Braz J Phys (2014) 44:356–360
DOI 10.1007/s13538-014-0205-y

SOCIEDADE BRASILEIRA DE FÍSICA

GENERAL AND APPLIED PHYSICS

Low-Energy Neutron Interaction with a Classical Electric Field

S. Bruce · M. D. Campos · J. Diaz-Valdes

Received: 15 November 2013 / Published online: 30 May 2014

Abstract We explore the feasibility of performing experiments to detect the interaction of low-energy neutrons with a given classical electric field. Bound states could be identified by means of approximate Aharonov–Casher configurations. It is shown that neutron-bound states are found under extremely low-energy conditions and very strong electric fields.

Keywords Supersymmetry · Neutron trapping · Relativistic wave equations · Solutions of wave equations

1 Introduction

Aharonov and Casher (AC) pointed out the existence of a quantum mechanical process [1] wherein the behavior of an uncharged magnetic moment is affected by the presence of a classical electric field. Let us consider a charge distribution with axial symmetry centered around the z-axis. The nearly point dipoles, e.g. neutrons, are completely polarized along, say, the positive y direction. It is easy to see that this system can be recast in a supersymmetric form [2–6] because in the first approximation, the system is easy to deal with. This 'hidden' symmetry helps us to uncovers many features of the system and to simplify its study.

In this article, we are concerned about the bound-state problem of a system consisting of a polarized neutron moving across two infinite parallel conducting planes possessing constant charge density per unit area. We study the bound-state problem of both a (supersymmetric) delta double-well potential and a delta double barrier potential, assuming confinement in the other directions similarly to the case of a cylindrical configuration [4]. This also leads us to examine the problem of a mixed potential composed of a delta barrier and a delta well. Finally, we present different schemes of possible *thought* experimental setups.

To be specific, let us consider the motion of a spin-$1/2$ chargeless particle with an anomalous magnetic moment κ_n in the presence of a classical electrostatic field. The Dirac equation can be written (see, for instance, [7, 8]) in a covariant form as

$$\left(\gamma_\mu \widehat{p}^\mu - \frac{\hbar e \kappa_n}{2M_n c^2}\mathcal{F}^{\mu\nu}\Sigma_{\mu\nu} - M_n c\right)\Phi(\mathbf{r},t) = 0, \qquad (1)$$

where $\mathcal{F}^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$ is the electromagnetic field tensor, with $e = -|e|$ and $\kappa_n = -1.913$ nm. In this article, $\mathcal{F}^{0j} = -\mathcal{E}_j$, $\mathcal{F}^{ij} = -\epsilon^{ijk}\mathcal{B}_k = 0$, and $\mathcal{A}^0$ is the only surviving term of $\mathcal{A}^\mu$. From (1), we find

$$\left(c\alpha\cdot\widehat{\mathbf{p}} - \frac{\hbar e\kappa_n}{M_n c} i\gamma^0\alpha\cdot\mathcal{E}(\mathbf{r}) + M_n c^2\gamma^0\right)\Phi(\mathbf{r},t) = c\widehat{p}^0\Phi(\mathbf{r},t) \qquad (2)$$

in terms of the Dirac $\alpha's$, $\gamma^0 = \beta$ matrices.

Here, we have used the fact that $\mathcal{F}^{0j} = -\mathcal{F}^{j0}$ and $\Sigma^{0j} = -\Sigma^{j0} = i\gamma^0\gamma^j$.

2 Motion in 1 + 1 Dimensions

As the electric field has only one nonvanishing component ($\mathcal{F}^{03}(z) = -\mathcal{E}_3(z) = -\mathcal{E}(z)$), then in $1+1$ dimensions (2) reads

$$\left(\alpha_3\widehat{p}^3 - \frac{\hbar e\kappa_n}{M_n c^2} i\gamma^0\alpha^3\mathcal{E}(z) + M_n c\gamma_0\right)\Phi(z,t) = \widehat{p}^0\Phi(z,t). \qquad (3)$$

S. Bruce · M. D. Campos · J. Diaz-Valdes (✉)
Department of Physics, University of Concepcion, P.O. Box 160c
Concepcion, Chile
e-mail: jdiaz@udec.cl

Using the fact that $\gamma^i = \gamma^0\alpha^i$, then $\{\alpha^3, i\gamma^0\alpha^3\} = \{\alpha^3, \gamma^0\} = \{\gamma^0, i\gamma^0\alpha^3\} = 0$, with $\alpha_3^2 = (i\gamma_0\alpha_3)^2 = \gamma_0^2 = I_{4\times4}$. Therefore, it is sufficient to represent this equation in terms of Pauli matrices τ's instead of Gamma matrices since this representation is easy to work with [9]. In fact, the total number of Hermitian 4×4 matrices (Dirac, etc.) is 15. Hence, for a one-dimensional problem like ours, they turn to be redundant. Therefore, one can use a convenient complete set of 2×2 Pauli matrices [5]. We choose the representation

$$\widehat{H}\varphi(z,t) = \left(-i\hbar c\tau^1\frac{\partial}{\partial z} - \frac{\hbar e\kappa_n}{M_n c}\tau^2\mathcal{E}(z) + M_n c^2\tau^3\right)\varphi(z,t) = i\hbar\frac{\partial}{\partial t}\varphi(z,t), \tag{4}$$

in which $-\alpha^3 \to \tau^1$, $i\gamma^0\alpha^3 \to \tau^2$, $\gamma^0 \to \tau^3$, with $\varphi(z,t)$ a spinor

$$\varphi(z,t) = \begin{pmatrix}\phi_u(z,t)\\ \phi_l(z,t)\end{pmatrix}. \tag{5}$$

Applying the operator $\widehat{H}$ on (4), we can write the differential equations for the upper and lower components, ϕ_u and ϕ_l, as follows

$$\left\{\left(\widehat{p}_z + i\frac{\hbar e\kappa_n}{M_n c^2}\mathcal{E}(z)\right)\left(\widehat{p}_z - i\frac{\hbar e\kappa_n}{M_n c^2}\mathcal{E}(z)\right) + M_n^2c^2\right\}\phi_u(z,t) = -\left(\frac{\hbar}{c}\right)^2\frac{\partial^2\phi_u(z,t)}{\partial t^2},$$
$$\left\{\left(\widehat{p}_z - i\frac{\hbar e\kappa_n}{M_n c^2}\mathcal{E}(z)\right)\left(\widehat{p}_z + i\frac{\hbar e\kappa_n}{M_n c^2}\mathcal{E}(z)\right) + M_n^2c^2\right\}\phi_l(z,t) = -\left(\frac{\hbar}{c}\right)^2\frac{\partial^2\phi_l(z,t)}{\partial t^2}, \tag{6}$$

with $\widehat{p}_z = -i\hbar d/dz$ and $\mathcal{E}(z) \equiv \mathcal{E}_3(z)$. Assuming that $\mathcal{A}^0$ is time-independent, we let the time dependence of φ be given by

$$\varphi_E(z,t) = \varphi_E(z)e^{-\frac{iE}{\hbar}t} = \begin{pmatrix}\phi_u(z)\\ \phi_l(z)\end{pmatrix}e^{-\frac{iE}{\hbar}t}. \tag{7}$$

The resulting set of differential equations can be straightforwardly rewritten in the supersymmetric form as:

$$\mathrm{H}_S\varphi_\varepsilon(z) = \varepsilon\varphi_\varepsilon(z), \tag{8}$$

with

$$\mathrm{H}_S = \left\{Q, Q^\dagger\right\}, \qquad [\mathrm{H}_S, Q] = \left[\mathrm{H}_S, Q^\dagger\right] = 0, \tag{9}$$

where $\varepsilon \equiv \left(E^2 - M_n^2c^4\right)/2M_nc^2 \geq 0$. Here,

$$Q \equiv \frac{1}{\sqrt{2M_n}}\tau^-\otimes\left(\widehat{p}_z - i\frac{\hbar e\kappa_n}{M_n c^2}\mathcal{E}(z)\right) \tag{10}$$

is the supersymmetric charge with $\tau^- = (1/2)(\tau_1 - i\tau_2)$, where the τ_1, τ_2 are Pauli matrices. The alternative eigenvalue (8) is indistinguishable from (1) (for stationary states) and emerges naturally from it.

From (8), (9) and (10) (or alternatively from (6)), we find that φ_ε satisfies the differential equation

$$\mathrm{H}_S\Phi_\varepsilon(z) = \frac{\hbar^2}{2M_n}\left\{-\frac{d^2}{dz^2} - \tau_3\frac{e\kappa_n}{M_nc^2}\frac{d\mathcal{E}(z)}{dz} + \left(\frac{e\kappa_n}{M_nc^2}\right)^2\mathcal{E}^2(z)\right\}\varphi_\varepsilon(z) = \varepsilon\varphi_\varepsilon(z). \tag{11}$$

Thus, the superpotential is proportional to the electric field. The space derivative of $\mathcal{E}(z)$ is crucial: From the second term of the left-hand side of (11), we see that the neutrons tend to move toward regions where the *gradient* of the electric field changes. The third term in this equation corresponds to the appearance of an induced electric dipole moment on the particle.

To begin with, let us consider two infinite parallel planes both with uniform charge density per unit area σ separated by a distance $2a$ [5]. The static electric field is given by

$$\mathcal{E}(z) = 4\pi\sigma\left[\theta(z+a) - \theta(a-z)\right], \tag{12}$$

with θ the Heaviside step function and $\sigma > 0$. Thus, the field derivative is

$$\frac{d\mathcal{E}(z)}{dz} = 4\pi\sigma\left[\delta(z+a) + \delta(a-z)\right], \tag{13}$$

which leads to a kind of Dirac double-well potential in (11).

From (11), we obtain two differential equations for stationary states [5]:

$$\left\{\frac{d^2}{dz^2} + k^2\right\}\varphi_\varepsilon(z) = 0, \quad |z| < a;$$
$$\left\{\frac{d^2}{dz^2} - k'^2\right\}\varphi_\varepsilon(z) = 0, \quad |z| > a, \tag{14}$$

with $\varepsilon \equiv \left(E^2 - M_n^2c^4\right)/2M_nc^2 \geq 0$ in which $k = \sqrt{2M_n\varepsilon/\hbar^2}$, $k' \equiv \sqrt{\zeta^2 - k^2}$ and $\zeta \equiv 4\pi e\sigma\kappa_n/M_nc^2$, with $E^2 = \hbar^2c^2k^2 + M_n^2c^4$. We now look for bound states φ_ε for which $k^2 < \zeta^2$. They are written in the form

$$\varphi_\varepsilon^{(+)}(z) \equiv \begin{pmatrix}\phi_\varepsilon^{(+)}(z)\\ 0\end{pmatrix}, \qquad \varphi_\varepsilon^{(-)}(z) \equiv \begin{pmatrix}0\\ \phi_\varepsilon^{(-)}(z)\end{pmatrix}. \tag{15}$$

From (14), we can infer the 'ansatz' for even solutions to be [5]:

$$\phi_\varepsilon^{(\pm)}(z) \propto \begin{cases} B\cos(kz), & |z| \leq a,\\ \exp\left(-k'|z|\right), & |z| \geq a.\end{cases} \tag{16}$$

Similarly, for odd solutions, the states $\phi_\varepsilon^{(\pm)}$ have the form

$$\phi_\varepsilon^{(\pm)}(z) \propto \begin{cases} C\sin(kz), & |z| \leq a,\\ \exp\left(-k'|z|\right), & |z| \geq a.\end{cases} \tag{17}$$

If $k = 0$ (i.e. $\varepsilon = 0$, $E = M_nc^2$), the ground state corresponds to $\varphi_{\varepsilon=0}^{(+)}$ in (16). The boundary continuity and discontinuity conditions at $z = a$ produce the complete spectra (see Fig. 1) for the even and odd solutions of the double-well [5]:

$$\tan(s) = -\frac{\zeta a}{s} + \sqrt{\left(\frac{\zeta a}{s}\right)^2 - 1},$$
$$\cot(s) = \frac{\zeta a}{s} - \sqrt{\left(\frac{\zeta a}{s}\right)^2 - 1}, \tag{18}$$

which are trascendental formulae for k, with $s = ka$. Similarly, the complete spectra for the even and odd solutions of the double-barrier are found from

$$\tan(s) = \frac{\zeta a}{s} + \sqrt{\left(\frac{\zeta a}{s}\right)^2 - 1},$$
$$\cot(s) = -\frac{\zeta a}{s} - \sqrt{\left(\frac{\zeta a}{s}\right)^2 - 1}. \tag{19}$$

Comparing (18) with (19), we find a double degeneracy in the spectra: the plus sign/minus sign solution of the former is directly related (same eigenvalues for k) with the plus sign/minus sign solution of the latter, except for the (even) nondegenerate ground state ($k = 0$, i.e. $\tan(s) = 0$) in the first (18) which does not exist in any other case (see Fig. 1). It is significant that when $k = 0$, the neutron detects the sign of the charge distribution: This is a purely quantum mechanical effect. When $k^2 > \zeta^2$, (15) becomes scattering states.

Next, we examine a mixed configuration. It consists of a potential which is neither symmetric nor antisymmetric under the interchange $z \to -z$. Hence, we do not expect

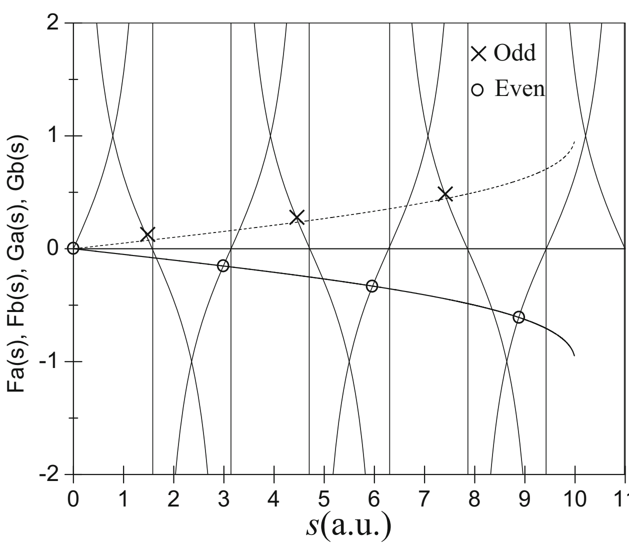

Fig. 1 Location of discrete eigenvalues for even and odd solutions (18) in *double-well* with $\zeta a = 10$ au. $F_a(s) = \tan(s)$, $F_b(s) = -(\zeta a/s) + \sqrt{(\zeta a/s)^2 - 1}$, $G_a(s) = \cot(s)$, $G_b(s) = (\zeta a/s) - \sqrt{(\zeta a/s)^2 - 1}$

that there will be solutions of definite parity rather a linear combination of them. Let us consider two infinite parallel planes with opposite charge density per unit area separated by a distance $2a$. The static electric field is defined by

$$\mathcal{E}(z) = 4\pi\sigma\left[\theta(z+a) + \theta(a-z) - 2\theta(a-z)\theta(z+a)\right]. \tag{20}$$

Hence,

$$\frac{d\mathcal{E}(z)}{dz} = 4\pi\sigma\left[\delta(z-a) - \delta(z+a)\right], \tag{21}$$

with $\mathcal{E}^2(z) = 4\pi\sigma\mathcal{E}(z)$.

From (14) and (20), it is sufficient to set the general solutions for the upper component of φ_ε to be:

$$\varphi_\varepsilon(z) \equiv \begin{pmatrix}\phi_\varepsilon(z)\\ 0\end{pmatrix}, \tag{22}$$

with

$$\phi_\varepsilon(z) \propto \begin{cases} A\cos(kz) + B\sin(kz), & |z| \le a,\\ \exp(-k'z), & z \ge a,\\ C\exp(k'z), & z \le -a.\end{cases} \tag{23}$$

The boundary continuity and discontinuity conditions at $z = \pm a$ provide four trascendental equations

$$\frac{\sin\left(s \pm \frac{\pi}{4}\right)}{\sin\left(s \mp \frac{\pi}{4}\right)} = \pm\frac{a\zeta}{s} + \sqrt{\left(\frac{a\zeta}{s}\right)^2 - 1}, \tag{24}$$
$$\frac{\sin\left(s \mp \frac{\pi}{4}\right)}{\sin\left(s \pm \frac{\pi}{4}\right)} = \pm\frac{a\zeta}{s} + \sqrt{\left(\frac{a\zeta}{s}\right)^2 - 1}. \tag{25}$$

Equation (24) is related through supersymmetry and then give the same spectra (double degeneracy). Similarly, double degeneracy is found from the remaining eigenvalue relations (25). Notice that a solution with $k = 0$ does not exist in this case, i.e. supersymmetry is broken. The energy levels are found from graphical solutions of (24) and (25) in which the number of discrete levels depends on both σ and a. As s and ζ are restricted to positive values, then the energies are found from the intersections in the first and fourth quadrant. Notice also, from (23), that for a given k, confinement increases only if σ (and correspondingly ζ) is larger.

3 Some Numerical Results

In order to apply this model to a thought physical configuration, let us consider an isolated thin (solid) conductor disk of thickness $2a$ (two basic capacitors) with a approximately constant charge density per unit area with its symmetry axis in the z direction (see Fig. 2a). The electric field within the conductor ($|z| < a$) vanishes while closely outside its surfaces ($|z| \gtrsim a$) is $+4\pi\sigma\hat{\mathbf{z}}$ for $z > a$, $-4\pi\sigma\hat{\mathbf{z}}$ for $z < -a$. This field is (roughly) modeled by the expression given in (12). Low-energy neutrons are to be confined along the z-axis by the action of the electric field and its gradient according to (11). Here, we are assuming confinement in the other spatial degrees of freedom, and we have not taken into account the very significant influence of gravity. Notice that the distance d between the internal and external conducting planes must be large ($d \gg a$) so that tunneling (through the external ones) be as small as possible.

For the mixed potential, a possible concrete physical configuration consists in a thin (solid) conductor disk of thickness $2a$ conductors between two solid plane conductors connected to a strong voltage as shown in Fig. 2b

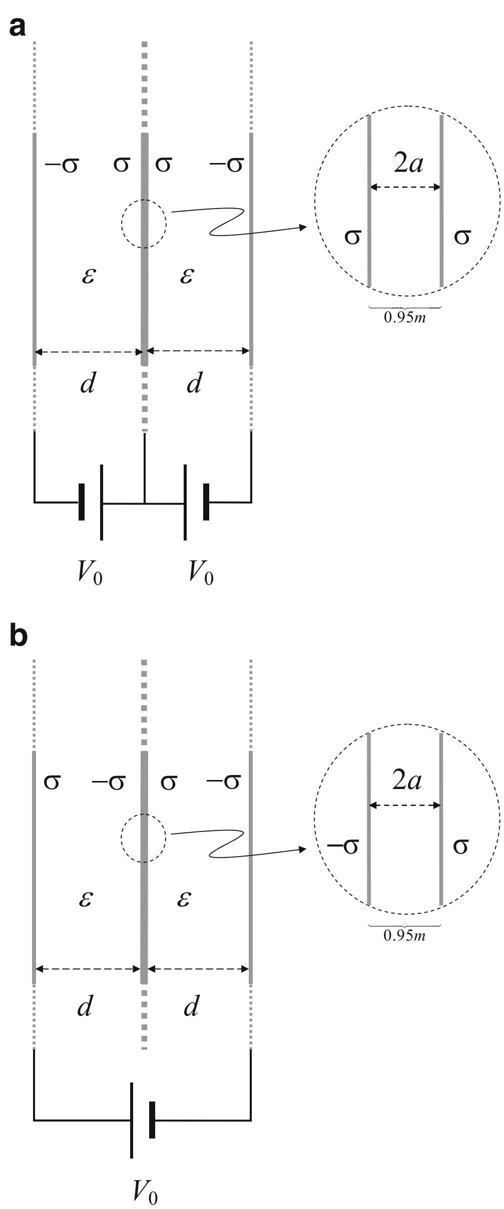

Fig. 2 **a** Scheme of a possible experimental setup for the double-well potential for $d \gg a$. **b** Scheme of a possible experimental setup for the mixed potential

(two serial capacitors). The electric field within the conductor ($|z| < a$) vanishes while closely outside its surfaces ($|z| \gtrsim a$) is $+4\pi\sigma\widehat{\mathbf{z}}$ for $z > a$, $+4\pi\sigma\widehat{\mathbf{z}}$ for $z < -a$. This field is (roughly) modeled by the expression given in (20). The phenomenological discussion is entirely similar to that of the symmetric potentials.

From (18), (19), (24) and (25), the term $\sqrt{(\zeta/k)^2 - 1}$ must be real for *bound* states. Then, $k \le \zeta = 4\pi\mu_n\sigma = \mu_n\mathcal{E}_0$, where $\mathcal{E}_0$ is the electric field between the conducting planes. If we utilize any of the more efficient dielectric materials at hand in order to obtain the strongest possible electric field ($\mathcal{E}_0 \sim 10^9$ V/m [10]), then the speed of the neutrons would be $\approx 8 \times 10^{-4}$ cm/s. This is an extremely low speed compared with that of ultra-cold neutrons. Furthermore, from the same conditions, we get $\tan(\zeta a) \sim 1$, i.e. the thickness of the conducting planes would be $2a \gtrsim 95$ cm.

To study a thought experiment to detect neutron scattering states in this kind of setups, we must take into account other conditions, for instance, the size of the configuration, the neutron stopping power of the dielectric materials and the reflectivity of these materials. In fact, for scattering states ($k > \zeta$), a straightforward calculation shows that the amplitude $r = e^{-2ika}\zeta^2/2k^2$ for the well, while a single delta potential has the value $r = \zeta/(2ik - \zeta)$. The result is that there is a non-zero probability $R = |r|^2$ for the particle to be reflected. This does not depend on the sign of ζ, i.e. a delta barrier has the same probability of reflecting the particle as a delta well. For ultra-cold neutrons ($\hbar k/M_n \sim 7.6 \times 10^2$ cm/s) $R \sim (1/4)\,(\zeta/k)^4$ for the well, while $R \sim (\zeta/2k)^2 \approx 7.2 \times 10^{-16}$ for the delta potential, which is a quite small number for an AC signal.

4 Conclusions

At present, the viability of performing experiments to detect (neutron) bound and scattering states is remote. In the future, an improvement in the laboratory conditions is expected and other configurations can be examined. For the moment, the possibility of such experiments remains as a far-reaching problem.

A physical configuration where the Aharonov-Casher effect could be detected is the low energy neutron-electron (positron) system, where the classical electric field of the electron is expressed as $\mathcal{F}^{\mu\nu}$ ($\mathbf{B} = \mathbf{0}$) in (1). In the past [11], this system has been studied as a spin-spin problem since both particles carry dipole magnetic moment. However, here-bound states could be found assuming Barut's hypothesis [12] wherein the charge of the electron is, on the average, located on a surface of radius $\lambda = \hbar/2m_ec$, the (1/2) Compton wavelength of the electron. The strength of the electrostatic field on this surface is $\sim 10^{15}$ V/m. This is a strong enough field that could support at least one stable bound state for a certain lifetime of the system.

At present, we are advocated to study this problem which will be published elsewhere.

Finally, it is interesting to note that other authors have discussed the solutions of the Dirac equation for cylindrical, three-dimensional [13, 14] and one-dimensional [15] delta potentials in the context of *minimal interactions*. However, these potentials do not share the property of supersymmetry (for *non-minimal* interactions) as in our case.

Acknowledgments This work was supported by Dirección de Investigación, Universidad de Concepción, through grant P.I. 210.011.049-1.0.

References

1. Y. Aharonov, A. Casher. Phys. Rev. Lett. **53**, 319 (1984)
2. S. Bruce, L. Roa. Europhys. Lett. **46**, 703 (1999)
3. S. Bruce. J. Phys. A: Math. Gen. **38**, 6999 (2005)
4. S. Bruce, et al., Nucl. Instr. and Meth. B. **266**, 988 (2008)
5. S. Bruce, et al. Ann. Phys. (Berlin). **523**, 939 (2011)
6. S. Bruce. Physica Scripta. **64**, 102 (2001)
7. C. R. Hagen. Rev, Phys. Lett. **64**, 2347 (1990)
8. X. He, B. McKellar. B., Phys. Lett. B. **256**, 250 (1991)
9. L. Schiff. *Quantum Mechanics*, 3rd edition (McGraw-Hill, New York, 1968), p. 474
10. Dielectric Materials. *McGraw-Hill Encyclopedia of Science and Technology*, 10th Edition, (2007)
11. G. R. Isaak. Phys. Lett. B. **32**, 81 (1970)
12. A. O. Barut, A. J. Bracken. Phys. Rev. D. **23**, 2454 (1981)
13. M. Loewe, F. Marquez, R. Zamora. J.Phys. A. **465303**, 45 (2012)
14. M. Loewe, M. Sanhueza. J.Phys. A. **23**, 553 (1990)
15. V. Villalba, L. A. González-Díaz. Eur. Phys. J. C. **61**, 519 (2009)