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Relativistic Approach to the Hydrogen Atom in a Minimal Length Scenario

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Abstract We show that relativistic contributions to the ground-state energy of the hydrogen atom from a minimal length introduced by a Lorentz-covariant algebra are more important than non-relativistic contributions; the non-relativistic approach is therefore unsuitable. We compare our result with experimental data to estimate an upper bound of the order 10^{-20}m for the minimal length.

Keywords Minimal length · Lorentz-covariant algebra · Dirac equation · Hydrogen atom

1 Introduction

The notion of a minimal length is not new. In view of the divergences arising from the advent of Quantum Field Theory, in the 1930s, Heisenberg concluded that a minimal length should exist, which would offer a natural cutoff for divergent integrals [1–3]. In 1947, Snyder proposed a Lorentz-covariant algebra of the position and momentum operators leading to a non-continuous space-time, which implemented a minimal length [4]. Battisti and Meljanac have analyzed several physical consequences of this non-commutativity [5].

All candidates to a quantum-gravity theory lead to the prediction of the a minimal length. This is hardly surprising, since all of them combine the fundamental constants G, c, and $\hbar$ of gravity, relativity and quantum mechanics, respectively, which define the fundamental constant

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1,6 \times 10^{-35}\text{m}, \quad (1)$$

known as the Planck length.

Clearly, constructing a constant with dimension of length is insufficient to establish the existence of a minimal length. A stronger argument is that the smaller the region of space-time we want to probe, the higher the energy of the incident particles necessary to probe it. The rising energy magnifies the gravitational field created by the incident particles and increasingly disturbs its trajectory, measurements of which become progressively more uncertain. Eventually, any such measurement becomes impossible.

Since the Planck length is very small, experimental probes of the minimal length are unavailable, presently and in the foreseeable future. It should be noted, however, that present models of extra dimensions yield an effective Planck length scale higher than the four dimensional value [6–8], which motivates experimental searches for a minimal length with current technologies [9].

As a result, interest in the search for experimental constraints on an upper bound for the minimal length value has steadily grown. Many papers have been dedicated to this issue, and a wide variety of results, of quite different magnitudes, have been obtained. Although most of papers work in a quantum context, Benczik and collaborators have considered the classical limit (Poisson bracket) of the deformation of the canonical commutation relations and its effect on the classical orbits of particles in a central-force potential [10]. They have found an upper bound for the minimal length of the order 10^{-68}m, which is 10^{33} order smaller than the Planck length. Reference [11] considered a treatment of many particles leading to an effective parameter

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related to the minimal length for the planet Mercury divided by the number of particles in the planet, which led to a more satisfactory upper bound of the order 10^{-17}m.

The effects of a minimal length on the spectrum of the hydrogen atom have been calculated by many authors, motivated by high-precision experimental measurement of the frequency of the radiation emitted during the transitions. In 1999, to our knowledge, Brau [12] was the first to calculate such effects, estimated a maximum value of the order of 10^{-17}m. A different result was obtained in 2003 by Akhoury and Yao [13], who worked in the momentum representation. In 2005, Benczik et al. [14] also considered the spectrum of the hydrogen atom in momentum representation, using perturbation theory. Their results were in agreement with those obtained by Brau, except in the case $l = 0$, where a numerical method and a cutoff procedure had to be introduced to deal with the divergent $\frac{1}{r^3}$ term [14]. In 2006, Stetsko and Tkachuk [15] proposed a modified, divergence-free perturbation theory, which yielded corrections to all energy levels, including the $l = 0$ levels [15], in agreement with Brau's results. In 2007, Nouicer obtained the exact energy eigenvalues and eigenfunctions for the Coulomb potential in one-dimensional space using path integrals in momentum space [16]. In view of the discrepancy with the results of Akhoury and Yao, in 2010, Bouaziz and Ferkous considered the same problem, but proposed another method to solve the s-wave Schroendiger equation in momentum space [17]. Their result for the upper bound for the minimal length is of the order of 10^{-24}m.

Most treatments of the hydrogen atom in the literature are non-relativistic. Nonetheless, we expect the relativistic effects associated with a minimal length to be more relevant than the non-relativistic ones, since the effects of a minimal length grow with energy. A non-relativistic approach may therefore neglect important terms that would emerge in a relativistic approach. Therefore, it seems appropriate to study relativistic effects on the hydrogen atom in a minimal length framework. This considered, we study the hydrogen atom using the Dirac equation with a central potential in a minimal length scenario.

Dirac equation treatments of the minimal length problem are found in the literature. Reference [8] obtained the Dirac equation in the minimal length scenario resultant from modifying the canonical functional relation between the momentum $\vec{p}$ and the wave vector $\vec{k}$. To preserve the symmetry between space and time, the authors have forced the energy E and the frequency ω to satisfy the same functional relation. In 2005, Nozari and Karani derived a modified Dirac equation for a free particle and claimed that, due to the quantum fluctuation of the background space-time, it is impossible to have free particles in a minimal length scenario [18]. In that same year, Quesne and Tkachuk exactly solved the Dirac oscillator in the momentum representation [19]. One year later, in 2006, Nouicer published an alternative solution of the same problem [20], while Quesne and Tkachuk used a Lorentz-covariant algebra to deal with the (1+1)-dimensional Dirac oscillator [21]. In 2011, Samar proposed a modified perturbation theory in momentum representation in order to deal with the hydrogen atom in a minimal length scenario introduced by a Lorentz-covariant deformed algebra and found an upper bound of the order of 10^{-19}m [22]. In 2013, Menculini, Panella and Roy derived exact solutions for the (2+1)-dimensional Dirac equation in a homogeneous magnetic field [23]. In the same year, Antonacci Oakes et al. calculated the ground state energy of the hydrogen atom via the Dirac equation using (non Lorentz-covariant) Kempf's algebra to recover Brau's result [24]. For more information on the minimal length literature, the interested reader is referred to Reference [3, 27–29].

There are several ways to implement a minimal length scenario. One of them is to modify the canonical commutation relations. The deformed commutation relations due to Kempf [25] are frequently used, which are given by the equalities[1]

$$[\hat{X}_i, \hat{P}_j] = i\hbar\left[\left(1+\beta\hat{\mathbf{P}}^2\right)\delta_{ij} + \beta'\hat{P}_i\hat{P}_j\right], \tag{2}$$

$$[\hat{X}_i, \hat{X}_j] = -i\hbar\frac{\left[2\beta-\beta'+\left(2\beta+\beta'\right)\beta\hat{\mathbf{P}}^2\right]}{\left(1+\beta\hat{\mathbf{P}}^2\right)} \times\left(\hat{X}_i\hat{P}_j-\hat{X}_j\hat{P}_i\right), \tag{3}$$

$$[\hat{P}_i, \hat{P}_j] = 0, \tag{4}$$

where β and β' are parameters related to the minimal length.

Keeping in mind that the Dirac equation is Lorentz-covariant, while the algebra proposed by Kempf is not, we prefer the Lorentz-covariant algebra proposed by Quesne and Tkachuk [21, 26], given by

$$[\hat{X}^\mu, \hat{P}^\nu] = -i\hbar\left[\left(1-\beta\hat{P}_\rho\hat{P}^\rho\right)g^{\mu\nu} - \beta'\hat{P}^\mu\hat{P}^\nu\right], \tag{5}$$

$$[\hat{X}^\mu, \hat{X}^\nu] = i\hbar\frac{\left[2\beta-\beta'-\left(2\beta+\beta'\right)\beta\hat{P}_\rho\hat{P}^\rho\right]}{\left(1-\beta\hat{P}_\rho\hat{P}^\rho\right)} \times\left(\hat{P}^\mu\hat{X}^\nu-\hat{P}^\nu\hat{X}^\mu\right), \tag{6}$$

$$[\hat{P}^\mu, \hat{P}^\nu] = 0, \tag{7}$$

to introduce a minimal length in the theory.

The rest of this paper is organized as follows. Section 2 uses a "position" representation, which satisfies

[1] We use boldface to denote a vector operator.

the Lorentz-covariant commutation relations of Quesne-Tkachuk in the special case $\beta' = 2\beta$, to modify the Dirac equation and therefore to introduce the hydrogen atom in a minimal length scenario. Section 3 calculates the ground state energy of the hydrogen atom in the minimal length scenario and roughly estimates an upper bound for the minimal length. Our conclusions are presented in Section 4.

2 Hydrogen Atom in a Minimal Length Scenario

In this section, in order to introduce the hydrogen atom in a minimal length scenario, we use the Quesne–Tkachuk Lorentz-covariant deformed algebra to modify the Dirac equation with a central potential. We consider the case $\beta' = 2\beta$. To first-order in β, (5), (6), and (7) then take the forms

$$[\hat{X}^\mu, \hat{P}^\nu] = -i\hbar\left[\left(1-\beta\hat{\mathbf{P}}_\rho\hat{\mathbf{P}}^\rho\right)g^{\mu\nu} - 2\beta\hat{\mathbf{P}}^\mu\hat{\mathbf{P}}^\nu\right], \tag{8}$$

$$[\hat{X}^\mu, \hat{X}^\nu] = 0, \tag{9}$$

$$[\hat{P}^\mu, \hat{P}^\nu] = 0. \tag{10}$$

The commutation relations in (9) and (10) lead to the minimum $\Delta X_i^{min} = \hbar\sqrt{5\beta}$.

It is not difficult to verify that the following non-trivial transformations of operators from x^μ and p^μ to X^μ and P^μ satisfy the Quesne-Tkachuk commutation relations (8–10) to first-order in β [30]:

$$\hat{X}^\mu = \hat{x}^\mu, \tag{11}$$

$$\hat{P}^\mu = \left(1-\beta\hat{p}^\nu\hat{p}_\nu\right)\hat{p}^\mu, \tag{12}$$

where $\hat{x}^i$, $\hat{p}^i \equiv -i\hbar\frac{\partial}{\partial x^i}$ $(i = 1, 2, 3)$ and $\hat{p}^0 \equiv i\hbar\frac{1}{c}\frac{\partial}{\partial t}$ are the position, momentum, and (except for the c factor) energy operators in ordinary quantum mechanics[2], respectively. In other words, $\hat{x}^i$ and $\hat{p}^i$ satisfy the relations

$$[\hat{x}^i, \hat{x}^j] = 0, \tag{13}$$

$$[\hat{p}^i, \hat{p}^j] = 0, \tag{14}$$

$$[\hat{x}^i, \hat{p}^j] = -i\hbar g^{ij}. \tag{15}$$

[2] We use the expression "ordinary quantum mechanics" in opposition to quantum mechanics with a minimal length.

The Dirac equation with the electrostatic central potential of the proton in ordinary quantum mechanics reads

$$\left[-c\gamma^0\gamma^\mu\hat{p}_\mu + \gamma^0 mc^2 - \frac{\hbar c\alpha}{r}\right]|\psi(t)\rangle = 0, \tag{16}$$

where α is the fine structure constant and $\gamma^\mu \equiv \left(\hat{\beta}, \hat{\beta}\vec{\alpha}\right)$, with

$$\hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \tag{18}$$

$\vec{\sigma}$ denoting the Pauli matrices.[3]

To find the Dirac equation for the hydrogen atom in the minimal length scenario, we substitute $\hat{P}_\mu$ for $\hat{p}_\mu$ in (16). Equation (11) keeps the central potential unmodified to the considered order, i. e., to $\mathcal{O}(\beta)$. The Dirac equation therefore takes the form

$$\begin{aligned}&\left[-i\hbar\frac{\partial}{\partial t} + c\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right) + \hat{\beta}mc^2 - \frac{\hbar c\alpha}{r}\right]|\psi^{ML}(t)\rangle \\ &\quad - \beta\left[\frac{i\hbar^3}{c^2}\frac{\partial^3}{\partial t^3} + i\hbar\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2\frac{\partial}{\partial t}\right. \\ &\qquad\left. - \frac{\hbar^2}{c}\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)\frac{\partial^2}{\partial t^2} + c\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^3\right]|\psi^{ML}(t)\rangle = 0,\end{aligned} \tag{19}$$

where $\langle\psi_{\vec{\xi}}^{ML}|\psi^{ML}(t)\rangle = \psi^{ML}(\vec{\xi}, t)$ are the "quasi-position states".[4]

3 Ground State Energy

To eliminate the time variable in (19), we try the following *ansatz*:

$$|\psi^{ML}(t)\rangle = e^{-\frac{i}{\hbar}\mathcal{E}t}|\varphi^{ML}\rangle, \tag{20}$$

[3] $\vec{\alpha}$ and $\hat{\beta}$ should not be confused with the fine structure constant α and the minimal length parameter β.

[4] Note that x^i is not the eigenvalue of the $\hat{X}^i$ operator. In fact, the minimal length implies that $\hat{X}^i$ cannot have any physical eigenstate, i. e., any eigenfunction within the Hilbert space. Consequently, we are forced to introduce the so-called "quasi-position representation", which consists of projecting the states $|\psi^{ML}(t)>$ onto the set of maximally localized states $|\psi_{\xi}^{ML}>$. Thus, $<\psi_{\vec{\xi}}^{ML}|\psi^{ML}(t)>= \psi^{ML}(\vec{\xi}, t)$ are the "quasi-position wave functions" [31–33].

where $\mathcal{E}$ describes the time evolution of the stationary state $|\psi^{ML}(t)\rangle$. Substituting (20) into (19) and neglecting terms of order $\mathcal{O}(\beta^2)$, we come to the result

$$\left[-\mathcal{E} + c\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right) + \hat{\beta}mc^2 - \frac{\hbar c\alpha}{r}\right]|\varphi^{ML}\rangle + \beta m^2c^2\left[\frac{1}{m^2c}\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^3 - \frac{\mathcal{E}}{m^2c^2}\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 - \frac{\mathcal{E}^2}{m^2c^3}\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right) + \frac{\mathcal{E}^3}{m^2c^4}\right]|\varphi^{ML}\rangle = 0. \tag{21}$$

For $\beta \to 0$, $\mathcal{E}$ is the ordinary energy E of the hydrogen atom. Therefore, if we assume the mass scale M_{ML} of the minimal length to be much larger than the electron mass $\left(\beta = \frac{c^2}{M_{ML}^2c^4}, \text{so} \beta m^2c^2 = \frac{m^2}{M_{ML}^2} \ll 1\right)$, then we can treat the second term on the right-hand side of (21) as a perturbation. Consequently, (21) indicates that

$$\mathcal{E} = E^{ML} = E + \beta m^2c^2E^1 + \mathcal{O}(\beta^2) \tag{22}$$

and

$$|\varphi^{ML}\rangle = |\varphi\rangle + \beta m^2c^2|\varphi^1\rangle + \mathcal{O}(\beta^2), \tag{23}$$

where E is the energy of the hydrogen-atom eigenstate $|\varphi\rangle$ obtained from the ordinary Dirac equation.

Substituting the right-hand sides of (22) and (23) for $\mathcal{E}$ and $|\varphi^{ML}\rangle$ on the left-hand side of (21) and neglecting terms of order $\mathcal{O}(\beta^2)$, we are led to the equality

$$E^{ML} = E + \beta\langle\varphi|\left[c\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^3 - E\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 - \frac{E^2}{c}\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right) + \frac{E^3}{c^2}\right]|\varphi\rangle. \tag{24}$$

In general, the computation of the expectation value on the right-hand side of (24) is very laborious. For the ground state, however, we are faced with the following relatively simple calculation. In this case, (24) takes the form

$$E_0^{ML} = E_0 + \beta\left[c\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^3|\varphi_0\rangle - E_0\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2|\varphi_0\rangle - \frac{E_0^2}{c}\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)|\varphi_0\rangle + \frac{E_0^3}{c^2}\right], \tag{25}$$

where $E_0 = mc^2\sqrt{1-\alpha^2}$ is the energy of the hydrogen-atom ground state $|\varphi_0\rangle$, obtained from the ordinary Dirac equation.

We now have to calculate the expressions

$$\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)|\varphi_0\rangle = \int\left(\phi_0^\dagger, \chi_0^\dagger\right)\begin{pmatrix}0 & \vec{\sigma}\cdot\hat{\mathbf{p}}\\ \vec{\sigma}\cdot\hat{\mathbf{p}} & 0\end{pmatrix}\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}d^3\vec{x}, \tag{26}$$

$$\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2|\varphi_0\rangle = \int\left(\phi_0^\dagger, \chi_0^\dagger\right)\begin{pmatrix}(\vec{\sigma}\cdot\hat{\mathbf{p}})^2 & 0\\ 0 & (\vec{\sigma}\cdot\hat{\mathbf{p}})^2\end{pmatrix}\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}d^3\vec{x}, \tag{27}$$

and

$$\langle\varphi_0|\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^3|\varphi_0\rangle = \int\left(\phi_0^\dagger, \chi_0^\dagger\right)\begin{pmatrix}0 & (\vec{\sigma}\cdot\hat{\mathbf{p}})^3\\ (\vec{\sigma}\cdot\hat{\mathbf{p}})^3 & 0\end{pmatrix}\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}d^3\vec{x}. \tag{28}$$

The two-component eigenspinors of the ground state, ϕ_0 and χ_0, are given by the expression

$$\langle\vec{x}|\varphi_0\rangle = \begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix} = \begin{pmatrix}F(r)Y_0^{1/2,m}(\theta,\phi)\\ -if(r)Y_1^{1/2,m}(\theta,\phi)\end{pmatrix} \tag{29}$$

where

$$F(r) = a_0br^\gamma e^{-ar}, \tag{30}$$

and

$$f(r) = b_0br^\gamma e^{-ar}, \tag{31}$$

with

$$\gamma = \epsilon - 1, \tag{32}$$

$$a = \left(\frac{mc}{\hbar}\right)\sqrt{1-\epsilon^2}, \tag{33}$$

$$b = \left(\frac{mc}{\hbar}\right)^\gamma, \tag{34}$$

$$a_0 = \left(\frac{2a}{b}\right)^{\gamma+1}\sqrt{\frac{(1+\epsilon)}{\Gamma(2\gamma+3)}}, \tag{35}$$

$$b_0 = \sqrt{\frac{1-\epsilon}{1+\epsilon}}a_0, \tag{36}$$

$$\epsilon = \frac{E_0}{mc^2}, \tag{37}$$

and $Y_{j\pm1/2}^{j,m}(\theta,\phi)$ are the common eigenspinors of $\hat{j}_z$ and $\hat{J}^2$ [34].

We now employ the identity

$$\vec{\sigma}\cdot\hat{\mathbf{P}} = \vec{\sigma}\cdot\vec{e}_r\left(-i\hbar\frac{\partial}{\partial r} + i\frac{\vec{\sigma}\cdot\hat{\mathbf{L}}}{r}\right), \tag{38}$$

with

$$\vec{\sigma} \cdot \vec{e}_r Y^{j,m}_{j\pm 1/2} = -Y^{j,m}_{j\pm 1/2}, \tag{39}$$

to show that

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})\,\phi_0 = i\hbar \frac{dF}{dr} Y^{1/2,m}_1, \tag{40}$$

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})\,\chi_0 = \left(\hbar \frac{df}{dr} + 2\hbar \frac{f}{r}\right) Y^{1/2,m}_0, \tag{41}$$

and that

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})^2\,\phi_0 = -\hbar^2 \left(\frac{d^2F}{dr^2} + 2\frac{1}{r}\frac{dF}{dr}\right) Y^{1/2,m}_0, \tag{42}$$

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})^2\,\chi_0 = i\hbar^2 \left(\frac{d^2f}{dr^2} + 2\frac{1}{r}\frac{df}{dr} - 2\frac{f}{r^2}\right) Y^{1/2,m}_1, \tag{43}$$

and

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})^3\,\phi_0 = -i\hbar^3 \left(\frac{d^3F}{dr^3} + 2\frac{1}{r}\frac{d^2F}{dr^2} - 2\frac{1}{r^2}\frac{dF}{dr}\right) Y^{1/2,m}_1, \tag{44}$$

$$(\vec{\sigma} \cdot \hat{\mathbf{p}})^3\,\chi_0 = -\hbar^3 \left(\frac{d^3f}{dr^3} + 4\frac{1}{r}\frac{d^2f}{dr^2}\right) Y^{1/2,m}_0. \tag{45}$$

After some algebra, we find that

$$\langle\varphi_0|\,(\vec{\alpha} \cdot \hat{\mathbf{p}})\,|\varphi_0\rangle = \frac{mc}{\epsilon}\left(1-\epsilon^2\right), \tag{46}$$

$$\langle\varphi_0|\,(\vec{\alpha} \cdot \hat{\mathbf{p}})^2\,|\varphi_0\rangle = m^2c^2 \frac{(2-\epsilon)\left(1-\epsilon^2\right)}{\epsilon\,(2\epsilon-1)}, \tag{47}$$

and

$$\langle\varphi_0|\,(\vec{\alpha} \cdot \hat{\mathbf{p}})^3\,|\varphi_0\rangle = m^3c^3 \frac{\left(1-\epsilon^2\right)^2}{\epsilon\,(2\epsilon-1)}. \tag{48}$$

Straightforward algebra then leads to the result

$$E^{ML}_0 = mc^2\epsilon + \beta m^3c^4 \left(\frac{1-2\epsilon-2\epsilon^4+4\epsilon^5}{2\epsilon^2-\epsilon}\right). \tag{49}$$

It proves instructive to expand E^{ML}_0 in power of the fine structure constant. Simple calculations then lead to the result

$$E^{ML}_0 \approx mc^2\left(1-\frac{\alpha^2}{2}-\frac{\alpha^4}{8}\right)+\beta m^3c^4\left(1-\frac{7\alpha^2}{2}+\frac{3\alpha^4}{8}\right). \tag{50}$$

Clearly, the sum of the terms independent of the fine-structure constant on the right-hand side of (50) is the electron rest energy, a result in agreement with Reference [30]. Hence, subtracting the rest energy of the electron,

$$\Delta E^{ML}_0 = E^{ML}_0 - mc^2 - \beta m^3c^4, \tag{51}$$

we are led to the equality

$$\Delta E^{ML}_0 \approx -mc^2\left(\frac{\alpha^2}{2}+\frac{\alpha^4}{8}\right) - \beta m^3c^4\left(\frac{7\alpha^2}{2}-\frac{3\alpha^4}{8}\right), \tag{52}$$

Equation (52) shows that the correction to the ground-state energy of the hydrogen atom is always negative and of $\mathcal{O}(\alpha^2)$, in agreement with Reference [22].

Naively, we might expect to recover Brau's result in the limit of small α [12]. Careful examination nonetheless reveals that the relativistic effects stemming from the minimal length introduced by the Lorentz-covariant algebra start at order $\mathcal{O}(\alpha^2)$, instead of starting at $\mathcal{O}(\alpha^4)$ as in the non-relativistic approaches. This shows that relevant terms may be lost when relativistic effects are disregarded.

To make an estimation of the minimal length, we compare our theoretical result with the experimental data for the $1S-2S$ hydrogen-atom energy splitting. To our knowledge, the most accurate result [2,466,061,413,187,035(10)Hz], correct to four parts in 10^{15}, has been obtained by Parthey et al. [35]. Given this result, to make a rough estimation of a maximum value for the minimal length, we have to note that the lowest-order correction to the $2S$ energy due to a minimal length must be of $\mathcal{O}(\alpha^2)$, because the $1S$ and $2S$ ordinary states have the same symmetry. Therefore, if the experimental error is entirely attributed to the minimal length corrections and we assume that the effects of the minimal length cannot yet be seen experimentally from (52) it results that

$$\Delta X^{min}_i \leq 10^{-20} m. \tag{53}$$

4 Summary and Conclusion

We have shown that the relativistic contributions to the ground state of the hydrogen atom due to a minimal length are more relevant than non-relativistic ones. The hydrogen atom has been treated in a minimal length scenario via a Dirac equation with central potential modified by the Lorentz-covariant deformed algebra of Quesne-Tkachuk, in the special case $\beta' = 2\beta$ — see (8), (9), and (10). To bypass the problem of substituting $\hat{X}_i$ for derivatives of $\hat{p}_i$ in the Coulomb potential ($\frac{1}{r}$), we have used the "position" representation given by (11) and (12). Under the assumption that the electron mass is much smaller than the mass scale of the minimal length, we have perturbatively calculated the energy shift of the hydrogen atom ground state.

By expanding the ground state energy in (49) in powers of the fine structure constant, we have found the energy shift to be of $\mathcal{O}(\alpha^2)$, two orders lower than the $\mathcal{O}(\alpha^4)$ shift found by Brau. This confirms our statement that the relativistic effects of a minimal length are more relevant than the non-relativistic ones.

If instead of a Lorentz-covariant algebra, we consider the Kempf algebra [24] [see (2), (3), and (4)], (25) takes the form

$$E_0^{ML} = E_0 + \beta c \langle \varphi_0 | \left(\vec{\alpha} \cdot \hat{\mathbf{p}}\right)^3 | \varphi_0 \rangle, \tag{54}$$

which yields the ground state energy

$$E_0^{ML} \approx mc^2 \left(1 - \frac{\alpha^2}{2} - \frac{\alpha^4}{8}\right) + \beta m^3 c^4 \alpha^4. \tag{55}$$

The lowest-order correction due to a minimal length is now of the same order as the non-relativistic effect. The Kempf algebra, which is not Lorentz-covariant, leads to terms of roughly the same size in the relativistic and non-relativistic approaches. However, the Dirac equation resulting from the Kempf algebra is not manifestly symmetric in space and time: same-level treatment of space and time calls for an ad hoc modification in the canonical functional relation between the energy operator and the generator of time translations [8, 24].

We emphasize that Lorentz covariance must be recovered in the presence of a minimal length. Recently, Ali, Das nd Vagenas proposed a generalized uncertainty principle (GUP) [36, 37] that is consistent with special relativity theories (Doubly Special Relativity) and includes a minimal length as fundamental limit for contractions of space. In the modified Dirac equation resulting from the GUP proposed in References [36, 37], we can find the $(\vec{\alpha} \cdot \hat{\mathbf{p}})^2$ term, which leads to terms of $\mathcal{O}(\alpha^2)$ in the ground state energy.

Comparing our result with experimental data we have roughly estimated an upper bound for the minimal length of order of $10^{-20}m$. The length scale of the Large Hadron Collider (LHC), of the order of $10^{-19}m$, is very close to our bound and lower than Brau's.

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References

1. H. Kragh, Arthur March, Werner Heisenberg and the search for a smallest length. Rewe d'Histoire des Sci. **8**(4), 401 (2012)
2. H. Kragh, Heisenberg's lattice world: the 1930 theory sketch. Am. J. Phys. **63**, 595 (1995)
3. S. Hossenfelder, Minimal length scale scenarios for quantum gravity. Living. Rev. Rel. **16**, 2 (2013)
4. H.S. Snyder, Quantized space-time. Phys. Rev. **71**(1), 38 (1947)
5. M.V. Battisti, S. Meljanac, Modification of Heisenberg uncertainty relations in noncommutative Snyder space-time geometry. Phys. Rev. **79**, 067505 (2009)
6. N. Arkani-Hamed, S. Dimopoulos, G. Dvali, The hierarchy problem and new dimensions at a millimeter. Phys. Lett. B. **429**, 263 (1998)
7. T. Appelquist, H.C. Cheng, B.S. Dobrescu, Bounds on universal extra dimensions. Phys. Rev. D. **64**, 035–002 (2001)
8. S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer, H. Stocker, Signatures in Planck regime. Phys. Lett. B. **575**, 85 (2003)
9. S. Das, E.C. Vagenas, Universality of quantum gravity correction. Phys. Rev. Lett. **101**, 221301 (2008)
10. S. Benczik, L.N. Chang, D. Minic, N. Okamura, S. Rayyan, T. Takeuchi, Short distance versus long distance physics: the classical limit of the minimal length uncertainty relation. Phys. Rev. D. **66**, 026–003 (2002)
11. C. Quesne, V.M. Tkachuk, Composite system in deformed space with minimal length. Phys. Rev. A. **81**, 012–106 (2010)
12. F. Brau, Minimal length uncertainty relation and hydrogen atom. J. Phys. A. **32**, 7691 (1999)
13. Akhoury, Y.P. Yao, Minimal length uncertainty relation and the hydrogen spectrum. Phys. Lett. B. **572**, 37 (2003)
14. S. Benczik, L.N. Chang, D. Minic, T. Takeuchi, Hydrogen-atom spectrum under a minimal-length hypothesis. Phys. Rev. A. **72**, 012104 (2005)
15. M.M. Stetsko, V.M. Tkachuk, Perturbation hydrogen spectrum in deformed space with minimal length. Phys. Rev. A. **74**, 012–101 (2006)
16. K. Nouicer, Coulomb potential in one dimension with minimal length: a path integral approach. J. Math. Phys. **48**, 112–104 (2007)
17. D. Bouaziz, N. Ferkous, Hydrogen atom in momentum space with a minimal length. **82**, 022–105 (2010)
18. K. Nozari, M. Karani, Minimal length and generalized Dirac equation. J. Phys. Stud. **15**(1), 1007 (2011)
19. C. Quesne, V.M. Tkachuk, Dirac oscillator with nonzero minimal uncertinty in position. J. Phys. A Mat. Gen. **38**, 1747 (2005)
20. K. Nouicer, An exact solution of the one-dimensional Dirac oscillator in the presence of minimal lengths. J. Phys. A, Math. Gen. **39**, 5125 (2006)
21. C. Quesne, V.M. Tkachuk, Lorentz-covariant deformed algebra with minimal length and application to the (1+1)-dimensional Dirac oscillator. J. Phys. A, Mat. Gen. **39**, 10909 (2006)
22. M.I. Samar, Modified perturbation theory for hydrogen atom in space with Lorentz-covariant deformed algebra with a minimal length. Mod. Phys. Lett. A. **20**(40), 3095 (2005)
23. L. Menculini, O. Panella, P. Roy, Exact solutions of the (2+1)dimensional Dirac equation in a constant magnetic field in the presence of a minimal length. Phys. Rev. D. **87**, 065–017 (2013)
24. T.L. Antonacci Oakes, R.O. Francisco, J.C. Fabris, J.A. Nogueira, Ground state of the hydrogen atom via Dirac equation in a minimal-length scenario. Eur. Phys. J. C. **73**, 2495 (2013)
25. A. Kempf, Non-pointlike particles in harmonic oscillators. J. Phys. A. **30**, 2093 (1997)
26. C. Quesne, V.M. Tkachuk, Lorentz-covariant deformed algebra with minimal length. Czech. J. Phys. **56**, 1269 (2006)

27. C.A. Mead, F. Wilczek, Walking the Planck length through history. Physics. Today. **54**, 15 (2001)
28. L.N. Chang, Z. Lewis, D. Minic, T. Takeuchi, On the minimal length uncertainty relation and the foundations of string theory. Adv. High Energy Phys. **2011**, 493–514 (2011)
29. M. Sprenger, P. Nicolini, M. Bleicher, Physics on the smallest scales: an introduction to minimal length phenomenology. Eur. J. Phys. **33**, 853 (2012)
30. S.K. Moyedi, M.R. Setare, H. Moayeri, Formulation of the Spinor field in the presence of a minimal length based on the Quesne-Tkachuk algebra. Int. J. Mod. Phys. A. **26**, 4981 (2011)
31. A. Kempf, G. Mangano, R.B. Mann, Hilbert space representation of the minimal length uncertainty relation. Phys. Rev. D. **52**, 1108 (1995)
32. A. Kempf, G. Mangano, Minimal length uncertainty relation and ultraviolet regularization. Phys. Rev. D. **55**, 7909 (1997)
33. G.C. Dorsch, J.A. Nogueira, Maximally localized states in quantum mechanics with modified commutation relation to all orders. Int. J. Mod. Phys. A. **27**, 125–0113 (2012)
34. E. Merzbacher. *Quantum mechanics*, 2nd edn. (Wiley, New York, 1970)
35. C.G. Parthey, Improved measurement of the hydrogen 1S-2S transition frequency. Phys. Rev. Lett. **107**, 203–001 (2011)
36. A.F. Ali, S. Das, E.C. Vagenas, Discreteness of space from the generalized uncertainty principle. Phys. Lett. B. **678**, 497 (2009)
37. S. Das, E.C. Vagenas, A.F. Ali, Discreteness of space from GUP II: Relativistic wave equations. Phys. Lett. B. **690**, 407 (2010)