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Wave Kinetic Description of Superfluidity

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Abstract We propose a wave kinetic description of superfluidity in Bose Einstein condensates. This is based on the traditional two-fluid model, where in contrast with current analysis, the two fluids are treated on similar grounds. Each of the fluids (the condensed gas and the phonon gas) is described by a similar wave kinetic equation. These two equations are coupled. Perturbations of the coupled fluids lead to the formation of excitons which, in the appropriate limits, tend to the first and second sound waves.

1 Introduction

Superfluidity is one of the most remarkable properties of ultra-cold matter close to the absolute zero of temperature. This is a state where the fluid viscosity goes to zero. It was first observed by Kapitza [1] and by Allen and Misener [2] in 1938 using liquid helium. The interest in the subject was revived in recent years after the discovery of Bose Einstein condensation in diluted alkaline gases. Condensate flows, which can persist for several seconds, have already been observed in condensates using toroidal traps, which can be seen as a consequence of superfluidity [3].

Our present knowledge of superfluity is solidly based on the theory of two fluids. The idea that inside a superfluid, two distinct fluids may coexist was first proposed by Tisza [4] soon after the discovery of Helium-4 superfluidity. One is the superfluid itself, which corresponds to the condensed phase. The other is a thermal or normal fluid with finite viscosity, which coexists with the first one. The two-fluid approach to superfluidity was refined by Landau, who proposed in alternative a phenomenological theory based on the concept of elementary quasi-particles, which would be phonons or rotons [5]. The Landau theory became with time a standard model for superfluidity [6, 7], although it was later recognized that the rotons are not independent new particles, but simply phonons with a modified dispersion relation. It was recently proposed that rotons can also occur in a non-condensed laser-cooled gas [8]. More elaborated microscopic quantum theories of superfluidity have also been developed [6, 9]. Our current view of the theory of two fluids is that the second fluid is made of quasi-particle excitations, or phonons, independent of the existence or not of rotons. In the diluted condensed gas, the roton properties are absent most of the time, but rotons can occur due to the enhanced dipole-dipole interactions between the atoms [10].

Here, we propose an alternative description of superfluidity in a condensed gas. It is based on the wave kinetic version of the theory of two fluids. It is known that the condensed gas can be described by a wave kinetic equation, which is equivalent to the Gross-Pitaevskii (GP) equation [11]. These two equations are both valid in the mean-field approximation. We will show here that the gas of phonons can also be described by a similar wave kinetic equation. Fluid equations for the phonon gas can be derived from such wave kinetic equation.

We will then study the excitation of long wavelength perturbations in this mixture of two coupled fluids. The corresponding dispersion relation will be established. These long wavelength perturbations correspond to the second sound and can be Landau damped by both the condensed atoms and the primary phonons. The case of (first and second sound) phonons excited by a particle moving through

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the condensate, or by the flow of the condensed itself, will lead to the so-called Landau criterium of superfluidity.

2 The Condensate Equations

We start with the quantum description of the condensed gas. In the mean field approximation, the condensed state is characterized by a wave function $\Phi(\mathbf{r}, t)$, which satisfies the GP equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r}, t) = H\Phi(\mathbf{r}, t)\,,$$
$$H \equiv \left[-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}}(\mathbf{r}, t) + g\,|\Phi(\mathbf{r}, t)|^2\right]. \quad (1)$$

This is a nonlinear Schrödinger equation, where the Hamiltonian H contains the confining potential $V_{\text{ext}}(\mathbf{r}, t)$, as well as a nonlinear term associated with low energy collisions between the atoms. This term is weighted by the parameter $g = 4\pi\hbar^2 a/M$, where a is the s-wave scattering length and M is the mass of the atoms. The mean field approximation neglects quantum fluctuations and is valid for condensates with a large number of atoms. An equivalent description is known to exist [11], where the GP equation is replaced by a wave kinetic equation of the form

$$i\hbar\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)W = \int\int V(\mathbf{q}, t)[W_- - W_+]$$
$$\exp(i\mathbf{q}\cdot\mathbf{r})\frac{d\mathbf{q}}{(2\pi)^3}\,, \quad (2)$$

where $W \equiv W(\mathbf{u}; \mathbf{r}, t)$ is a Wigner function, or quantum quasi-distribution, and $\mathbf{u}$ is the atom velocity. This new function is related to the wave function $\Phi(\mathbf{r}, t)$ by the Fourier transformation

$$W(\mathbf{u}; \mathbf{r}, t) = \int \Phi^*(\mathbf{r}-\mathbf{s}/2, t)\Phi(\mathbf{r}+\mathbf{s}/2, t)\exp(-i\mathbf{k}\cdot\mathbf{s})\,d\mathbf{s}\,, \quad (3)$$

where $\mathbf{k} = M\mathbf{u}/\hbar$. In (2), we have introduced $V(\mathbf{q}, t)$ as the Fourier transform of the total (confinement plus nonlinear) potential and the auxiliary quantities $W_\pm \equiv W(\mathbf{u} \pm \hbar\mathbf{q}/2M)$. The two different descriptions of the condensate given by (1) and (2) are exactly equivalent, one in terms of the wave function $\Phi(\mathbf{r}, t)$ and the other in terms of the quasi-distribution W, or in other words, of the quantum correlations. Equation (2) is sometimes called a Wigner-Moyal equation and also a von Neumann equation.

From any of the two equations above, (1) or (2), we can derive a set of fluid equations, describing the evolution of the mean density n, and mean velocity $\mathbf{v}$ of the quantum fluid. They are the continuity equation of the quantum fluid

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\,, \quad (4)$$

and the momentum conservation equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\frac{1}{M}\nabla\left(gn + V_{\text{ext}} + V_B\right). \quad (5)$$

The quantity V_B is a quantum potential, usually called the *Bohm potential*, and defined by

$$V_B = -\frac{\hbar^2}{2M^2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\,. \quad (6)$$

If we start from the GP (1), we can derive these equations by using the Madelung transformation

$$\Phi(\mathbf{r}, t) = \sqrt{n}\exp(i\varphi)\,, \quad \mathbf{v} = \nabla\varphi\,. \quad (7)$$

From here, it results the well-known property that a condensate is irrotational, $\nabla\times\mathbf{v} = 0$. Alternatively, if we start from the wave kinetic (2), we have to use the following definitions

$$n = \int W(\mathbf{u}; \mathbf{r}, t)\,d\mathbf{u}\,, \quad \mathbf{v} = \frac{1}{n}\int \mathbf{u}\,W(\mathbf{u}; \mathbf{r}, t)\,d\mathbf{u}\,. \quad (8)$$

This completes our presentation of the quantum equations describing the evolution of a Bose Einstein condensate in the mean field approximation. In the following section, we show that the fluid of elementary excitations or phonons can also be described by similar equations, both in their wave kinetic and fluid versions.

3 Phonon Kinetics

Let us start from the quantum fluid equations for the condensate as stated above. We can easily show that by linearizing (4) and (5) with respect to density perturbations $\tilde{n} = n - n_0$, where n_0 is the equilibrium density and $|\tilde{n}| \ll n_0$, we obtain a wave equation of the form

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 + \frac{\hbar^2}{4M^2}\nabla^4\right)\tilde{n} = 0\,. \quad (9)$$

Here, we have used $c_s = \sqrt{gn_0/M}$, which is the Bogoliubov sound speed. For perturbations evolving in space and time as $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, we can then derive the dispersion relation for sound waves in the condensate, as

$$\omega^2 = c_s^2 k^2 + \frac{\hbar^2 k^4}{4M^2}\,. \quad (10)$$

This solution is valid for elementary perturbations in a uniform and stationary condensate, but we can extend it to the case of a slowly varying condensate, such that its density evolves in space and time scales much larger and slower than $1/k$ and $1/\omega$. For this purpose, we make the replacement $n_0 \to n_0 + n'(\mathbf{r}, t)$, where n' represents the slowly varying part. Under such conditions, the dispersion relation

(10) becomes a local relation, where the sound speed now evolves in space and time according to

$$c_s^2(\mathbf{r}, t) = (g/M)[n_0 + n'(\mathbf{r}, t)]. \tag{11}$$

Here, the quantity $n'(\mathbf{r}, t)$ is, for instance, associated with the long wavelength density perturbations characterizing the second sound, which will become apparent later.

In the present context, it is important to notice that the wave (9), with a space and time varying sound speed, as given by (11), can be reduced to a wave kinetic equation of the Wigner-Moyal type, very similar to (2). This will be of paramount importance for our model of superfluidity. For this purpose, we consider the density auto-correlation function, $K_{12} = \tilde{n}(\mathbf{r}_1, t_1)\tilde{n}(\mathbf{r}_2, t_2)^*$. Defining two new space time variables, $\mathbf{s}$ and τ, such that $\mathbf{r}_1 = \mathbf{r} + \mathbf{s}$, $t_1 = t + \tau/2$ and $\mathbf{r}_2 = \mathbf{r} - \mathbf{s}$, $t_2 = t - \tau/2$, and introducing the double Fourier transformation of K_{12}, we obtain

$$F(\mathbf{r}, t, \omega, \mathbf{k}) = \int d\mathbf{s} \int d\tau K_{12} \exp(-i\mathbf{k} \cdot \mathbf{s} + i\omega\tau). \tag{12}$$

This is the Wigner function for the phonon field. Using the phonon wave equation (9), we can then derive an evolution equation for $F(\mathbf{r}, t, \omega, \mathbf{k})$ of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla\right) F = -\frac{i}{2\omega}\frac{g}{M} \int \frac{d\mathbf{q}}{(2\pi)^3} \int \frac{d\Omega}{2\pi} n'(\Omega, \mathbf{q})[F^- - F^+] \exp(i\mathbf{q} \cdot \mathbf{r} - i\Omega t). \tag{13}$$

Here, we have used the Fourier components of the spectrum of density fluctuations, as given by

$$n'(\mathbf{r}, t) = \int \frac{d\mathbf{q}}{(2\pi)^2} \int \frac{d\Omega}{2\pi} n'(\Omega, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{r} - i\Omega t), \tag{14}$$

where the wavevectors $\mathbf{q}$ and frequencies Ω describe the slow evolution of the background density. It should be emphasized that n' is distinct from the density perturbations associated with the phonon field, $\tilde{n}$ considered before. We have also defined $F^\pm \equiv F(\omega \pm \Omega/2, \mathbf{k} \pm \mathbf{q}/2)$. In (13), we have used the group velocity of the phonon modes, as defined in accordance with the dispersion relation (10), by

$$\mathbf{v}_k \equiv \frac{\partial\omega}{\partial\mathbf{k}} = \left(c_s^2 + \frac{\hbar^2 k^2}{2M}\right)\frac{\mathbf{k}}{\omega}. \tag{15}$$

At this point, it is useful to introduce the so-called quasi-particle approximation, which assumes that the phonon dispersion relation stays locally valid in the slowly changing background. This allows us to write

$$F(\mathbf{r}, t, \omega, \mathbf{k}) = F_k(\mathbf{r}, t)\delta(\omega - \omega_k), \tag{16}$$

where ω_k is the local solution of (10). The new Wigner function $F_k \equiv F_k(\mathbf{r}, t)$ will now satisfy a simplified kinetic equation of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla\right) F_k = -\frac{i}{2n_0}\frac{c_{s0}^2}{\omega_k} \int n'(\mathbf{q}, t)[F_k^- - F_k^+] \exp(i\mathbf{q} \cdot \mathbf{r}) \frac{d\mathbf{q}}{(2\pi)^3}. \tag{17}$$

Here, the sound speed c_{s0} is defined with the unperturbed density n_0, and we have used the notation $F_k^\pm \equiv F(\mathbf{k} \pm \mathbf{q}/2)$. This equation is formally similar to our previous wave kinetic (2). It provides a kinetic description for the phonon gas in the same way as (2) provides a kinetic description for the condensate. Therefore, they both describe, at the same wave kinetic level, the evolution of the two co-existing fluids: the condensate and the phonon gas.

4 Phonon Fluid Equations

A step further in the description of the phonon field, seen as a gas of quasi-particle excitations, is taken with the derivation of phonon fluid equations. They can be obtained from the kinetic (17) in the following way. First, we take the quasi-classical limit, where the inequality $|\mathbf{q}| \ll |\mathbf{k}|$ stays valid. This allows us to assume that

$$F_k^\pm \simeq F_k \pm \frac{\mathbf{q}}{2} \cdot \frac{\partial F_k}{\partial \mathbf{k}} \tag{18}$$

This quasi-classical assumption is not strictly necessary for our derivation of phonon fluid equations, but it considerably simplifies our discussion. Replacing this approximation in (17), we can easily derive the Vlasov equation for the phonon gas of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla + \mathbf{f}_k \cdot \frac{\partial}{\partial \mathbf{k}}\right) F_k = 0\,. \tag{19}$$

Here, the quantity $\mathbf{f}_k$ is the force acting on the phonons, as due to the density gradients of the condensate density and defined by

$$\mathbf{f}_k = \frac{1}{2n_0}\frac{c_{s0}^2}{\omega_k} \nabla n' \tag{20}$$

At this point, we should notice that the characteristics of the Vlasov equation coincide with the equations of motion for the individual phonons described as quasi-classical particles.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_k\,, \quad \frac{d\mathbf{v}_k}{dt} = \mathbf{f}_k \tag{21}$$

Integration of (19) over the momentum spectrum $\mathbf{k}$ leads to the continuity equation for phonons, as

$$\frac{\partial n_\nu}{\partial t} + \nabla \cdot (n_\nu \mathbf{u}_\nu) = 0. \tag{22}$$

where the mean phonon density n_ν and velocity $\mathbf{u}_\nu$ are defined by

$$n_\nu(\mathbf{r}, t) = \int F_k(\mathbf{r}, t) \frac{d\mathbf{k}}{(2\pi)^3}\,, \quad \mathbf{u}_\nu(\mathbf{r}, t) = \frac{1}{n_\nu} \int \mathbf{v}_k F_k \frac{d\mathbf{k}}{(2\pi)^3}\,. \tag{23}$$

We now multiply (4.2) by $\mathbf{v}_k$ and integrate over $\mathbf{k}$. The resulting momentum conservation equation can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\nu \cdot \nabla\right) \mathbf{u}_\nu = -\frac{\nabla P_\nu}{n_\nu} + \mathbf{F}_\nu \,, \tag{24}$$

where we have introduced the phonon gas pressure P_ν and the average force acting on the phonons $\mathbf{F}_\nu$. The first of these quantities was defined as

$$P_\nu = \frac{n_\nu}{3} \int (\mathbf{v}_k \cdot \mathbf{u}_\nu) F_k \frac{d\mathbf{k}}{(2\pi)^3} \equiv \frac{n_\nu}{3} u_{th}^2 \,, \tag{25}$$

where u_{th} is an effective thermal velocity for the phonon gas. The factor of $(1/3)$ in the definition of the pressure results from having assumed gas isotropy. As for the average force, we have defined it as

$$\mathbf{F}_\nu = \frac{1}{n_\nu} \int F_k \left[\frac{\partial \mathbf{v}_k}{\partial t} + \nabla \cdot (\mathbf{v}_k \mathbf{v}_k) + \frac{\partial}{\partial \mathbf{k}} \cdot (\mathbf{f}_k \mathbf{v}_k)\right] \frac{d\mathbf{k}}{(2\pi)^3} \,. \tag{26}$$

This force will play an important role in the dispersion properties of the second sound, as discussed later. The formal analogy between the phonon fluid (22) and (24) and those for the condensate, as given by (4) and (5), should be stressed here. The only qualitative difference is the absence of a quantum force in (24), which results from having taken the quasi-classical approximation (18), but a new term, similar to the Bohm potential term in (5), could also have been included in (24) if we had not taken this quasi-classical assumption.

5 Second Sound

Let us now consider a perturbation in the phonon fluid, as described by the above fluid equations. We assume $n_\nu = n_{\nu 0} + \tilde{n}_\nu$, where $n_{\nu 0}$ is the unperturbed phonon density. Linearizing (22) and (24) with respect to the perturbations and neglecting the mean force $\mathbf{F}_\nu$, we get

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{3} u_{th}^2 \nabla^2\right) \tilde{n}_\nu = 0, \tag{27}$$

For perturbations evolving as $\tilde{n}_\nu \propto \exp(i\mathbf{q} \cdot \mathbf{r} - i\Omega t)$, this leads to the dispersion relation

$$\Omega^2 = \frac{1}{3} u_{th}^2 q^2. \tag{28}$$

This is usually called the *second sound*. It corresponds to oscillations of the phonon number density alone, where the interactions with the background condensed gas were not taken into account. In general, however, we expect these oscillations in n_ν to be coupled with those of the condensed gas density n. Such a coupling is provided by the microscopic force $\mathbf{f}_k$ and by the corresponding average force $\mathbf{F}_\nu$, as defined by (26). This force can be roughly estimated by noting that $\omega_k \simeq c_s k$ and $v_k \simeq c_s$ if we account only the contribution in the long wavelength (purely acoustic) limit. This allows us to write

$$\frac{\partial}{\partial \mathbf{k}} \cdot \mathbf{f}_k \mathbf{v}_k \simeq -\frac{g}{2M} \nabla n, \tag{29}$$

where n is the density of the condensed gas. On the other hand, for isotropic phonon distributions F_k, the first term in (26) is zero, and neglecting nonlinear terms, we obtain a simple expression for the average force, as

$$\mathbf{F}_\nu = -\frac{1}{n_{\nu 0}} \frac{g}{2M} \nabla n. \tag{30}$$

Including this force in the wave equation for the perturbations of the phonon gas, we get

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{3} u_{th}^2 \nabla^2\right) \tilde{n}_\nu = \frac{g}{2M} \nabla^2 n', \tag{31}$$

where n' is the perturbed density of the condensate. Assuming that $\tilde{n}_\nu$ and n' are related with each other and therefore both can evolve in space and time as $\exp(i\mathbf{q} \cdot \mathbf{r} - i\Omega t)$, we obtain the following relation between the two density perturbations

$$n' = \tilde{n}_\nu \frac{2M}{q^2 g} \left(\Omega^2 - \frac{1}{3} u_{th}^2 q^2\right) \,. \tag{32}$$

Let us now turn to the fluid equations of the condensed fraction. Coupling between the condensate and the phonon gas can be described by nonlinear contributions, which were ignored in the derivation of the density wave (27). These nonlinear contributions are mainly due to slow ponderomotive effects associated with the spectrum of phonons. Neglecting the confining potential $V_0 = 0$, which is irrelevant to our present discussion, we can derive from (22) and (24) the following nonlinear equation

$$\frac{\partial^2 n}{\partial t^2} - \frac{1}{M} \nabla \cdot [n \nabla (V_B + gn)] = \nabla \cdot [n(\mathbf{v} \cdot \nabla \mathbf{v}) + \nabla \cdot (n\mathbf{v})\mathbf{v}] \,. \tag{33}$$

Let us then consider a slow perturbation of the condensed gas $(n', \mathbf{v}')$ in the presence of an arbitrary phonon spectrum. We can then use

$$n = n_0 + n' + n_{ph} \,, \quad \mathbf{v} = \mathbf{v}' + \mathbf{v}_{ph} \,, \tag{34}$$

where n_{ph} and $\mathbf{v}_{ph}$ represent the total density and velocity perturbations associated with the entire phonon spectrum. They can therefore be defined as

$$n_{ph}(\mathbf{r}, t) = \int n_k \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega_k t) \frac{d\mathbf{k}}{(2\pi)^3} \,, \tag{35}$$

and

$$\mathbf{v}_{ph}(\mathbf{r}, t) = \frac{1}{n_0} \int \frac{\omega_k \mathbf{k}}{k^2} n_k \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega_k t) \frac{d\mathbf{k}}{(2\pi)^3} \,. \tag{36}$$

The frequency ω_k is determined, for a given wave vector $\mathbf{k}$, by the local dispersion relation

$$\omega_k^2 = \frac{g}{M}(n_0 + n')k^2 + \frac{\hbar^2 k^4}{4M^2} . \tag{37}$$

The quantity n_{ph}, which represents the total phonon density perturbations, should not be confuse with the quantity n_ν introduced above, which is the density of the phonon quasi-particles. A simple relation exists between these two quantities, as will be stated below. Linearizing (33) with respect to the slow perturbations, we get

$$\left(\frac{\partial^2}{\partial t^2} - \frac{g n_0}{M}\nabla^2 + \frac{\hbar^2}{4M^2}\nabla^4\right) n' = n_0 \nabla^2 |\mathbf{v}_{ph}|^2 + \frac{g}{2M}\nabla^2 |n_{ph}|^2 . \tag{38}$$

Putting (35) and (36) together, we obtain

$$|n_{ph}|^2 = \int F_k \frac{d\mathbf{k}}{(2\pi)^3} = n_\nu , \tag{39}$$

where $F_k \equiv n_k n_{-k} = |n_k|^2$ and

$$|\mathbf{v}_{\rm ph}|^2 = \frac{1}{n_0}\int \frac{\omega_k^2}{k^2} F_k \frac{d\mathbf{k}}{(2\pi)^3} \equiv \frac{c_0^2}{n_0^2} n_\nu . \tag{40}$$

The quantity c_0^2 introduced here can be seen as the square phase velocity averaged over the phonon spectrum. Replacing this in (38), we obtain

$$\left(\frac{\partial^2}{\partial t^2} - \frac{g n_0}{M}\nabla^2 + \frac{\hbar^2}{4M^2}\nabla^4\right) n' = \left(\frac{c_0^2}{n_0} + \frac{g}{2M}\right)\nabla^2 n_\nu . \tag{41}$$

We can see that the density perturbations of the condensate are again coupled with the phonon spectrum. Assuming as above that $n_\nu = n_{\nu 0} + \tilde{n}_\nu$, we get for perturbations evolving as $\exp(i\mathbf{q}\cdot\mathbf{r} - i\Omega t)$, a new relation between n' and $\tilde{n}_\nu$, of the form

$$n' = \tilde{n}_\nu \frac{c_1^2}{n_0} \frac{q^2}{\left(\Omega^2 - \Omega_q^2\right)} \tag{42}$$

where we have used

$$\Omega_q^2 = c_{s0}^2 q^2 + \frac{\hbar^2 q^4}{4M^2} , \quad c_{s0}^2 = \frac{g n_0}{M} \tag{43}$$

and

$$c_1^2 = c_0^2 + \frac{1}{2}c_{s0}^2 . \tag{44}$$

Replacing this in (32), we finally arrive at the dispersion relation for the slow perturbations in the superfluid, as given by

$$\left(\Omega^2 - \Omega_q^2\right)\left(\Omega^2 - \frac{1}{3}u_{th}^2 q^2\right) = \frac{g}{2M}\frac{c_1^2}{n_0} q^4 . \tag{45}$$

This describes a generalized sound wave which includes the combined oscillations of two fluids, the condensed and the thermal fluids, where the later has been described as a gas of phonons. In the absence of coupling provided by the quantity c_1^2, this generalized mode would brake into two distinct wave modes. One is that of Bogoliubov sound waves which can be excited in the condensate, in the absence of any phonon spectrum, and is determined by the dispersion relation $\Omega^2 = \Omega_q^2$. The other is that of second sound waves, which involve the phonon gas alone, in an unperturbed background condensate, and is described by the dispersion relation $\Omega^2 = u_{th}^2 q^2/3$. This means that, according to the generalized dispersion relation (45), the first and the second sound can become coupled.

6 Superfluid Currents

Let us consider the transport properties of a superfluid by taking into account the contribution of the two different fluids, the condensate and the phonon fluid. We can first define the total mass density ρ as

$$\rho = M(n_0 + n_x) \equiv M n , \tag{46}$$

where n_0 is the density of the condensed phase, n is the total density, and $n_x = n - n_0$ is the density of the thermal fluid. It is plausible to assume that n_x is proportional to the density of phonons n_ν because these quasi-particles represent the thermal fraction of the medium. The above definition of ρ derives from our assumption of two different fluids, considered on equal grounds, and described by similar kinetic and fluid equations. We can then define $n_x = \alpha\, n_\nu$, where α is a constant of proportionality still to be determined.

Now, by adding the two conservation equations for both the condensate and the phonon gas, (4) and (22), we can derive an equation of conservation for the total mass of the form

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 , \quad \mathbf{J} = M(n_0\mathbf{v} + \alpha\, n_\nu \mathbf{u}_\nu), \tag{47}$$

where $\mathbf{J}$ is the total mass flow, $\mathbf{v}$ is the mean velocity of the condensate, and $\mathbf{u}_\nu$ the mean velocity of the phonon gas. We can then write, taking the definition of $\mathbf{u}_\nu$ into account, the total mass flow as

$$\mathbf{J} = M n_0 \mathbf{v} + M\alpha \int \mathbf{v}_k F_k \frac{d\mathbf{k}}{(2\pi)^3} . \tag{48}$$

However, this quantity should also be identified with the total momentum transported by the medium, $\mathbf{S}$. By definition, the total momentum associated with both the condensed atoms and the phonon quasi-particles is

$$\mathbf{S} = M n_0 \mathbf{v} + M \int \hbar\mathbf{k} F_k \frac{d\mathbf{k}}{(2\pi)^3} , \tag{49}$$

where $\hbar\mathbf{k}$ is the momentum of an individual phonons and F_k is the phonon distribution. On the other hand, we know that the phonon group velocity is determined by (33), allowing us to write

$$\mathbf{S} = Mn_0\mathbf{v} + \int M_k \mathbf{v}_k F_k \frac{d\mathbf{k}}{(2\pi)^3}, \tag{50}$$

where we have introduced the quantity M_k, as defined by

$$M_k = \frac{\hbar\omega_k}{c_s^2 + (\hbar^2 k^2/2M)}, \tag{51}$$

This new quantity can be clearly identified with the *phonon effective mass*. It obviously depends on the momentum $\mathbf{k}$ of the corresponding phonon mode. Notice that phonons, just as photons in vacuum, have no finite rest mass because this effective mass tends to zero with the frequency. Equating the expressions of $\mathbf{J}$ and $\mathbf{S}$, as given by (48) and (50), we can determine the constant of proportionality α between the thermal atoms and the phonons as

$$\alpha \equiv \frac{n_x}{n_0} = \frac{\mathbf{u}}{n_\nu u^2} \cdot \int \frac{M_k}{M} \mathbf{v}_k F_k \frac{d\mathbf{k}}{(2\pi)^3}. \tag{52}$$

It is interesting to consider the limit of very small phonon wavevectors or long wavelengths. In this limit, the quantum dispersion term proportional to k^4 can be neglected, and we get the approximate expressions for the phonon mass and group velocity as

$$M_k \simeq \frac{\hbar k}{c_s}, \quad \mathbf{v}_k \simeq c_s \frac{\mathbf{k}}{k}. \tag{53}$$

In the same limit, the mass flow (48) reduces to $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_\nu$, with $\mathbf{J}_0 = Mn_0\mathbf{v}$, and the thermal current reduces to

$$\mathbf{J}_\nu \simeq M\alpha c_s \int \frac{\mathbf{k}}{k} F_k \frac{d\mathbf{k}}{(2\pi)^3} \tag{54}$$

The constant of proportionality α is of the order of $\hbar/Mu$, as shown by (52).

Another important question is related to the ratio between the thermal and the condensed density, n_x/n_0 or n_ν/n_0. This can also be studied with the help of the wave kinetic formalism. It is known that an atom with velocity v can emit of absorb phonons with phase velocity $\omega_k/k \sim v$ by a Cherenkov process. The balance between emission and absorption depends on the details of the atom quasi-distribution and leads to Landau damping or growth of phonon modes, as shown by the kinetic theory [12]. A systematic kinetic approach to the problem of phonon turbulence in a condensate is however still missing.

7 Conclusions

We have shown here that the two-fluid model of superfluidity can be described using the wave kinetic approach. In this approach, both the condensate and the phonon fields are described by two Wigner distributions which satisfy two similar wave kinetic equations. In the quasi-classical limit, these two Wigner distributions tend to the classical distribution functions for both the condensed atoms and the phonon quasi-particles. From each of the two wave kinetic equations, we can derive a different sets of fluid equations which describe, at the macroscopic level, the average behavior of both the condensate and the phonon gas.

The dispersive properties of the second sound were also derived. The second sound is a long wavelength perturbation of the two coupled fluids, the condensed fluid of ultra-cold atoms, and the noncondensed fluid associated with phonon quasi-particles. The Landau criterion for superfluidity and the total flow inside the two coupled fluids could also be established from our description. Landau damping of the second sound by both the condensed atoms and the phonon quasi-particles can also be described by the present wave kinetic approach, as will be shown in a future publication.

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