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GENERAL AND APPLIED PHYSICS

Locating Cantori for Symmetric Tokamap and Symmetric Ergodic Magnetic Limiter Map Using Mean-Energy Error Criterion

S. M. Jazayeri · A. R. Sohrabi

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Abstract We use a method based on the conservation of energy, the mean-energy error criterion, to approximately locate the place of a cantorus by locating the series of its convergents. The mean-energy error curve has nearly stationary parts in the vicinity of elliptic (minimax) orbits, the so-called magnetic islands. Stable minimax orbits converge to orbits homoclinic to a cantorus. By tracing the island series, we limit the cantorus to a narrow region. A near-critical perturbation parameter is used so that, while the cantorus may be destabilized, its high-order minimax orbits remain intact. As illustrations, we consider two symplectic maps, systematically derived from the Hamilton–Jacobi equation and Jacobi's theorem, in the context of the magnetically confined plasmas in a tokamak: a symmetric tokamap realistically reproduces the main features of a tokamak, and a symmetric ergodic magnetic limiter (EML) map is defined to describe the action of EML rings on the magnetic field lines in the tokamak.

Keywords Cantori · Symplectic maps · Ergodic magnetic limiter map · Tokamap · Tokamak

1 Introduction

Resonant magnetic perturbations create chaotic magnetic fields at the edge of tokamak plasmas. The dynamical properties of this chaotic layer can be studied by plotting symplectic maps for the Poincare surface of a section of field lines. Besides chaotic regions, there are magnetic islands, Kolmogorov–Arnold–Moser (KAM) tori and cantori in the Poincare plot. Cantori are the "remnants" of KAM tori when the latter are destroyed as the nonlinearity parameter grows. They are Cantor-like sets of zero measure composed of infinite sets of points that are invariant under the Poincare map [1].

A cantorus forms a countable infinity of gaps, and chaotic orbits can only cross it through its gaps [2]. In the weak-perturbation regime, these gaps are so small that a chaotic orbit spends typically a long time before escaping through the cantorus. The cantori therefore form partial barriers [3, 4] that inhibit local chaotic diffusion (radial transport of plasma particles in the tokamak), although they cannot block diffusion. Global chaotic diffusion is also limited by the existence of several consecutive cantori in the chaotic region.

A cantorus has a rotation number equal to the rotation number of the KAM torus that has generated it. In fact, both robust tori and cantori are characterized by irrational winding numbers. These irrationals can be approximated by successive truncations of their continued fraction representations, i.e., cantori and robust tori can be approached by high-order periodic orbits. As pointed out by MacKay, the flux through the cantorus gaps is minimal for a cantorus with noble rotation numbers [5]. Hence, only noble cantori are important for the diffusion of field lines.

Localization of high-order periodic orbits for maps and flows can be obtained numerically by searching hyperbolic and/or elliptic periodic points with a minimization method analogous to solving a set of nonlinear equations [6]. For continuous flows, high-order periodic orbits can also be determined via the Lagrangian variational method [7]. Although a general Lagrangian and Hamiltonian action-based formulation for defining and calculating almost-invariant tori exists, in discrete-time area preserving dynamical systems, no unified Hamiltonian and Lagrangian formulation has heretofore been presented [8].

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Symplectic maps are rigorously derived from the Hamilton–Jacobi equation and Jacobi's theorem [9]. One of the advantages of the Hamilton–Jacobi method is to produce maps that are reversible under time reversal and hence preserve one of the main features of the systems obeying Hamiltonian dynamics. Remarkably, such maps appear to have one particular symmetry line, called the dominant line, containing a point of minimax orbit for every rational frequency [10]. The symmetry can be exploited to search for periodic points on the dominant line in the phase space.

We use a numerical method to highly accurately locate the cantorus of a symplectic map. This method is simply based on the conservation of energy through mean-energy error (MER) [9], which is approximately stationary around stable elliptic-fixed points. We start from an initial point on the dominant line of the map, iterate the map, and look for the stationary part of the MER curve around elliptic-fixed points.

The minimizing and minimax orbits related to the convergents of the noble winding number of a given cantorus constitute island chains that approach the cantorus for higher-order periodic points. In fact, stable minimax orbits converge to orbits homoclinic to the cantorus [10]. In near-critical perturbation, a cantorus may be destabilized, but its high-order elliptic orbits remain intact [11].

We have used the method of mean-energy error criterion to locate approximately the place of a cantorus for the symmetric tokamap and symmetric ergodic magnetic limiter (EML) map which are Hamiltonian systems with one and a half degrees of freedom describing chaotic magnetic fields in tokamaks. The tokamap has been proved to realistically describe toroidal magnetic field lines in tokamaks. We restrict ourselves mainly to the case of a monotonic safety factor profile in the perturbed configuration. Specifically, a symmetric version of a tokamap could be deduced rigorously from Hamilton–Jacobi equation and Jacobi's theorem [9].

The most robust cantorus at the edge of tokamak, i.e., the most resistant barrier, is considered. This cantorus is of crucial importance since it is the last barrier preventing diffusion of magnetic field lines toward the tokamak wall. For the systems with shear, such as those represented by the tokamap, the golden mean cantorus was found to be not the most robust barrier [12].

The second example concentrates on tokamaks with EML rings, which perturb the equilibrium Hamiltonian of the magnetic field lines so that the perturbed Hamiltonian is nearly integrable [13, 17]. To determine the dynamics, we construct a symplectic map for the Hamiltonian flow, a robust procedure. The symmetric EML map [14] is, therefore, systematically derived from the continuous field-line equations in the tokamak. We truncate the locally most noble cantorus near the edge of the plasma for typical tokamak parameters and finally, on the basis of the MER criterion, determine the approximate position of this cantorus with high accuracy.

The paper is organized as follows. In Section 2, the cantori are defined and the MER criterion for locating cantori is introduced. Sections 3 and 4 are devoted to the tokamap and EML maps, respectively, and locate the main cantori in the plasma edges for two examples of magnetically confined plasmas. Finally, the conclusions in Section 5 outline our main results.

2 Locating a Cantorus

In this section, after reviewing cantori concepts, we introduce the MER criterion for locating the cantori of the symplectic maps.

2.1 Cantori

Cantori are striking examples of the complexity of naturally ordered structures embedded in a sea of chaos. Notwithstanding leakiness, they may constitute relevant barriers to global phase-space diffusion [4]. The most important cantori are those with noble winding numbers, which can be regarded as infinite rational series. Rational approximations to a noble number are obtained by truncating such infinite series at high levels; the higher the level, the better the approximation. These rational approximations are called convergents. The usual way to locate a cantorus in phase space is to trace the periodic orbits corresponding to the rationals resulting from successive truncations of irrational winding number $\omega=[a_1,a_2\ldots]$, namely, $1/a_1$, $1/(a_1+1/a_2)$, etc. These periodic orbits are successively below and above the cantorus, i.e., approach it from both sides.

2.2 MER Criterion for Locating Cantori

The perturbed Hamiltonian may be presented as a sum of a completely integrable part and a periodic perturbation

$$H(\theta, I, t; \varepsilon) = H_0(I) + \varepsilon H_1(\theta, I, t). \quad (1)$$

The MER criterion uses the invariance of the total Hamiltonian along the orbits in phase space and measures the deviation of the Hamiltonian on the mapped orbit from its initial value

$$MER(I_0\theta_0) = \frac{1}{n}\sum_{k=1}^{n}[H(I_k\theta_k) - H(I_0, \theta_0)], \quad (2)$$

where n is the number of iterations. By plotting the phase-space (I,θ) for the map at constant time $t=$ const, we are able to calculate the MER for any initial point. While it is considerably variable in chaotic regions, the MER is approximately stationary in the vicinity of periodic elliptic points.

The Hamiltonian maps, which are reversible under time reversal, have one particular symmetry line, called the dominant line, on which an elliptical orbit point is found for every rational frequency [10]. If we select N equally spaced initial points on the dominant line in an interval starting below the cantorus and ending above it, the plot of the MER for each point will identify the location of high-order elliptic orbits of the cantorus. In fact, any flat part of the MER curve corresponds to a convergent to the cantorus irrational frequency. The length of any flat segment is the size of the corresponding island.

Zooming in on smaller intervals reveals more stationary segments on the MER curve, related to higher-order convergents of the cantorus, which yield better approximations for the cantorus location. Finally, we have two high-order convergents below and above the cantorus. The cantorus lies between these convergents, but its exact location can never be found.

The perturbation part of the Hamiltonian (1) being a function of time t, we cannot expect the Hamiltonian to remain constant. However, the MER can provide useful information when it varies only slightly along the orbits, i.e., when $|dH/d\varphi|=|\partial H/\partial\varphi|\ll 1$, as it happens, e.g., with perturbed magnetic field Hamiltonians in tokamaks. The MER criterion could also be applied to rapidly varying Hamiltonians H_1 in such regular orbits as regions around elliptical orbits, i.e., where we need it [9].

To numerically determine the cycle number, i.e., the denominator of the rational winding number for elliptical orbits, we only have to select an initial point on it and measure the number of toroidal turns that are needed to return appropriately close to the chosen point.

3 Application to a Symmetric Tokamap

In this section, we locate the position of high-order elliptical orbits of inverse golden mean tori in the phase space of a symmetric tokamap to highly accurately locate the cantorus via the MER criterion.

3.1 Symmetric Tokamap

The tokamap [12] is an important perturbed symplectic twist map describing the global behavior of magnetic field lines in toroidal systems such as tokamaks. A symmetric tokamap can be constructed by the Hamilton–Jacobi method, which describes the continuous system more accurately than the conventional tokamap [15]. The Hamilton–Jacobi method is based on a canonical transformation that makes the evolution unperturbed during a time period, while all perturbations act instantaneously during one kick per period [16]. The solutions before and after the kicks are matched to establish a symplectic map that describes the evolution exactly. The generating function associated with this map satisfies the Hamilton–Jacobi equations. The perturbed Hamiltonian may be written as the sum of a completely integrable part and a periodic perturbation, $H(\theta,I,\varphi;\varepsilon)=H_0(I)+\varepsilon H_1(\theta,I,\varphi)$, in which the toroidal angle φ plays the role of time. The equilibrium Hamiltonian function is described by the following equality [12]:

$$H_0(I) = -\frac{1}{10}I^4 + \frac{1}{3}I^3 - \frac{3}{4}I^2 + I, \tag{3}$$

with the safety factor $q(I)=1/\omega(I)=4/(2-I)(2-2I+I^2)$, where $\omega(I)$ is the equilibrium winding number.

The perturbation has the form

$$H_1(\theta,I,\varphi) = -\frac{1}{(2\pi)^2}\frac{I}{1+I}\sum_n \cos(\theta - n\varphi). \tag{4}$$

Here, the summation goes to infinity as the magnetic field line turns around the torus. The construction of the map is based on the Hamilton–Jacobi method for integrating Hamilton equations. The symmetric mapping is composed of three steps, the final result for symmetric tokamap being expressed as follows [15]:

$$\begin{cases} \bar{I}_k = I_k - \dfrac{\varepsilon}{2}\dfrac{\bar{I}_k}{1+\bar{I}_k}\sin\theta_k \\ \Theta_k = \theta_k - \dfrac{\varepsilon}{2}\dfrac{1}{2\left(1+\bar{I}_k\right)^2}\cos\theta_k \end{cases} \tag{5}$$

$$\bar{\Theta}_k = \Theta_k + \frac{2\pi}{q(\bar{I}_k)} \tag{6}$$

$$\begin{cases} I_{k+1} = \bar{I}_k - \dfrac{\varepsilon}{2}\dfrac{\bar{I}_k}{1+\bar{I}_k}\sin\theta_{k+1} \\ \theta_{k+1} = \bar{\Theta}_k - \dfrac{\varepsilon}{2}\dfrac{1}{2\left(1+\bar{I}_k\right)^2}\cos\theta_k \end{cases}. \tag{7}$$

The Poincare surface of a section of the symmetric tokamap is shown in Fig. 1 for the perturbation ε=0.851. Many magnetic islands around elliptic-fixed points along with

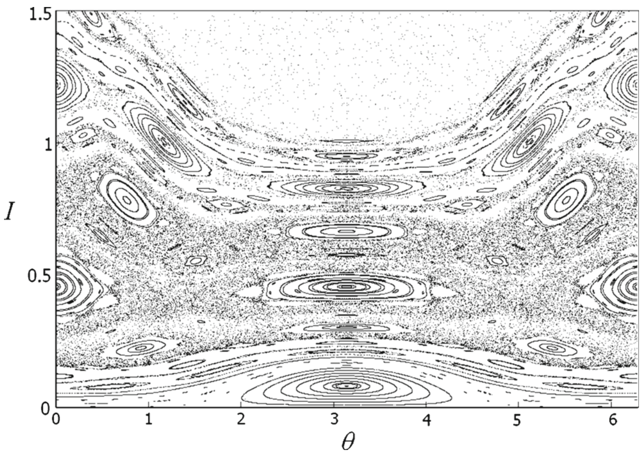

Fig. 1 Poincare surface of a section for the symmetric tokamap with the perturbation ε=0.851

intact KAM surfaces and chaotic regions are visible. The cantorus we are looking for is approximately located at $I\approx 0.88$ on the dominant (symmetry) line $\theta=\pi$. This perturbation destructs most KAM surfaces, so that, like a barrier, the cantorus separates two chaotic zones

3.2 Locating the Cantorus with Most Noble Winding Number for the Symmetric Tokamap

The most important cantorus in the plasma edge of the tokamap has the winding number [4,2,1,1,…] with the convergents 1/5, 1/4, 2/9, 3/13, 5/22, 8/35, 13/57, 21/92, 34/149, 55/241, 89/390, 144/631, 233/1,021, 377/1,652, …. These periodic orbits are alternately above and below the cantorus and approach it from both sides. Figure 2 shows the [4,2,1,1,…] surface along with its convergents for a symmetric tokamap under a perturbation ε=0.851. In this figure, the islands are plotted up to 5/22.

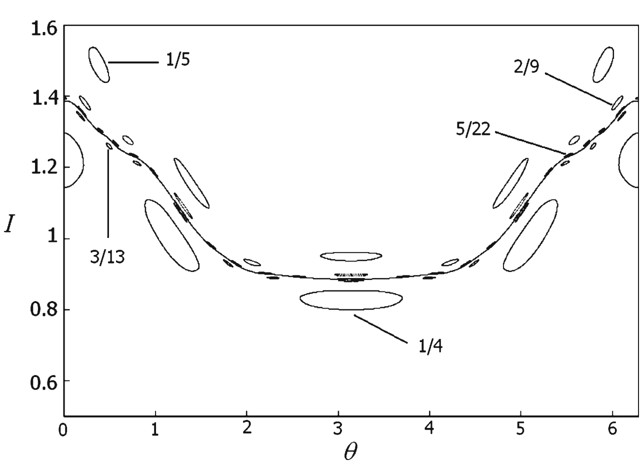

Fig. 2 Convergents to the [4,2,1,1,…] surface under the perturbation ε=0.851 for the symmetric tokamap. The high-order islands converge to a solid line, which approximately determines the location of the [4,2,1,1,…] surface

To identify more accurately the location of the surface, the higher-resolution plot in Fig. 3 shows results up to 34/149 islands for the same perturbation. It is very difficult to find the location of the higher-order elliptical orbits. In Fig. 3, we have used the MER criterion to find the location of elliptical orbit 34/149.

To plot the approximate [4,2,1,1,…] surface in Fig. 3, we have chosen an initial point between the rationals 21/92 and 34/149 on the dominant line at $\theta=\pi$.

In Fig. 4, we have used the MER criterion to locate the high-order elliptical orbits around the [4,2,1,1,…] surface of the symmetric tokamap under the perturbation ε=0.851. We have chosen 500 initial points on line and an action parameter between I=0.88545, below the surface, and I=0.88560, above it.

Each initial point has been iterated for 1,000 times, i.e., for n=1,000 in Eq. (2). The figure shows a few stationary segments corresponding to elliptical orbits, the lower-order ones being confirmed by the phase-space portrait. The stationary segments for higher-order periodic points are smaller. As Fig. 4 shows, the plateaus in the MER curve are slightly inclined, due to the radial variation of the total Hamiltonian.

Section 2 has described a method to determine the cycle number of each periodic point related to a stationary segment. Five periodic points are shown in Fig. 4. The [4,2,1,1,…] surface is located somewhere between the 89/390 and 34/149 rationals. Care must be taken because certain stationary segments showing periodic points are not convergents to the [4,2,1,1,…] winding number. An example is the boxed continued fraction in Fig. 4, equal to 47/206. A closer view of Fig. 4 is shown in Fig. 5 for an action parameter between I=0.88504 and I=0.88506, so that we consider the MER for regions between the 144/631 and 89/390 periodic points. The map in Fig. 5 has been iterated 5,000 times. The [4,2,1,1,…] surface is now located somewhere between the 377/1,652 and 233/1,021 periodic orbits, with an accuracy of the order of 5×

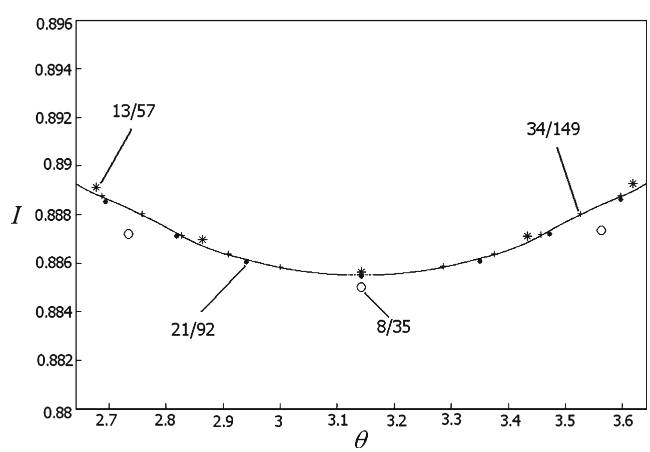

Fig. 3 Higher-order convergents to [4,2,1,1,…] surface under the perturbation ε=0.851. The high-order islands converge to a solid line, which approximately marks the location of the [4,2,1,1,…] surface

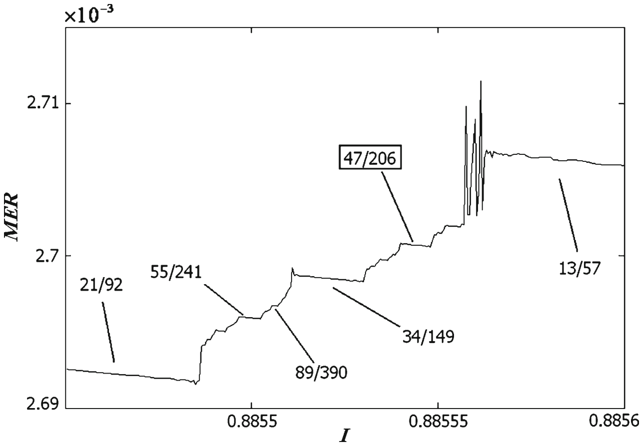

Fig. 4 MER for the region around [4,2,1,1,…] surface for symmetric tokamap under the perturbation ε=0.851. The [4,2,1,1,…] surface is located somewhere between the 89/390 and 34/149 island series

10^{-6}. The same procedure can be continued on and on to locate higher-order elliptical orbits.

4 Application to Symmetric EML Map

The plasma-wall interactions needed to protect wall components constitute one of the major technical problems in long runs of magnetically confined plasmas in tokamaks. To face this problem, EML rings are used to create a chaotic region at the plasma edge. In this section, we determine the position of the locally most important cantorus near the edge of plasma.

4.1 Symmetric EML Map

The EML is a circular ring wound around the tokamak in the poloidal direction, which generates a net poloidal magnetic field, as Fig. 6 shows. Adding an EML magnetic field to the equilibrium magnetic field converts the integrable system into a nearly integrable system. The equations which govern the magnetic field lines in tokamak are similar to canonical Hamiltonian equations. The total Hamiltonian function is defined as the poloidal flux of the magnetic field $H=H_0+\varepsilon H_1$. The equilibrium part H_0 is analytically obtained by solving the Grad–Shafranov equation [13, 17]

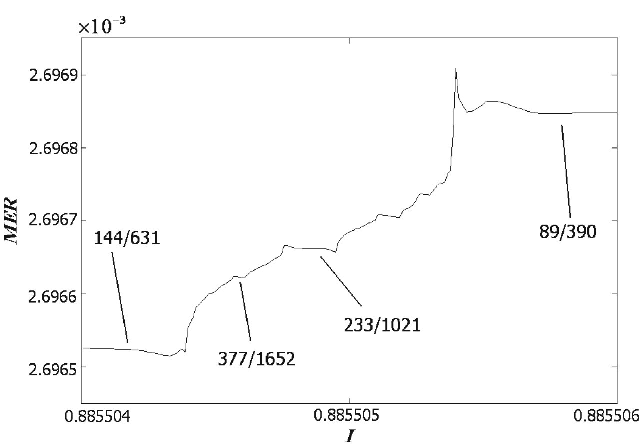

Fig. 5 Closeup view of Fig. 4. The [4,2,1,1,…] surface is located between the 377/1,652 and 233/1,021 island series

$$H_0 = \frac{I_p}{2I_e}\sum_{k=1}^{\gamma+1}\frac{1}{k}\left[1-\left(1-\frac{r_t^2}{a^2}\right)^k\right], \tag{8}$$

in which I_p is the toroidal plasma current, I_e is the current in external coils, a is plasma radius, and γ is a positive constant. The perturbation H_1, produced by the EML rings, is derived from the analytical solution of the Laplace equation with proper boundary conditions

$$H_1(J,\nu,\varphi) = \sum_{m=0}^{2m_0}\left|J_{m-m_0}(m_0\lambda)\left(\frac{r_t}{b}\right)^m\right|\cos(m\theta_t(\nu,J)-n_0\varphi), \tag{9}$$

where J_k is the Bessel function of order k, the integers n_0 and m_0 are positive, and λ is a current-modulation parameter. Here, (J,ν) are action–angle variables, defined in polar toroidal coordinates (r_t,θ_t) by the equalities [13, 17]

$$J(r_t) = \frac{1}{4}\left[1-\left(1-4\frac{r_t^2}{R_0'^2}\right)^{1/2}\right] \tag{10}$$

$$\nu(r_t,\theta_t) = 2\arctan\left[\frac{1}{\Lambda}\tan\left(\frac{\theta_t}{2}\right)\right] \tag{11}$$

$$\Lambda(r_t) = \left(1-2\frac{r_t}{R_0'}\right)^{1/2}\times\left(1-2\frac{r_t}{R_0'}\right)^{-1/2}. \tag{12}$$

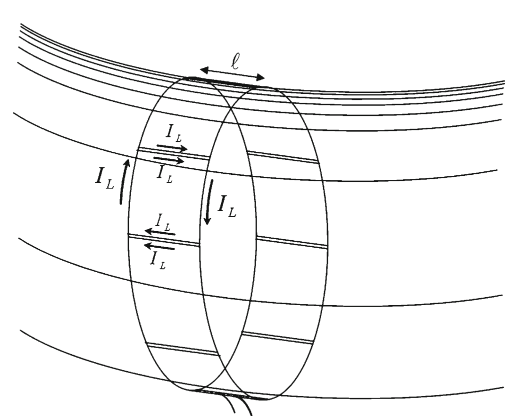

Fig. 6 An ergodic magnetic limiter ring

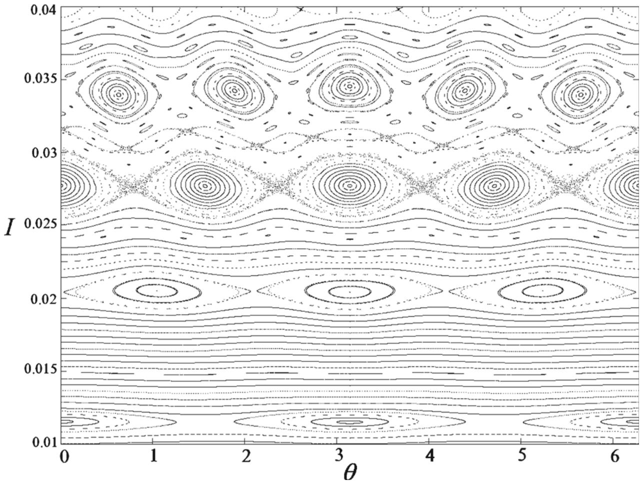

Fig. 7 Phase portrait for the symmetric EML map with N=4 EML rings and ε=−1.17×10^{-4}

The perturbation parameter is defined as

$$\varepsilon = -\frac{1}{\pi}\left(\frac{l}{2\pi R_0'}\right)\left(\frac{I_L}{I_e}\right), \tag{13}$$

where I_L is the current in the EML rings, R_0' is the shifted radius of magnetic flux surfaces, and R_0 is the geometrical major radius so that $R_0'-R_0$ is the so-called Shafranov shift. A safety factor, derived from the equilibrium-field equations, takes the form [18]

$$q = \frac{I_e}{I_p}\frac{r_t^2}{R_0'^2}\left[1-\left(1-\frac{r_t^2}{a^2}\right)^{\gamma+1}\right]^{-1}\left(1-4\frac{r_t^2}{R_0'^2}\right)^{-1/2}. \tag{14}$$

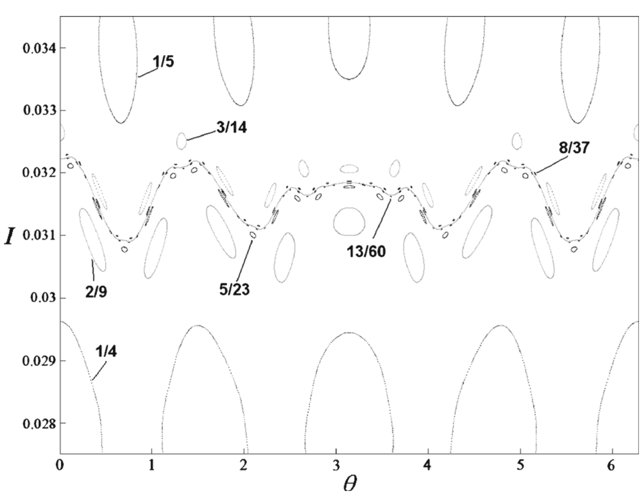

Fig. 8 Elliptic orbits convergent to the surface [4,1,1,…] under the perturbation ε=−1.17×10^{-4}. The high-order islands converge to a solid line, approximately the location of the [4,1,1,…] surface

Here, the toroidal angle φ plays the role of time. The symmetric EML map in polar toroidal coordinate is written in the following form:

$$\begin{cases} \theta_{k+1} = \theta_k + \dfrac{2\pi}{Nq(J_k)} + \varepsilon\dfrac{\partial S(\theta_k, J_k)}{\partial J_k} + \varepsilon\dfrac{\partial S(\theta_{k+1}, J_k)}{\partial J_k} \\ I_{k+1} = I_k - \varepsilon\dfrac{\partial S(\theta_k, J_k)}{\partial \theta_k} - \varepsilon\dfrac{\partial S(\theta_{k+1}, J_k)}{\partial \theta_k} \end{cases} \tag{15}$$

$$J_k = I_k - \varepsilon\frac{\partial S(\theta_k, J_k)}{\partial \theta_k}, \tag{16}$$

where N is the number of EML rings that wind around the tokamak and S is the generating function [9]. The phase-space portrait of this map for the perturbation ε=−1.17×10^{-4} and mode number (m_0,n_0)=(4,1) with N=4 EML rings and the current-modulation parameter λ=0.4827 is illustrated in Fig. 7.

4.2 Locating the Locally Most Important Cantorus in the Plasma Edge for a Symmetric EML Map

With the parameters we have chosen, the 1/5 periodic orbits are located near the plasma edge. The cantori between the 1/4 and 1/5 periodic orbits are very important for controlling the loss of field lines at the edge because the cantori play the role of partial barriers and limit radial diffusion of chaotic field lines to the tokamak wall. Two cantori with the winding numbers [4,1,1,…] and [4,2,1,…] are in this region. We will focus on the [4,1,1,…] surface because it is closer to the plasma edge and is also more noble. This surface can be

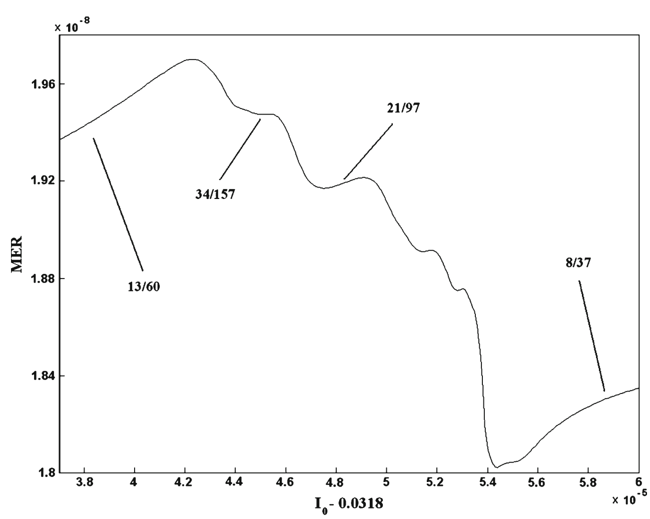

Fig. 9 MER for the region around the [4,1,1,…] cantorus for the symmetric EML map under the perturbation ε=−1.17×10^{-4}. The [4,1,1,…] surface is located between 34/157 and 21/97 island series

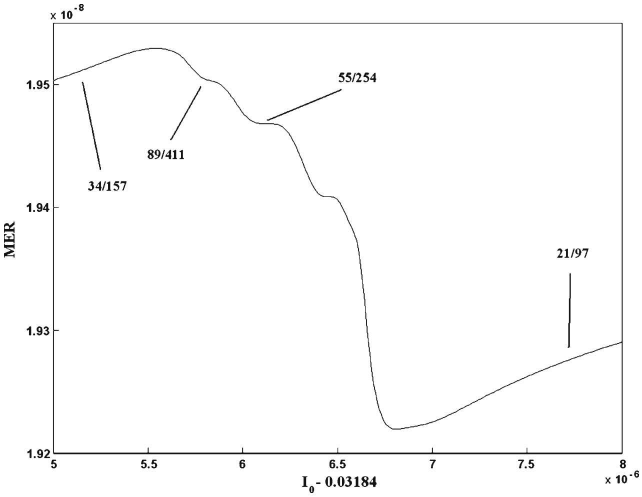

Fig. 10 Closeup view of Fig. 9. The [4,1,1,…] surface is located between the 89/411 and 55/254 island series

characterized by the convergents 1/4, 1/5, 2/9, 3/14, 5/23, 8/37, 13/60, 21/97, 34/157, 55/254, 89/411, 144/665, 233/1,076, …. Here, we use the MER criterion to locate the high-order elliptical orbits associated with this cantorus. In Fig. 8, the [4,1,1,…] surface is plotted along with the elliptic orbits approaching it from both sides. The perturbation parameter is $\varepsilon=-1.17\times10^{-4}$, and the other parameters are the same as Fig. 7. Figure 9 shows the MER for the [4,1,1,…] surface versus the action parameter for the perturbation parameter $\varepsilon=-1.17\times10^{-4}$.

To plot this figure, we have chosen 400 initial points for the constant angle $\theta=\pi$ on the symmetry line and an action parameter between I=0.031837, below the surface, and I=0.031836, above it. Each initial point is iterated 1,000 times. The figure shows a few stationary segments, which are related

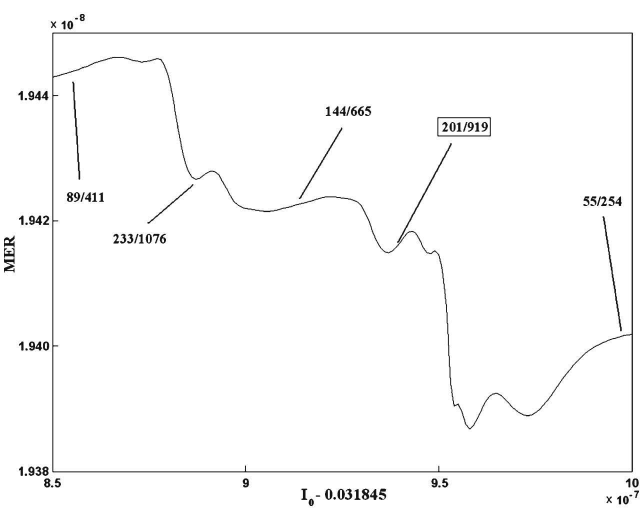

Fig. 11 Closeup view of Fig. 10. The [4,1,1,…] surface is located between the 233/1,076 and 144/665 island series

to elliptical orbits. The cycle number of each segment can be determined as described in Section 2. In this figure, the [4,1,1,…] cantorus is limited to the regions between 21/97 and 34/157 elliptical orbits.

Figure 10 shows a closer view of Fig. 9 for an action parameter between I=0.031845, above the surface, and I=0.031848, below it. Here, the symmetric EML map has been iterated 3,000 times. The [4,1,1,…] surface is somewhere between the 55/254 and 89/411 elliptical orbits. A closer view of Fig. 10 is plotted in Fig. 11 for an action parameter between I=0.03184585 and I=0.031846. The map is iterated 6,000 times. Now, the [4,1,1,…] cantorus is limited to the regions between 144/665 at $I\approx0.03184591$ and 233/1,076 at $I\approx0.03184589$, an interval approximately 2×10^{-8} wide. In other words, the MER criterion locates the [4,1,1,…] cantorus with an accuracy of the order of 10^{-8}. Figure 11 shows certain stationary segments showing elliptical orbits that are not convergents of [4,1,1,…]. The winding number for one of them is boxed in the figure.

5 Conclusion

The location of a cantorus in symplectic maps can be approximately determined from the high-order islands converging to it. Higher-order islands have smaller size and are not revealed in the phase-space portrait. Here, we have used the MER criterion, which stems from energy conservation, to highly accurately determine the place of a cantorus on the phase-space portrait. Our procedure evaluates the total perturbation Hamiltonian, while the dynamical system evolves in time. The MER has nearly a constant value in the area inside islands around elliptic points, while it varies significantly in chaotic regions. If the MER is plotted along the dominant line of a symplectic map, on which an elliptical orbit point corresponds to every rational frequency, all the islands will be precisely distinguishable in the resulting figure. Closeup views of the plot reveals the convertgents, i.e., the higher-order islands that converge from above and from below the cantorus. The cycle number of any convergent, i.e., the denominator of the rational winding number of the elliptical orbits, can be easily determined numerically by selecting an initial point on any plateau and measuring the number of toroidal turns that are needed to return appropriately close to the chosen point. Eventually the cantorus can be located, since it lies somewhere between the convergents approaching it from above and those approaching from below.

Symplectic maps can be rigorously derived from the Hamilton–Jacobi equation and Jacobi's theorem. We applied the procedure on two symplectic maps in fusion plasmas: a symmetric tokamap and a symmetric EML map. These maps describe the dynamical behavior of the chaotic magnetic field lines in a tokamak. In the first example, the location of the most robust cantorus at the plasma edge was determined by the MER criterion. In the second, the location of the locally most noble cantorus near the plasma edge in a tokamak with EML rings was determined.

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