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PARTICLES AND FIELDS

Quantization of Horizon Entropy and the Thermodynamics of Spacetime

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Abstract This is a review of my work published in the papers of Skakala (JHEP 1201:144, 2012; JHEP 1206:094, 2012) and Chirenti et al. (Phys. Rev. D 86:124008, 2012; Phys. Rev. D 87:044034, 2013). It offers a more detailed discussion of the results than the accounts in those papers, and it links my results to some conclusions recently reached by other authors. It also offers some new arguments supporting the conclusions in the cited articles. The fundamental idea of this work is that the semiclassical quantization of the black hole entropy, as suggested by Bekenstein (Phys. Rev. D 7:2333–2346, 1973), holds (at least) generically for the spacetime horizons. We support this conclusion by two separate arguments: (1) we generalize Bekenstein's lower bound on the horizon area transition to a much wider class of horizons than only the black-hole horizon, and (2) we obtain the same entropy spectra via the asymptotic quasi-normal frequencies of some particular spherically symmetric multi-horizon spacetimes (in the way proposed by Maggiore (Phys. Rev. Lett. 100:141301, 2008)). The main result of this paper supports the conclusions derived by Kothawalla et al. (Phys. Rev. D 78:104018, 2008) and Kwon and Nam (Class. Quant. Grav. 28:035007, 2011), on the basis of different arguments.

1 Introduction

This paper is a review of some of the work done during my postdoctorial fellowship in Brazil. It summarizes the basic results from some of the papers I published during that period [1–4]. It gives a more detailed discussion of the results than the accounts in those papers, and it connects the results in [1–4] to certain conclusions recently reached by other researchers. It also presents some new results, such that provide additional support for the basic idea presented in this work. The fundamental idea of this work is that the semiclassical quantization for black-hole entropy, as suggested by Bekenstein [5], holds (at least) generically for the spacetime horizons and that the asymptotic quasi-normal frequencies carry information about the quantized spectra (as suggested by Maggiore [6]).

The structure of this paper goes as follows: first, we offer a brief introduction to some of the main results from the area of spacetime thermodynamics. Then, we give some arguments in support of the connection between the asymptotic quasi-normal frequencies and the quantization of entropy, as suggested by Maggiore [6], who refined Hod's original conjecture [9]. In the following section, we show that the lower bound on the area/entropy increase derived by Bekenstein in his seminal work [5] can be applied to a much larger class of spacetime horizons. These results are followed by the analysis of the behavior of asymptotic quasi-normal frequencies of some of the spherically symmetric multi-horizon spacetimes and interpreting the behavior along the route suggested by Maggiore. In all the sections, we offer a relatively detailed discussion of our results.

1.1 The Brief Overview of the Spacetime Thermodynamics

The first origins of the spacetime thermodynamics trace down to the period more than 40 years ago, to the formulation of black-hole no-hair conjecture and the discovery of the Penrose process [10], which shows a certain amount of energy can be extracted from a rotating black hole. The fact

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that the black hole energy can decrease came as a surprise, and the natural question that came up after Penrose's discovery was whether there are and what are the parameters of the isolated black hole that cannot be decreased in any classical physical process. The answer to this question came with the work in [11–13], which showed the black-hole horizon area (or, consequently, the irreducible mass) cannot be decreased during the classically allowed physical processes.

The result of [11] was subsequently followed by a very interesting insight by Bekenstein [5], suggesting that the black-hole horizon area behaves in the same way as one would expect from the black hole entropy (proposing their proportionality, such that a preferable option is linear dependence $S \sim A$). Bekenstein's suggestion corresponded to the four laws of black hole mechanics derived in the work of [14]. (Let us mention that the nature of what provides the degrees of freedom for the black-hole entropy has been an intense ongoing debate since then.) The four laws of black hole mechanics provided a complete analogy with the classical four laws of thermodynamics. They suggested, if taken seriously, that a black hole has both the entropy proportional to the area of its horizon, as proposed by Bekenstein, and a temperature proportional to the black hole surface gravity. The four laws of black hole mechanics were first seen just as a curious analogy, until Hawking showed that the analogy is, in fact, an identity by proving that black holes semiclassically radiate [15, 16]. Hawking's result fixed the proportionality of the black hole temperature and the surface gravity to be (in Planck units) $T = \kappa/2\pi$, and the entropy with the horizon area to be $S = A/4 + \text{const}$. (Note that everywhere, in what follows, we will use the Planck units. The reader will be reminded at few more times in different places in the text.)

Since the early years, the field of spacetime thermodynamics conceptually developed in various directions. First, a generalized second law of black-hole thermodynamics was proposed [17]. Second, it has been shown by different authors that not only the black-hole horizon but, generally, spacetime horizons radiate with a temperature proportional to their surface gravity (as related to a suitably normalized Killing field). This was, for example, shown for the accelerated observers with the Rindler horizons [18], for the cosmological horizon [19], and recently, it was suggested that even Cauchy horizons could radiate [20]. This means that the quantum radiation is a characteristic feature of spacetime horizons in general (independent of whether the horizons are *absolute* or *apparent* and independent on the general relativity theory [21]), rather than being exclusively a property of the black-hole horizon. (For general results related to static spherically symmetric spacetimes, see for example [21].)

This way of thinking was further extended in the works of [21–24] which suggest that not only the concept of temperature but also the concept of horizon entropy[1] can be understood more generally as a property of spacetime horizons. (This is again independent of whether the horizon is observer dependent or not.) This simply means that it is more appropriate to speak about "spacetime thermodynamics," rather than exclusively about "black-hole thermodynamics." A lot of support to such ideas is given by the fact that the first law of thermodynamics can be formulated reasonably generally by using only quasi-local concepts [21–23] (without assumptions such as asymptotic flatness). For example, Padmanabhan [22] had shown that Einstein equations when evaluated at a horizon in a static spherically symmetric spacetime reduce to a thermodynamical identity. (This can be generalized to stationary axisymmetric spacetimes [21].)

The interesting question to be answered was as follows: How does the classical general relativity know about the thermodynamics of spacetime? The interesting answer to this question was suggested in [23]. The general theory of relativity could be just a continuum statistical description of fundamentally different degrees of freedom. Just like sound waves in a medium statistically relate (in a thermodynamic equilibrium) to the atoms and molecules, those being the real degrees of freedom on the more fundamental level. Taking such a viewpoint, it is not so surprising that a general relativity inherently contains some information about parameters related to the equation of state of those degrees of freedom.

1.2 Bekenstein's Semiclassical Quantization of Horizon Entropy

One can try to "dig out" even more subtle information about the thermodynamical variables. This relates to the third important direction in which the early ideas evolved –Bekenstein's ideas about the semiclassical quantization of the black hole horizon area (and, consequently, black hole entropy). Bekenstein in his early work [5] noticed that Christodulou's reversible process, describing a particle being radially dropped under the horizon, such that the particle's classical turning point is put infinitesimally close to the horizon, leads to a conclusion that a black hole horizon area behaves as a classical adiabatic invariant. One can then use the Ehrenfest principle from the early days of quantum mechanics. The Ehrenfest principle states that the classical adiabatic invariants should have semiclassically discrete quantum spectra, and therefore, Bekenstein proposed that the black hole horizon area have a semiclassically discrete spectrum.

[1] Let us also mention here that the generalization of the concept of black hole entropy to arbitrary gravity theory given by a diffeomorphism invariant Lagrangian was given by Wald [25].

Furthermore, by considering a quantum modification of the classical reversible Christodulou process, Bekenstein concluded that there is, in fact, a nonzero lower bound on the horizon area increase. This means that placing the particle's classical turning point infinitesimally close to the horizon contradicts the quantum principle of uncertainty, and the reversible processes are not allowed. Bekenstein obtained the lower bound for the horizon area increase in Planck units as $\Delta A_{min} = 8\pi$. The universality of this bound (independence on the black hole parameters) led Bekenstein [5] to the conclusion that the preferred option is the *equispaced* area spectrum. The quantum black hole horizon area spectrum can be simply conjectured from the following: (1) the existence of a universal, black-hole parameter independent lower bound on the area change, such that the minimum area change can be in certain limiting processes always approximated, and (2) the spectrum is reasonably simple, and therefore, the area quanta in the spectrum do not form an irregular or oscillatory sequence (sequence of numbers approximately around the value of the bound). Considering these two requirements on the area spectrum and Bekenstein's bound, one uniquely obtains Bekenstein's [5, 26] semiclassically equispaced spectrum given in Planck units as

$$A_n = 8\pi\gamma \cdot n, \qquad \gamma \in O(1). \tag{1}$$

(The uncertainty in the value of γ represents some uncertainties in the derivation of the lower bound.)

The suggestion given by (1) means that one obtains (also) the entropy of the black hole horizon semiclassically quantized and equispaced, but claiming that such an equispaced spectrum of entropy should be trusted beyond the semiclassical approximation would have significant consequences; the statistical interpretation of entropy (logarithm of the microscopic degeneracy of a macroscopic state) means the quantum of entropy must be of the form $\Delta S = \ln(k), \quad k \in \mathbb{N}_+$ [27]. On the other hand, over the years, different black-hole quantization schemes appeared (see for example [28–32]), most of them in favor of the equispaced area/entropy spectrum, but largely agreeing on fixing the γ value in (1) as $\gamma = 1$. (This would mean that the bound was fixed in some precise sense.) Such results might indicate that one should be probably less ambitious and keep the suggestions for the equispaced entropy spectra only within the realms of the semiclassical approximation [6].

One could then naturally ask if the Bekenstein semiclassical quantization scheme is only a property of the black hole horizons or a more general property of the spacetime horizons, and indeed, even the Bekenstein semiclassical entropy quantization law is suggested to apply generally to spacetime horizons [7]. It had been shown in [7] that for the general Lanczos-Lovelock gravity theory one can impose certain reasonable physical assumptions and obtain an equispaced semiclassical entropy spectrum for general spacetime horizons. The spectrum is, for every horizon, of what seems to be the most popular form $S_n = 2\pi \cdot n$ (in Planck units). Outside the realm of general relativity, equispaced entropy spectrum does not necessarily give the equispaced horizon area spectra, but considering only Einstein's gravity, the quantization of the spacetime horizon area becomes $A_n = 8\pi \cdot n$.

The reasoning (for more details, see [7]) goes as follows: Any effective action of the observer constrained to the accessible region must depend only on the information accessible to that observer. Then, consider any kind of spacetime horizon (horizon, as perceived by the observers, linked to the coordinates singular at the given horizon). The requirement that the observer constrained by such a horizon must depend only on the information accessible to the observer, leads in the semiclassical WKB approximation to the fact that an effective action of such an observer contains a surface term, which, when evaluated on the horizon, encodes all the information hidden to the observer. Such a boundary term is not generally covariant and should not have any effect on the quantum effects described by the observer. For the boundary term to fulfill this condition, it has to be semiclassically quantized as

$$A_{sur} = 2\pi n, \qquad n \in \mathbb{Z}.$$

In Einstein's theory, this term is just the entropy of the horizon and therefore gives the uniform quantization of entropy with the quantum $\Delta S = 2\pi$. (However, this entropy spectrum turns out to be general within the Lanczos-Lovelock gravity theories [7].)

1.3 The Conjectured Connection to the Asymptotic Quasi-Normal Modes

Additional evidence for the equispaced horizon area/entropy spectra appeared around 15 years ago in the works of Bekenstein [33] and Hod [9]. Take some specific field (scalar, electromagnetic, gravitational, etc.) and let it scatter (for example) on the Schwarzschild black hole spacetime. The scattering amplitude has a discrete, infinite number of nonreal complex poles, the quasi-normal frequencies, such that are labeled by the wave mode number ℓ and an overtone number n. (The frequencies are most conveniently computed by decomposing the perturbation in the tensor spherical harmonics.) Some combination of the frequencies dominates the time evolution of an arbitrary perturbation (coming from compactly supported initial data) within a characteristic time scale. This is the reason why the quasi-normal frequencies are often called the "ring-down" frequencies or "the characteristic sound of black holes." (For more details of what are the quasi-normal modes, see for example the reviews [34–36].)

If one fixes ℓ, the quasi-normal frequencies still form an infinite discrete set, with an unbounded imaginary part (they describe arbitrarily highly damped oscillations). Let us focus on the asymptotic sequence of the frequencies with the damping going to infinity (this is what we further call *asymptotic* frequencies). These frequencies show for the Schwarzschild black hole, perturbation with spin s, and the wave mode number ℓ the following behaviour [36–38]:

$$\omega_{s,n,\ell} = (\text{offset})_s + i \cdot 2\pi T \cdot n + O\left(n^{-1/2}\right). \quad (2)$$

By T, we mean here the Hawking temperature of the Schwarzschild black hole horizon. This is a remarkable universal feature. The spacing of the asymptotic sequence of the modes is completely independent of the type of perturbation or the wave mode number. (Note that this is, in fact, a general feature of spherically symmetric, static asymptotically flat spacetimes, not even constrained by the general relativity [39, 40].) Furthermore, the offset is some complex number, such that it is also independent of the wave-mode number and depends only on the spin of the perturbation. The fact that the spacing between the asymptotic frequencies is universally converging to a constant given as $2\pi T$ suggests the link of the frequencies to the Euclidean gravity [37]. (The Euclidean metric is periodic with the period given by the inverse temperature.)

It was suggested in the works of Bekenstein [33] and Hod [9] that by using Bohr's correspondence principle: "the transition frequencies at high quantum numbers equate to the classical oscillation frequencies," the asymptotically highly damped modes could link to the quantum black hole properties (in the semiclassical regime). In particular, Bohr's correspondence principle links the asymptotic quasi-normal frequencies to the quanta of energies emitted in the black hole state transitions. (We give a more detailed discussion later in the paper in the Section 4.1.) However, the original Hod's conjecture [9] was later refined by a suggestion of Maggiore [6]. Maggiore proposed [6] to treat the asymptotic quasi-normal frequencies as a collection of damped oscillators with proper frequencies given as (let us further use the notation $\omega = \omega_R + i\omega_I$)

$$\sqrt{\omega_{nR}^2 + \omega_{nI}^2}.$$

(As mentioned before, we give a more detailed reasoning why this connection might be relevant in the Section 4.1. For some further discussion, we recommend also the original papers [6, 9]. Also for some alternative, but complementary viewpoint about the connection of the asymptotic modes to the entropy/area spectra, see [41].) As we will see, Maggiore's link of the hypothetical quanta of energy emitted or absorbed in a black-hole state transition to the asymptotic quasi-normal modes reads as

$$\Delta M = \lim_{n\to\infty} \Delta_{(n,n-1)}\sqrt{\omega_{nR}^2 + \omega_{nI}^2}.$$

For the Schwarzschild black hole, this turns into

$$\begin{aligned}\Delta M &= \lim_{n\to\infty} \Delta_{(n,n-1)}\sqrt{\omega_{nR}^2 + \omega_{nI}^2} \\ &= \lim_{n\to\infty} \Delta_{(n,n-1)}\omega_{nI} = 2\pi T.\end{aligned} \quad (3)$$

Take the first law of black-hole mechanics and consider Bekenstein's suggestion for the quantum of entropy, $\Delta S = 2\pi\gamma$. Then, the mass quantum corresponding to the entropy change is given as follows:

$$T\Delta S = 2\pi\gamma \cdot T = \Delta M. \quad (4)$$

Equation (4) together with Maggiore's conjecture[2] suggests that the asymptotic spacing between the quasi-normal frequencies of the Schwarzschild black hole should be given as $2\pi\gamma T$. This is, as we have seen, precisely the case, if one fixes γ in the most popular way, $\gamma = 1$. Hence, Maggiore's suggestion gives the same universal result for the area quantum as Bekenstein's lower bound on the area transition. This can be seen as a strong additional support for the black hole area spectrum of the form $8\pi \cdot n$ (entropy quantization $2\pi \cdot n$). At the same time, it explains the "$2\pi T$" universality in the spacing of the asymptotic frequencies. (It matches those two seemingly unrelated results.)

1.4 The Results Presented in This Paper

In this paper, we are particularly interested in the spacetimes with multiple horizons, but we keep things sufficiently simple and will deal only with the spherically symmetric static spacetimes. This work is supposed to result in support of three basic ideas that were presented in this introductory section: (1) the entropy spectra of the spacetime horizons are all quantized with a quantum given as $\Delta S = 2\pi$; (2) the asymptotic quasi-normal modes play the role in determining the spectra, as suggested by Maggiore; and (3) most importantly, there is no fundamental difference between the spacetime horizons in terms of their thermodynamical properties. The first statement just confirms the result of [7]. We further discuss some of the consequences of such a universal entropy quantization. The second point means that we extend the use of Maggiore's conjecture to the multi-horizon case (in an analogy to the Schwarzschild case), giving an additional support for the conjecture. Via Maggiore's conjecture, we explain the observed behavior of the

[2]This can be called a modified Hod's conjecture, but we will keep it for simplicity under the name "Maggiore's conjecture."

quasi-normal frequencies in the multi-horizon cases, and at the same time, we argue in favor of the entropy spectra proposed by [7]. The third statement is a very natural result, strongly suggested by the work of different authors [21, 23]. The fact that certain thermodynamical ideas appeared first in the context of the black-hole horizons is therefore more of a coincidence, rather than a consequence of fundamental properties.

2 The Thermodynamical Concepts in Static Spherically Symmetric Spacetimes

In this section, we will, for the convenience of the reader, briefly introduce the basic equations and results that are of key importance for our work. (The results are standard and can be found over the literature. The reader who is well familiar with the concepts can skip most of the details.)

Take a specific type of static spherically symmetric line element in the fixed "suitable" coordinates

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{5}$$

in some region where $f(r) > 0$. (Assume also that $f(r)$ is a well-behaved smooth function in this region.) Let us take in this region a space-like hypersurface of the fixed time $\Sigma_t \doteq (r, \theta, \phi)$ and consider the Killing field given as $K \doteq \partial_t$. The Killing field fulfills the following relation:

$$K^{a;b}_{\ \ ;b} = R^{ab} K_b. \tag{6}$$

Then, by applying Stoke's theorem, one obtains

$$\int_{\Sigma_t} K^{a;b}_{\ \ ;b} d\Sigma_a = \int_{\partial\Sigma_t} K^{a;b} d\Sigma_{ab}. \tag{7}$$

One can use then the Einstein equations to conclude that:

$$\begin{aligned}\int_{\partial\Sigma_t} K^{a;b} d\Sigma_{ab} &= \int_{\Sigma_t} R^{ab} K_b d\Sigma_a \\ &= 8\pi \int_{\Sigma_t} \left(T^{ab} - \frac{1}{2} T g^{ab}\right) K_b d\Sigma_a \\ &\quad + \Lambda \int_{\Sigma_t} g^{ab} K_b d\Sigma_a.\end{aligned} \tag{8}$$

Choose the boundary $\partial\Sigma_t$ to be at $r = R_1, R_2$ with $R_1 < R_2$. Then, from equation (8), we can deduce

$$\begin{aligned}-4\pi\tilde{M}_1 + 4\pi\tilde{M}_2 &= 8\pi \int_{\Sigma_t} \left(T^{ab} - \frac{1}{2} T g^{ab}\right) K_b d\Sigma_a \\ &\quad + \Lambda \int_{\Sigma_t} g^{ab} K_b d\Sigma_a.\end{aligned} \tag{9}$$

Here, $\tilde{M}_1$ and $\tilde{M}_2$ represent masses of the regions inside the chosen boundaries. These masses correspond to a quasi-local concept of mass in general relativity, also known as Komar mass. (More generally, one speaks about Komar's conserved quantities.)

Write equation (9) explicitly with the stationary spherically symmetric electromagnetic field

$$\begin{aligned}\tilde{M}_2 &= 2\int_{\Sigma_t} \left(T_M^{ab} - \frac{1}{2} T_M g^{ab}\right) K_b d\Sigma_a \\ &\quad + 2\int_{\Sigma_t} T_{EM}^{ab} K_b d\Sigma_a + 2 \cdot \frac{\Lambda}{8\pi} \int_{\Sigma_t} g^{ab} K_b d\Sigma_a + \tilde{M}_1.\end{aligned} \tag{10}$$

Here, T_M^{ab} is some matter field stress energy tensor, and T_{EM}^{ab} is the electromagnetic stress energy tensor. (The trace of the electromagnetic stress tensor is $T_{EM} = 0$.) We clearly see that the explicitly written integrals on the right side of equation (10) represent the contribution to the mass from the region between the boundaries.

To have more insight into equation (10), let us partly confirm it by calculating the relevant integrals on both sides. The integral with the electromagnetic stress energy tensor can be performed explicitly $\left(T_{EM}^{tt} = \frac{1}{8\pi}\frac{Q^2}{r^4} f^{-1}\right)$ as

$$\begin{aligned}\tilde{M}_{EM} &= 2\int_{\Sigma_t} T_{EM}^{ab} K_b d\Sigma_a = \int_{R_1}^{R_2} \frac{Q^2}{r^4} f^{-3/2} \cdot f^{3/2} \cdot r^2 dr \\ &= \int_{R_1}^{R_2} \frac{Q^2}{r^2} dr = -Q^2 \left[\frac{1}{R_2} - \frac{1}{R_1}\right].\end{aligned}$$

The second integral on the right side can be also easily computed as

$$\begin{aligned}\tilde{M}_\Lambda &= (4\pi)^{-1} \Lambda \int_{\Sigma_t} g^{ab} K_b d\Sigma_a \\ &= -(4\pi)^{-1} \Lambda \int_{\Sigma_t} f^{1/2} \cdot f^{-1/2} r^2 \sin(\theta) \cdot dr d\theta d\phi \\ &= -(4\pi)^{-1} \Lambda \frac{4\pi}{3} \left(R_2^3 - R_1^3\right) = -\frac{\Lambda}{3} \left(R_2^3 - R_1^3\right) \\ &= -2 \cdot \frac{\Lambda}{8\pi} \cdot [V(R_2) - V(R_1)].\end{aligned} \tag{11}$$

Here, $V(R) = \frac{4\pi}{3} R^3$ is a flat space volume contained in a sphere with radius R, and $-\Lambda/8\pi$ is the energy density of the cosmological constant term. The mass $\tilde{M}$ of a region inside the boundary $\partial\Sigma_t : r = R$ can be calculated as

$$\begin{aligned}\tilde{M} &= (4\pi)^{-1} \int_{\partial\Sigma_t} \Gamma^t_{rt} \cdot f(R) \cdot R^2 dS \\ &= (4\pi)^{-1} \int_{\partial\Sigma_t} \frac{1}{2} f^{-1}(R) \cdot f_{,r} \cdot f(R) \cdot R^2 dS = \frac{f_{,r}}{2} R^2.\end{aligned} \tag{12}$$

Consider the Unruh radiation observed by the accelerated observers. The four-acceleration of the Unruh stationary "floating" observers (world lines $r =$ const.) is

$$a^b = u^c u^b_{;c} \quad \rightarrow \quad a^{(t,r,\theta,\phi)} = \left(0, \frac{f_{,r}}{2}, 0, 0\right).$$

This means that the mass of the given boundary is equal to

$$\tilde{M} = \pm f^{1/2}|a|R^2 = \pm T_a \frac{A}{2}$$

where $T_a = f^{1/2}|a|/2\pi$ is the Davies-Unruh temperature, a red-shifted temperature measured by the Unruh observer floating at the boundary (moving along the world line $r = R$), and A is area of the spherical boundary. The $\pm$ sign depends on the sign of the $f_{,r}$ function. In the limiting case of the boundary being a horizon, one obtains from T_a the Hawking temperature T_H, and this is a temperature of the radiation measured by a stationary observer with the zero acceleration (moving along the geodesics). (In case of asymptotically flat spacetime $\Lambda = 0$, this is a stationary observer at the infinity, and in case of $\Lambda > 0$, the position of such a floating observer is at a finite radius given by $f_{,r}(R) = 0$.)

Let us consider that locally around the boundary, there is the Reissner-Nordström-deSitter (R-N-dS) geometry. In this case, the metric function f reads

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2.$$

Then, one obtains the following result:

$$\tilde{M} = M - \frac{Q^2}{R} - \frac{\Lambda}{3}R^3 \tag{13}$$

where the special cases of Schwarzschild, deSitter, Schwarzschild-deSitter (SdS) and Reissner-Nordström (R-N) spacetimes are obtained simply by setting $M = 0$, $\Lambda = 0$, and $Q = 0$ as appropriate. Also, let us derive for the Reissner-Nordström-deSitter geometry and a horizon with the area A

$$\begin{aligned} \delta A &= 8\pi \cdot R\delta R \\ &= -8\pi \cdot R\left[\frac{f_{,M}}{f_{,r}}\delta M + \frac{f_{,\Lambda}}{f_{,r}}\delta\Lambda + \frac{f_{,Q}}{f_{,r}}\delta Q\right] \\ &\rightarrow \pm T\delta S = \delta M - \frac{R^3}{6}\delta\Lambda + \frac{Q}{R}\delta Q. \end{aligned} \tag{14}$$

(Here, we used the fact that for a boundary, such that it is a spacetime horizon, we can relate the area of the boundary to the concept of entropy.)

At the end of this section, let us mention one key result. By varying (9) and taking the boundaries to be the black hole and the cosmological horizon (for a spacetime where the two horizons are present), one gets the quasi-local form of the first law of thermodynamics as

$$\int_{\Sigma_t} \delta T_{ab} K^a d\Sigma^b + T_{BH}\delta S_{BH} = -T_{CH}\delta S_{CH}. \tag{15}$$

Here, index "BH" means the black hole horizon, and "CH" means the cosmological horizon. The gauge chosen to compare the solutions is such that $\delta K^a = 0$.

3 The Generalization of the Bekenstein's Lower Bound on the Horizon Area Transition

Let us note that a Killing energy absorbed or emitted by the horizons can be (in a non-trivial sense) arbitrarily small, which leads to the conclusion that the horizon area behaves in certain cases as a classical adiabatic invariant. To clarify this, let us show an example. A neutral point particle can be described by the following stress energy tensor:

$$T^{ab} = m \cdot \delta\{r - R(t)\} \cdot (u^0)^{-1} u^a u^b.$$

(m is the rest mass of the particle.) Integrating such a distribution of energy along a hypersurface Σ_t leads to the following Killing energy:

$$E = m\sqrt{f(R_p)}$$

where R_p is a classical turning point, at which the point particle's four-velocity is orthogonal to Σ_t. The closer the turning point lies to the horizon, the smaller is the Killing energy by which the particle contributes to the area change of the horizon (as can be seen from equation (15)). As a result, one can regard the horizon area as a classical adiabatic invariant.

This argument was originally put forth for the area of the Kerr black hole horizon [5], but as we can easily see, it holds much more generally. As an example, we can take pure deSitter spacetime and drop a particle from a turning point infinitesimally close to the cosmological horizon. Then, also the cosmological horizon area would behave as a classical adiabatic invariant. Bekenstein used the classical adiabatic invariance and the Ehrenfest principle to deduce that the black-hole horizon area must have semiclassically discrete quantum spectrum, but then, it is reasonable to suggest that discrete spectra should be a general property of spacetime horizons.

Furthermore, as already mentioned, Bekenstein argued that going beyond the classical physics and employing the quantum principle of uncertainty modifies the classical results. This is because to keep the horizon's area unchanged, one would need to know in the same time the exact particle's momentum and position (to place the turning point arbitrarily close to the horizon). If one attributes to a particle with the rest mass m, a nonzero proper radius b, the minimal area change is for the Kerr black hole given as $\delta A = 8\pi mb$. Using this equation, one can reason as follows [5]: the proper radius has to be larger as the reduced Compton wavelength of the particle or its Schwarzschild radius, whichever is larger. The Compton wavelength is larger for $m < 2^{-1/2}$ in Planck units, and the Schwarzschild radius is larger for $m > 2^{-1/2}$ in Planck units. For the case of a larger reduced Compton wavelength, it holds that $b > \lambda$, and the reduced Compton wavelength relates to the rest mass as $m = \lambda^{-1}$. This means that the following must be hold:

$$m \cdot b \geq 1 \qquad \rightarrow \qquad \delta A \geq 8\pi$$

(in Planck units). For the case of a larger Schwarzschild radius, the following relation holds in Planck units:

$$\delta A \geq 8\pi \cdot mb \geq 16\pi m^2 \geq 8\pi.$$

This means that one obtains a lower bound on the Kerr black hole horizon area transition, in Planck units, as $\delta A \geq 8\pi$. As stated in Section 1, if the horizon area spectrum is discrete and there is a lower bound on the area quanta, the most natural spectrum to impose is an equispaced area spectrum $8\pi\gamma \cdot n$ with $\gamma \in O(1)$.

What we will show is that the same line of reasoning that Bekenstein originally used for the Kerr black hole horizon can be repeated for other spacetime horizons. By using the Bekenstein arguments, we arrive at the same lower bound for the area change for all the horizons of the geometry given by the line element (5). This suggests that also a lower bound on the area transition is a *fairly general* result. The universality of the lower bound was Bekenstein's argument for an equispaced area spectrum with the quantum being approximately given by the value of the bound. One can then use the same argument to support the suggestion by [7], to generally impose the same (and equispaced) entropy spectra for all the spacetime horizons. Let us first explicitly calculate the area transition lower bounds for a couple of examples and then sketch a general proof for the horizons of the line element (5).

3.1 deSitter Spacetime

One simple example is the pure deSitter spacetime (with $f(r) = 1 - \Lambda r^2/3$). Let us repeat the reasoning of Bekenstein. It is easy to observe that the area change is minimalized when the particle falls below the horizon radially. Therefore, let us drop the particle from its classical turning point slightly under the cosmological horizon. δ is the radial position of the center of mass of the particle (with the rest mass m), and the center of mass is supposed to follow the geodesics of the deSitter spacetime. The proper radius b of the particle is related to δ through the relation (R is the radial position of the deSitter cosmological horizon)

$$b = \int_{R-\delta}^{R} \sqrt{g_{rr}}\, dr = \int_{R-\delta}^{R} f^{-1/2}(r)\, dr. \tag{16}$$

This turns into the result

$$\delta = \sqrt{\frac{3}{\Lambda}} \left[1 - \cos\left(\sqrt{\frac{\Lambda}{3}} \cdot b \right) \right].$$

The Killing energy corresponding to a minimal area change is then

$$E = m \cdot \sqrt{1 - \frac{\Lambda}{3}\{R - \delta(b)\}^2} = m \cdot \sin\left(\sqrt{\frac{\Lambda}{3}} \cdot b \right).$$

Taking this relation up to the linear order in b turns into

$$E = m \cdot b \cdot \sqrt{\frac{\Lambda}{3}},$$

and in terms of horizon area transition, this gives

$$\frac{1}{4} \cdot T \cdot \delta A_{min} = \frac{1}{8\pi} \sqrt{\frac{\Lambda}{3}} \cdot \delta A_{min} = E \quad \rightarrow \quad \delta A_{min} = 8\pi \cdot m \cdot b.$$

Repeating the argumentation from the previous paragraph, one obtains precisely the same lower bound as Bekenstein for the Kerr black hole horizon area transition.

3.2 The Inner and the Outer Horizons of the Reissner-Nordström Spacetime

Let us further take the maximally analytically extended Reissner-Nordström geometry and consider both horizons (black hole and the inner Cauchy horizon). At this stage, let us skip the discussion about the physical meaning of such geometry (especially the discussion about instability of the inner, Cauchy horizon). We believe that *as a matter of principle*, this geometry still provides an important and interesting example, and it is not necessary to take physically seriously the model of the black hole interior (and even exterior) provided by the maximally analytically extended R-N spacetime. In other words, the principal issues that we are investigating are believed to survive for the physically relevant situations.

Consider dropping an *uncharged* particle from above the R-N black hole horizon or from below the white hole inner horizon in the maximally extended geometry. (Conveniently forget that the particle passing through the inner horizon could eventually destroy the geometry.) Consider the upper horizon to be at the radial coordinate r_+ and the inner horizon at the radial coordinate r_-. (The $\pm$ signs in the notation mean that the relevant quantities relate to the black hole outer/inner horizon case.) Then again, by relating the proper particle radius b and the coordinate radius $\delta_\pm$ through the integral (16), one obtains (in the linear order in $\sqrt{\delta_\pm}$)

$$b = \frac{2 \cdot r_\pm}{\sqrt{r_+ - r_-}} \sqrt{\delta_\pm}.$$

Then, the Killing energy becomes (in the same order $\sqrt{\delta}$)

$$E_\pm = \frac{m\sqrt{\delta_\pm}\sqrt{r_+ - r_-}}{r_\pm} = \frac{mb \cdot (r_+ - r_-)}{2 \cdot r_\pm^2}.$$

The area change corresponds to ($\kappa_\pm$ are surface gravities of the two horizons)

$$\begin{aligned}\pm\frac{1}{8\pi}\cdot\kappa_\pm\delta A_{(min)\pm} &= \frac{1}{8\pi}\frac{r_+-r_-}{2\cdot r_\pm^2}\delta A_{(min)\pm}\\ &= E_\pm \quad\rightarrow\quad \delta A_{(min)+}\\ &= \delta A_{(min)-} = 8\pi\cdot mb.\end{aligned}$$

Again, by proceeding as before, this gives the same lower bounds $\delta A_{(min)\pm} = 8\pi$, for both the black hole and the inner Cauchy horizons.

3.3 The Rindler Horizon and a Sketch of a General Proof

Let us take the Rindler line element in a suitable coordinates. It can be expressed as [21]

$$ds^2 = -2\kappa x\cdot dt^2 + (2\kappa x)^{-1}dx^2 + dL_\perp^2. \tag{17}$$

(The κ parameter is defined to be the surface gravity of the horizon, $\kappa = [(2\kappa x)_{,x}/2]_{x=0}$.) Then, let us place the particle's center of mass at a coordinate $x = \delta$ and drop it in the x direction below the horizon. The following holds:

$$b = \int_0^\delta \frac{1}{\sqrt{2\kappa x}}dx = \sqrt{\frac{2}{\kappa}}\cdot\sqrt{\delta}.$$

Then, the suitable Killing energy associated to the boost isometry is

$$E = m\sqrt{2\kappa\delta} = \kappa\cdot mb,$$

and the minimum area change reads

$$\frac{\kappa}{8\pi}\cdot\delta A_{min} = E \qquad\rightarrow\qquad \delta A_{min} = 8\pi\cdot mb. \tag{18}$$

Again, by repeating the same analysis as before, we arrive at the lower bound for the horizon area change $\delta A_{min} = 8\pi$. Note that the Rindler line element (17) *near the horizon* generally approximates the line element (5) (by using the expansion $f(r)\approx f_{,r}(R)\cdot[r-R] = 2\kappa\cdot[r-R]$). The proof for Rindler spacetime can therefore count as a general proof for horizons of the metric with the line element (5).

The lower bound on the area transition turned up to be *the same* and *completely independent* on the spacetime parameters, irrelevant on whether it was the inner, the outer black hole, or the cosmological horizon. The horizon area transition bound therefore seems to be even more universal as suggested by Bekenstein; it is independent both on the spacetime parameters and the particular horizon. If we accept (for each of the horizons) that the horizon area semiclassical spectrum is equispaced, it is natural to relate it to the lower bounds in Planck units as

$$A_{in} = 8\pi\gamma_i\cdot n, \qquad \gamma_i\in O(1) \tag{19}$$

where i labels the different horizons. It seems to be also natural to claim that γ_i is the same for all the horizons. The entropy quantization suggested by [7] gives (for general relativity) precisely the spectrum (19) with the universal choice $\gamma_i = 1$, which certainly fits the bill.

4 Maggiore's Conjecture and the Information from the Asymptotic Quasinormal Modes

4.1 Some Reasons Supporting the Conjectured Connection Between the Spacetime Thermodynamics and the Asymptotic Quasi-Normal Modes

Let us discuss some basic reasons to suspect a connection between the asymptotic quasi-normal modes and the thermodynamics of horizons. We will discuss it in a more generalized setting as is done in the original papers [6, 9], where the discussion is related to the Schwarzschild spacetime.

The dynamics of fields near horizons of the line element (5) can be generally approximated by a free field dynamics. This can be trivially shown for the simplest example of a massless scalar field, where the dynamics is given by

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Psi) = 0. \tag{20}$$

By decomposing the scalar field through multipoles $Y_{\ell m}$ as

$$\Psi = \sum_{\ell,m}\frac{\psi_{\ell m}}{r}Y_{\ell m},$$

and turning into the tortoise coordinate

$$x = \int^r\frac{dr'}{f(r')},$$

equation (20) turns into

$$-\frac{\partial^2\psi_{\ell m}}{\partial t^2} + \frac{\partial^2\psi_{\ell m}}{\partial x^2} - V(\ell,x)\psi_{\ell m}(x) = 0, \tag{21}$$

with the potential

$$V\{\ell, r(x)\} = f(r)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2f_{,r}(r)}{r}\right].$$

We see that near the horizons where from the definition $f(r)\approx 0$, the potential becomes $V(r)\approx 0$, and the field behaves as a free field. The perturbation equations for more complicated perturbative fields is more difficult to derive, but eventually, all of the fields can be reduced to degrees of freedom following the dynamics given by equation (21), with some suitable potential $V(x)$. Moreover, the potential is, in every case, vanishing as one approaches the horizon. (Therefore, the scalar field is, as in many situations, a good simple model for physically much more realistic fields.)

Now let us consider that with some approximation (the quasi-normal modes usually do not form a complete system

[34]), one can decompose the field into discrete quasi-normal modes. Considering this, the situation near the horizon resembles a classical emission by a system that can oscillate around some configuration only with certain discrete frequencies (such as a string). Furthermore, the oscillations decay due to the radiation of energy. Let us take this picture seriously; it allows us to associate with the horizon a set of discrete frequencies related to the quasi-normal modes of the field.

Now let us follow the Bohr correspondence principle: "In the limit of high quantum numbers the transition frequencies are equal to the classical oscillation frequencies." What was said in the previous paragraph suggests that one can attempt to identify the horizon transition frequency in the semiclassical limit with the quasi-normal modes, but we have a discrete infinity of quasi-normal modes, so which quasi-normal modes shall we pick? There is no information about the underlying quantum theory that could provide guidance, but there is one basic observation from the known physics: the transition between the systems' quantum levels in the semiclassical limit shall have relaxation times close to zero. Therefore, it is reasonable to look for the "correct" frequencies in the limit of the high damping. Remarkably enough, the behavior of the frequencies in this limit is independent on the wave mode number ℓ. Hod [9] suggested to associate the classical oscillation frequency simply with the real part of the quasi-normal modes as

$$\lim_{n\to\infty} \omega_{Rn} \doteq \omega_{R\infty}.$$

In particular Hod considered quasi-normal modes of gravitational perturbations and since for such perturbations, holds

$$\lim_{n\to\infty} \omega_{Rn} = T\ln(3);$$

Hod [9] obtained the Bekenstein entropy quantization of the "statistical" form [27]

$$S = \ln(3)n.$$

However, over the period of time, certain arguments were raised against Hod's reasoning [6, 36]. One very important objection is that the $\omega_{R\infty}$ quantity is not universal even for the Schwarzschild spacetime but depends on the spin of the perturbation. Even worse, when turning to a more complicated spacetime, like Reissner-Nordström, Schwarzschild-deSitter, Reissner-Nordström-deSitter or to nonspherically symmetric ones (like Kerr, Kerr-Newman, Kerr-Newman-deSitter), there is no indication that the quantity $\omega_{R\infty}$ is reasonably defined even for a specific field. (The formulas for the quasi-normal frequencies for more general spherically symmetric spacetimes are derived in [37, 38, 42–44]. These formulas seem to suggest that the $\omega_{R\infty} = T\ln(k),\ k \in \mathbb{N}_+$ result is, in fact, a fortuitous coincidence of the fact the spacetime has a single horizon.)

However, Maggiore [6] made a very interesting observation, which can be shown to resolve all the main objections against Hod's conjecture; since the quasi-normal oscillations are damped oscillations, it is not reasonable to identify the classical oscillation frequencies with the real part of the quasi-normal frequency, but rather with the proper frequency of the damped oscillator, given as

$$\omega_{0n} = \sqrt{\omega_{Rn}^2 + \omega_{In}^2}. \tag{22}$$

Let us further give our account on what is happening. One can preserve the idea that the only relevant perturbations are those that represent transitions with very low relaxation times, and these are the perturbations composed (with some approximation) only from highly damped quasi-normal modes. Then, via Bohr's principle, it is natural to see the sequence of frequencies (22) to describe the transition frequencies from some background state with quantum number N to a state with $N+n$. The frequency corresponding to a transition from $n \to n-1$ is then naturally

$$\Delta_{(n,n-1)}\omega_{0n}. \tag{23}$$

Since we deal with the highly damped frequencies, it is reasonable to take the expression (23) with a limit $n \to \infty$. This limit should give a sensible answer even if one considers that the transitions between N and $N+n$ are upper bounded by the fact that we are considering only "small" perturbations around background spacetime with some particular values of parameters. (Indeed, one can see from the numerical results that (23) is very well approximated by its value at $n \to \infty$ long time before one would expect that the transition is outside the scope of the linearized theory.) However, it should be stressed that at no point one can be self-confident enough to claim that this is more than a suggestive way of thinking. The perturbation cannot be decomposed in quasi-normal modes exactly, and many subtle points in the logic could become potentially fatal. This is the reason why the upper statement, despite being useful and interesting, is still only a *conjecture*. On the other hand, as we will see, the conjecture is very fruitful and has a potential to explain a great universality in the results obtained over the years.

There is another evidence that supports the connection between the quasi-normal modes and the horizon thermodynamics as suggested by Maggiore. It can be reasonably generally observed that the structure of the spacing in the imaginary part of the asymptotic frequencies depends only on the tails of the scattering potential, in other words, only on the characteristics of the horizons. Adding a matter field between the horizons, such that modifies the geometry between the horizons but keeps the structure of the horizons unchanged, does not affect the structure of the imaginary part of the asymptotic frequencies. (Of course the field whose quasi-normal frequencies we are computing must interact with the field that was added only via gravity.) (All this can be explicitly seen in the results derived in the papers [39, 40, 44], and one does not have to even worry about the general relativity theory.)

As another indirect support for the conjecture, one can see the results from the paper written by me and my collaborators [2]. (See also [4].) We have tried to answer the question of what happens with the asymptotic frequencies, if a generic spherically symmetric static asymptotically flat spacetime does not have an (absolute) horizon. The answer we found was that in such case, there will be *no* asymptotic quasi-normal modes. (For a fixed ℓ, there will exist a bound on the frequencies.) This means that there seems to be some type of general connection between the existence and properties of the asymptotic quasi-normal modes and the existence and properties of (absolute) horizons as could be assumed from the Maggiore's (or Hod's) conjecture. (Note that the nonexistence of the asymptotic frequencies was shown, assuming spherical symmetry and staticity, but if one believes that the zero angular momentum limit is for the most fundamental characteristics of the frequencies non-singular, then one can see it as an evidence that such a result survives for a more general, axially symmetric spacetimes.)

Hod's/Maggiore's conjecture, in fact, suggests that the classical general relativity matches the classical limit of some underlying quantum theory in a more "sensitive" way than that is usual in the standard examples of classical theories. The classical description becomes better and better approximation as the black hole grows, and the energy quanta emitted by the black hole go to zero more and more approximately as $\sim T$. Instead of approximating this picture by a continuum (assuming that Maggiore's conjecture is correct), general relativity matches the sequence of quanta already before reaching zero. Therefore, the information that is carried by the general relativity about thermodynamics is, in such case, deeper than expected, since the general relativity would carry some information about the discrete structure that appears under the continuous spacetime.

4.2 The Behavior of the Asymptotic Frequencies for the Spherically Symmetric Multi-Horizon Spacetimes

Here we will consider three spherically symmetric spacetimes: Reissner-Nordström-deSitter (R-N-dS), Schwarzschild-deSitter (S-dS), and Reissner-Nordström (R-N) spacetimes. One might wonder why we are considering these three spacetimes separately and not the latter ones as a special case of the R-N-dS spacetime. The reason is that the zero limits for M, Q, and Λ parameters are for the quasi-normal frequencies sometimes singular. (There are also reasons to believe that any Maggiore-like interpretation for the frequencies could be, in this limit, singular, as we will argue later.) The transcendental formulas for the asymptotic quasi-normal frequencies for all the three R-N, S-dS, and R-N-dS spacetimes can be written (for uncharged scalar, gravitational, and electromagnetic fields) as [38, 42, 43] follows:

$$\sum_{A=1}^{K} Z_A \exp\left(\sum_{i=1}^{N} B_{Ai} \cdot \frac{\omega}{T_i}\right) = 0. \tag{24}$$

Here, Z_A and B_{Ai} are $K \times 1$ and $K \times N$ matrices composed of integers, N is the number of horizons in the spacetime ($N = 2, 3$ in our cases), K is some integer that depends on the particular formula, and T_i are the temperatures of the horizons. (Note that we consider here also the inner, Cauchy horizon, assuming the horizon temperature from the periodicity of the Euclidean solution; hence, $T = |\kappa|/2\pi$. Here, κ is the surface gravity of the horizon.)

The formulas (24) were known for some time (since [38, 42, 43]), but it is known that they do not have analytic solutions (unlike simpler Schwarzschild case). On the other hand, despite of the fact that they do not have analytic solutions, we have shown [45, 46] that a lot of *important* analytic information can be extracted about the behavior of the solutions. (Important from the point of view of Maggiore's conjecture.)

We have shown [45] that the solutions for all the three cases (for a field with spin s) follow a simple pattern. If the ratio of all the horizon temperatures is a *rational* number, the frequencies split into a *finite* number of families labeled by a, given as

$$\omega_{asn} = (\text{offset})_{as} + in \cdot 2\pi \cdot \text{lcm}(T_1, T_2, \ldots, T_N) + O(n^{-1/2}). \tag{25}$$

Here, *lcm* means "the least common multiple" of the temperatures; hence,[3]

$$\text{lcm}(T_1, T_2, \ldots, T_N) = p_1 T_1 = p_2 T_2 = \ldots = p_N T_N,$$

[3] The formulas (24) derived in [38, 42, 43] are generally accepted (and numerically confirmed) as the exact formulas for the asymptotic sequence of the highly damped frequencies. There are also known results for the asymptotic quasi-normal frequencies of the S-dS spacetime obtained via Born approximation [40, 47] or approximation by analytically solvable potentials [48, 49]. Although, on the first sight, these results might seem to be inconsistent with the behavior described by (24), one has to realize that both the methods of Born approximation and of analytically solvable potentials allow for each frequency an error of the order of unity. Considering this, the results obtained by Born approximation and by analytically solvable potentials can, at least in some cases, describe some form of averaged behavior of the solutions of (24) and, hence, be fully consistent with the results of [40, 47–49].

where $p_1, \ldots, p_N$ are relatively prime integers. It can be also proven that *in case ratio of two of the horizon temperatures is irrational, there is no infinite periodic subsequence of the sequence of asymptotic quasi-normal frequencies.*

This behavior shows almost the same universality as in the Schwarzschild case and is fascinating: How can the frequencies depend (in case of R-N and R-N-dS spacetimes) also on the inner Cauchy horizon, which means on the horizon that is in the causally disconnected region? The rather formal sounding explanation is that the frequencies must depend on all the Λ, M, and Q parameters, and one can always transform the Λ, M, and Q variables into the T_1, T_2, and T_3 variables. (Or for R-N spacetime, the M and Q variables to T_1 and T_2 variables.) This means that the frequencies must depend also on the temperature of the Cauchy horizon, but why does one observe such a simple and universal structure, which is, in fact, symmetric under the exchange of[4] $T_i \leftrightarrow T_j$? Note that this cannot be explained by the unit analysis. A general unit analysis just tells you that the gap in the spacing of the imaginary part of the families is (for R-N-dS, for example) given as

$$T_1 \cdot F\left(\frac{T_1}{T_2}, \frac{T_2}{T_3}, n, s, \ell\right),$$

where F is an arbitrary function. (Let us mention that [50] offers some answers to the question of why there is the observed dependence of the asymptotic frequencies on the temperature of the Cauchy horizon. The authors of [50] use the Kerr-CFT correspondence, and the conclusions seem to be consistent with the conclusions we make in this paper.)

Consider (in the imaginary part) the monotonically ordered union of the families of the asymptotic frequencies. In [1], we have shown that it is reasonable to expect the limit in the spacing of the imaginary part of such a sequence *not* to exist. The limit can be shown not to exist also in case the ratio of some of the temperatures is an irrational number, but despite of this unpleasant property, there is a way of how to interpret the observed behavior of the frequencies along the lines suggested by Maggiore.

4.3 The Interpretation of the Frequencies for the Schwarzschild-deSitter (S-dS) Spacetime

Let us first [2] consider the field propagating in the maximally analytically extended S-dS spacetime. The field does not change the cosmological constant and propagates through the white hole horizon and penetrates eventually the black hole horizon. The first law of thermodynamics (15) turns into

$$T_{BH}\delta S_{BH} + T_{CH}\delta S_{CH} = 0. \tag{26}$$

(We are labeling the two horizons in the same way as before.) If both the horizons have equispaced entropy spectra of the form $S_{i,n_i} = 2\pi\gamma \cdot n_i$, then the first law of thermodynamics gives

$$\frac{T_{CH}}{T_{BH}} = -\frac{\Delta S_{BH}}{\Delta S_{CH}} = -\frac{n_1}{n_2},$$

which means that the temperatures of the two horizons must have a rational ratio. As we have seen, the quasi-normal modes represent the quanta of Killing energy, and from equation (14), we see that the energy which flowed through the horizons (sufficiently slowly, as we are in thermodynamics) corresponds to δM. Therefore, by applying Maggiore's conjecture, one obtains

$$\begin{aligned}\delta M = T_{BH}\delta S_{BH} = -T_{CH}\delta S_{CH} = T_{BH}\cdot m_{BH}\cdot 2\pi\gamma \\ = T_{CH}\cdot m_{CH}\cdot 2\pi\gamma = \lim_{n\to\infty}\Delta_{(n,n-1)}\omega_{nI}.\end{aligned} \tag{27}$$

Here, m_{BH} and m_{CH} are some integers and express the fact that the entropies can change only by the integer multiples of $2\pi\gamma$. Considering the fact that δM has to be the smallest possible quantum, such that, in the same time, fulfills the equation (27), one obtains

$$\delta M = 2\pi\gamma \cdot \mathrm{lcm}(T_{BH}, T_{CH}) = \lim_{n\to\infty}\Delta_{(n,n-1)}\omega_{nI}. \tag{28}$$

This is precisely what we observe in the spacing in each of the finite number of families, if we fix $\gamma = 1$. It means that only the equispaced families of modes should carry the physical information about the mass quanta emitted in the horizon state transition.[5] This is what we consider to be a generalization of Maggiore's original hypothesis, which was used for the Schwarzschild spacetime.

4.4 The Interpretation of the Frequencies for the Reissner-Nordström (R-N) and the Reissner-Nordström-deSitter (R-N-dS) Spacetimes

Let us further consider maximally analytically extended R-N and R-N-dS spacetimes. Furthermore, let us consider that there is an uncharged field (also Λ stays fixed) flowing from the white hole horizon towards and through the cosmological horizon of the R-N-dS spacetime (or towards the asymptotic infinity of the R-N spacetime). (Alternatively, let

[4]This is again a general feature of static spherically symmetric two horizon spacetimes[40, 44], independent on general relativity. On the other hand Maggiore's conjecture depends on the general relativity.

[5]The fact that this particular information is encoded in the families of frequencies, rather than the asymptotic sequence itself, is a necessary consequence of the fact that the frequencies are continuous functions of the spacetime parameters.

us consider the Hawking radiation coming from the black hole horizon.) We claim again that these considerations have meaning *as a matter of principle*, despite of the fact that, physically, the models are unrealistic for various different reasons: (a) there are no charged astrophysical macroscopic objects, so one cannot consider the charged geometry seriously[6]; (b) even if one considered astrophysical charges, the white hole cannot appear in a solution arising from a stellar collapse; and (c) the collapse scenario is supposed to lead to a mass inflation singularity replacing the inner horizon of the R-N (or R-N-dS) geometry[7]. Note that the instability of Cauchy horizons is a lucky consequence for those who want a general relativity to be predictable; in that case, it is necessary to exclude the possibility that time like singularities could form. (Or in other words, the strong cosmic censor conjecture holds.)

However, let us consider the previous example and take seriously (at least on some limited time scale) also the inner Cauchy horizon. Consider explicitly the R-N-dS spacetime. (It is straightforward to modify this considerations for the R-N spacetime, so we will not explicitly do it). Then, equation (26) still holds, and from (14), we see that the energy flowing through the horizons corresponds again to δM, but since $\delta\Lambda = \delta Q = 0$, the variation in δM has to fulfill also

$$\delta M = -T_{IH}\delta S_{IH}.$$

Here, T_{IH} and S_{IH} are the entropy and temperature of the inner, Cauchy horizon. (For the other two horizons, we keep the notation from the previous section.) This means that by applying the same logic as in the previous section (and assuming that all the horizons have the same, discrete, and Bekenstein-type spectra), one comes to the following conclusion: If one allows processes, such that change only the M parameter, all the temperatures must have *rational* ratios[8]. Further, by using the same logic as before, one obtains

$$\begin{aligned}\delta M &= -T_{IH}\delta S_{IH} = T_{BH}\delta S_{BH} = -T_{CH}\delta S_{CH}\\ &= T_{IH}\cdot m_{IH}\cdot 2\pi\gamma = T_{BH}\cdot m_{BH}\cdot 2\pi\gamma\\ &= T_{CH}\cdot m_{CH}\cdot 2\pi\gamma = \lim_{n\to\infty}\Delta_{(n,n-1)}\omega_{nI}.\end{aligned} \tag{29}$$

[6]However, the (interior) Reissner-Nordström geometry is often considered as a good qualitative approximation for the Kerr geometry. The external Kerr geometry is, of course, the one that is supposed to be physically relevant.

[7]On the other hand, let us mention here the following two points: (1) there is a regime in the R-N-dS spacetime in which the Cauchy horizon is stable [51], and (2) the mass inflation singularity is only a weak singularity [52, 53]. The tidal forces experienced by an infalling observer are finite, and it can be suggested that an infalling observer could still cross the singularity[52, 53]. This would mean that it can still act as some kind of horizon.

[8]For the R-N spacetime, the rational ratios follow directly from the fact that $T_{IH}S_{IH} = T_{BH}S_{BH}$.

($m_{IH}, m_{BH}, m_{CH} \in \mathbb{Z}$.) Consider that we are again interested in the smallest allowed quantum, such that fulfills the equation (29). Then, one obtains again

$$\delta M = 2\pi\gamma\cdot \mathrm{lcm}(T_{IH}, T_{BH}, T_{CH}) = \lim_{n\to\infty}\Delta_{(n,n-1)}\omega_{nI}. \tag{30}$$

This is again the observed behavior for the families of the asymptotic frequencies, such that one (again) fixes the γ parameter in the usual way $\gamma = 1$. (As in case of S-dS spacetime, this provides the same generalization of Maggiore's conjecture for the multi-horizon spacetimes.) (Let us add here that the same result; the universal entropy (area) spectra given as $2\pi\cdot n$ $(8\pi\cdot n)$ was obtained for the multi-horizon spacetimes in [8]. The authors used different argumentation as we did here, but their reasoning was also partly related to the asymptotic quasi-normal modes.)

4.5 Discussion

Let us briefly discuss the condition for the rational ratios of the horizon temperatures. It is used as an important middle step in the argumentation, connecting the behavior of the frequencies with the quantization of entropy for the different horizons, but one, in fact, does not have to mention the rational ratios at all and one can use directly the formula (29) leading to equation (30), where the rational ratio of temperatures follows implicitly from the result. (The least common multiple must be a well-defined function for the temperatures.) Since irrational numbers are dense in the set of rational numbers, it would seem as a miracle, if the rational ratios would say something fundamental about the nature (something like the general theory of relativity carrying information about some quantization of temperatures). Let us keep this question open, but let us mention that a similar rational ratio condition for the temperatures of the two horizons in the S-dS spacetime had been derived in [54]. In particular, the rational ratios follow from the condition of existence of a global thermodynamical equilibrium in the S-dS spacetime.

Let us also mention that although the gap in the spacing between the frequencies is a continuous function of the Λ, Q, and M parameters, our interpretation of the frequencies is singular as $Q \to 0$ or $\Lambda \to 0$. Is this a problem? I would like to argue that this is certainly not. We are constrained to the semiclassical regime, and the interpretation is dependent on us taking the properties of all the horizons seriously. By taking such limits as $Q \to 0$ and $\Lambda \to 0$, one of the horizons becomes increasingly small, and its entropy spectrum eventually deviates from the semiclassical regime. Hence, it has to be expected that the thermodynamical interpretation for the frequencies is for the limits $Q \to 0$ or $\Lambda \to 0$ singular.

As a last point, let us discuss whether our suggestion that areas of all the different horizons are semiclassically equispaced can appear troublesome from the point of view of some known physics. Is there some basic constraint by the well-known physics that can interfere horizon area quantization rules? Such constraint might be the quantization of charge. Any black hole's charge should be given as a multiple of the elementary charge; hence, $Q = N \cdot e$. For the Reissner-Nordström black hole, the relation between areas of the horizons and the charge is simple

$$A_1 A_2 = K \cdot Q^4 = \left(k_\epsilon^2 G^2/c^8\right) \cdot Q^4.$$

(The value of K is approximately of the order $\sim 10^{-68}$ m^4/C^4.) This means

$$n_1 n_2 = \frac{1}{\left(8\pi l_p^2\right)^2} K e^4 N^4. \tag{31}$$

The dimensionless constant is

$$\frac{1}{\left(8\pi l_p^2\right)^2} K e^4 \sim 10^{-6}.$$

Take the weakest possible interpretation of what the equispaced spectra of the two horizons mean, and that is, the areas of the horizons can be given only as $A_i = 8\pi n_i$, and for each relevant couple of natural numbers n_1 and n_2, there is a quantum state in which the area values are obtained. (The "relevance" of the numbers is constrained by the condition that both $n_1, n_2 >> 1$ and correspond to a black hole far from extremality), but then, the condition (31) hardly represents any problem as it just tells that for arbitrary N, you have to find with a "good enough" approximation some natural numbers n_1 and n_2, such that fulfill the upper condition. In fact, the right side of the equation is always an integer with the error $\leq 1/2$, and for large $n_1 \cdot n_2$, this always gives good approximation. (Note that an entirely different thing would be to impose that the quantum states can be labeled *independently* by two different quantum numbers labeling the area spectra of the two horizons. This is a *stronger* condition, as we impose, and trivially contradicts our results,[9] considering that an electromagnetic charge has to be quantized in the physically relevant way.)

5 Conclusions

As we already discussed our results in several places in the paper, let us briefly conclude that the key statement of this review of our work is to extend the idea that black hole horizon has an entropy with a semiclassically equispaced spectrum to much more general spacetime horizons. Such a generalization was already proposed by using different arguments in [7]. We support the idea of [7] by adding two basic arguments: (1) we generalize Bekenstein's lower bound on the horizon area transition to much more general horizons than only the black hole horizon, and (2) we obtain the same suggestion for the spectra via the information provided by the asymptotic quasi-normal frequencies of the Reissner-Nordström, Schwarzschild-deSitter, and Reissner-Nordström-deSitter spacetimes (in the way proposed by Maggiore).

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[9] We agree that this would be closer to Bekenstein's proposal from [26].

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