}\rangle$ overestimate its total correlation. This fact indicates that, in this last case, there must exist redundant information shared among the subsystems. That is to say, if correlation is seen as shared information, then a subsystem must share the same information with two or more others parties of the whole physical system.

Monogamy relations for bipartite correlations via bipartite correlations were first noticed for entanglement measures [46, 48] and for non-local quantum correlations [49]. This kind of inequality was shown recently to be not generally applicable for separable-state quantum correlations [44]. Here, continuing the program initiated in Ref. [35], we showed that monogamy relations are restored for bipartite mutual informations if we employ a relative entropy-based measure of total multipartite correlation. These general monogamy inequalities led to a tight lower bound for the mutual information of three-partite mixed states in terms of

the respective correlations of its bipartite reductions. It is important to mention here that the main point of the original monogamy relations was to differentiate quantum (non-separable or non-local) from classical correlations [48]. As generalized, monogamy relations via multipartite correlations hold for total correlations, we notice that this kind of inequality cannot be generally used to characterize the quantumness of the correlations in a physical system.

Looking for a possible interpretation for the equalities and inequalities obtained in this article in terms of erasure of correlation by added noise [22] is an interesting topic for future investigations.

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Appendix: Proof of the Equality in (4)

Let us first prove some lemmas to be used subsequently.

Lemma 1 Let $\chi_s \in \mathcal{H}_s$ and $\xi_{s'} \in \mathcal{H}_{s'}$ be any pair of density operators for the systems s and s' , respectively. It follows that

$$\log_2(\chi_s \otimes \xi_{s'}) = \log_2(\chi_s) \otimes \mathbb{I}_{s'} + \mathbb{I}_s \otimes \log_2(\xi_{s'}). \quad (29)$$

Proof A function $f : \mathbb{C} \rightarrow \mathbb{C}$ from the complex numbers to the complex numbers is applied to normal operators O (with eigenvalues o_i and eigenvectors $|o_i\rangle$) as follows: $f(O) = \sum_i f(o_i)|o_i\rangle\langle o_i|$. Now let the eigen-decompositions of the density operators regarded above be $\chi_s = \sum_i a_i|a_i\rangle\langle a_i|$ and $\xi_{s'} = \sum_i b_i|b_i\rangle\langle b_i|$. So

$$\log_2(\chi_s \otimes \xi_{s'}) = \sum_{i,j} \log_2(a_i b_j) |a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j| \quad (30)$$

$$= \sum_i \log_2(a_i) |a_i\rangle\langle a_i| \otimes \sum_j |b_j\rangle\langle b_j| + \sum_i |a_i\rangle\langle a_i| \otimes \sum_j \log_2(b_j) |b_j\rangle\langle b_j| \quad (31)$$

$$= \log_2(\chi_s) \otimes \mathbb{I}_{s'} + \mathbb{I}_s \otimes \log_2(\xi_{s'}) \quad (32)$$

In order to obtain the last equality, we used the completeness relations for the state spaces \mathcal{H}_s and $\mathcal{H}_{s'}$. \square

Lemma 2 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be any function from the complex numbers to the complex numbers and let $\xi_{s'} \in \mathcal{H}_{s'}$ be any state of system s' . Then it follows that

$$f(\mathbb{I}_s \otimes \xi_{s'}) = \mathbb{I}_s \otimes f(\xi_{s'}). \quad (33)$$

Proof To prove this lemma, we need only to use the eigen-decomposition of $\xi_{s'} = \sum_i b_i|b_i\rangle\langle b_i|$ and the closure relation $\mathbb{I}_s = \sum_i |a_i\rangle\langle a_i|$ in order to write:

$$f(\mathbb{I}_s \otimes \xi_{s'}) = \sum_{i,j} f(b_j) |a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j| \quad (34)$$

$$= \sum_i |a_i\rangle\langle a_i| \otimes \sum_j f(b_j) |b_j\rangle\langle b_j| \quad (35)$$

$$= \mathbb{I}_s \otimes f(\xi_{s'}). \quad (36)$$

\square

Lemma 3 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be any function from the complex numbers to the complex numbers, let $\chi_{ss'} \in \mathcal{H}_{ss'}$ be any global state for the systems s and s' , and let $\xi_{s'} \in \mathcal{H}_{s'}$ be any state for the system s' . Then it follows that

$$\text{tr}(\chi_{ss'} f(\mathbb{I}_s \otimes \xi_{s'})) = \text{tr}_{s'}(\chi_{s'} f(\xi_{s'})), \quad (37)$$

with $\chi_{s'} = \text{tr}_s(\chi_{ss'})$.

Proof First, we use a basis $\{|a_i\rangle \otimes |b_j\rangle\}$ for $\mathcal{H}_{ss'}$ to write

$$\chi_{ss'} = \mathbb{I}_{ss'} \chi_{ss'} \mathbb{I}_{ss'} = \sum_{i,j,k,l} \chi_{ss'}^{ijkl} |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l|, \quad (38)$$

where we defined $\chi_{ss'}^{ijkl} = \langle a_i| \otimes \langle b_j| \chi_{ss'} |a_k\rangle \otimes |b_l\rangle$. For this global state, the reduced density operator of system s' is

$$\chi_{s'} = \text{tr}_s(\chi_{ss'}) = \text{tr}_s \left(\sum_{i,j,k,l} \chi_{ss'}^{ijkl} |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l| \right) \quad (39)$$

$$= \sum_{i,j,k,l} \chi_{ss'}^{ijkl} \text{tr}_s(|a_i\rangle\langle a_k|) |b_j\rangle\langle b_l| \quad (40)$$

We now make use of these expressions and of Lemma 3 to get

$$\text{tr}(\chi_{ss'} f(\mathbb{I}_s \otimes \xi_{s'})) = \text{tr}(\chi_{ss'} \mathbb{I}_s \otimes f(\xi_{s'})) \quad (41)$$

$$= \text{tr}_{s'} \text{tr}_s \left(\sum_{i,j,k,l} \chi_{ss'}^{ijkl} |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l| f(\xi_{s'}) \right) \quad (42)$$

$$= \text{tr}_{s'} \left(\sum_{i,j,k,l} \chi_{ss'}^{ijkl} \text{tr}_s(|a_i\rangle\langle a_k|) |b_j\rangle\langle b_l| f(\xi_{s'}) \right) \quad (43)$$

$$= \text{tr}_{s'}(\chi_{s'} f(\xi_{s'})). \quad (44)$$

\square

Now, we have the tools we need to prove the equality in (4).

Proposition 5 Let $\rho_{1\dots n} \in \mathcal{D}(\mathcal{H}_{1\dots n})$ be any n -partite state with marginals density operators ρ_s for its subsystems

$s = 1, \dots, n$. Let $\bigotimes_{s=1}^n \sigma_s$ be any n -partite product state. It follows that

$$S\left(\rho_{1\dots n} \parallel \bigotimes_{s=1}^n \sigma_s\right) = S\left(\rho_{1\dots n} \parallel \bigotimes_{s=1}^n \rho_s\right) + \sum_{s=1}^n S(\rho_s \parallel \sigma_s). \quad (45)$$

Proof Let us start by using the definition of quantum relative entropy in (3) to write

$$S\left(\rho_{1\dots n} \parallel \bigotimes_{i=1}^n \sigma_i\right) = -S(\rho_{1\dots n}) - \text{tr}\left(\rho_{1\dots n} \log_2 \bigotimes_{i=1}^n \sigma_i\right). \quad (46)$$

Utilizing Lemmas 1 and 3, we can express the last term on the right hand side of the last equation as

$$\begin{aligned} \text{tr}\left(\rho_{1\dots n} \log_2 \bigotimes_{i=1}^n \sigma_i\right) &= \text{tr}(\rho_{1\dots n} \log_2(\sigma_1) \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_n) \\ &+ \dots + \text{tr}(\rho_{1\dots n} \mathbb{I}_1 \otimes \dots \otimes \mathbb{I}_{n-1} \otimes \log_2(\sigma_n)) \end{aligned} \quad (47)$$

$$= \text{tr}(\rho_1 \log_2(\sigma_1)) + \dots + \text{tr}(\rho_n \log_2(\sigma_n)) \quad (48)$$

$$= \sum_{i=1}^n \text{tr}(\rho_i \log_2 \sigma_i). \quad (49)$$

In a similar manner, we have

$$\text{tr}\left(\rho_{1\dots n} \log_2 \bigotimes_{i=1}^n \rho_i\right) = \sum_{i=1}^n \text{tr}(\rho_i \log_2 \rho_i). \quad (50)$$

Using these two relations we get

$$\begin{aligned} S\left(\rho_{1\dots n} \parallel \bigotimes_{i=1}^n \sigma_i\right) &= -S(\rho_{1\dots n}) - \text{tr}\left(\rho_{1\dots n} \log_2 \bigotimes_{i=1}^n \rho_i\right) \\ &+ \sum_{i=1}^n \text{tr}(\rho_i \log_2 \rho_i) - \sum_{i=1}^n \text{tr}(\rho_i \log_2 \sigma_i) \end{aligned} \quad (51)$$

$$= S\left(\rho_{1\dots n} \parallel \bigotimes_{i=1}^n \rho_i\right) + \sum_{i=1}^n S(\rho_i \parallel \sigma_i), \quad (52)$$

concluding thus the proof of the proposition. \square

We observe that the last equality can be expressed as

$$S\left(\rho_{1\dots n} \parallel \bigotimes_{i=1}^n \sigma_i\right) = S\left(\rho_{1\dots n} \parallel \bigotimes_{i=1}^n \rho_i\right) + S\left(\bigotimes_{i=1}^n \rho_i \parallel \bigotimes_{i=1}^n \sigma_i\right). \quad (53)$$

So, in this case, the triangle inequality is satisfied (and saturated) by quantum relative entropy.

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