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luizno.bjp@gmail.com
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Ulhoa, S. C.
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On the Quasinormal Modes for Gravitational Perturbations of the Bardeen Black Hole

S. C. Ulhoa

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Abstract We investigate gravitational perturbations on a regular black hole described by the Bardeen solution. Bardeen's black hole is a solution of Einstein's equations with no singularity at the origin of the radially symmetric system. Notwithstanding this regularity, the Bardeen solution still has event horizons dependent on its characteristic parameters. When a black hole is perturbed, it oscillates and gives rise to damped vibrational modes known as quasinormal modes. Here, we compute the quasinormal frequencies of a regular black hole to third order in the WKB approximation for gravitational perturbations.

Keywords Quasinormal frequencies · Gravitational perturbations · Regular black holes · Bardeen space-time

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1 Introduction

In 1957, Regge and Wheeler explored the stability of the Schwarzschild solution under gravitational perturbations [1]. Their analysis lead to a Schröedinger-like equation with a specific effective potential and a frequency associated with the temporal dependence of the perturbation. Since then, other perturbations, for different systems, have been analyzed and lead to similar equations [2]. The frequencies in such equations, which can be calculated by means of semi-analytic or numerical methods, are discrete, complex numbers. To obtain them, one imposes such specific boundary conditions as the existence of ongoing waves at the spatial infinity and ingoing waves at the event horizon. Each frequency corresponds to a *quasinormal mode*, a damped oscillation of the space-time geometry that can be used to investigate fundamental features of the gravitational field. As usual, the real part is related to the oscillation itself, while the imaginary part defines the damping rate. Besides disclosing unveiling physical aspects of gravitational waves, the discrete frequencies of the quasinormal modes may open an inroad for the study of gravity quantization.

A regular black hole was first derived from Einstein's equations by Bardeen, who found an approximate expression [3]. Later, it was discovered that Bardeen's solution could be viewed as the gravitational collapse of a magnetic monopole, with a nonlinear electromagnetic energy-momentum tensor as the source of field equations [4], a discovery that promoted the Bardeen metric to the category of exact solutions of Einstein's equations. Other regular solutions were obtained by coupling certain matter fields, such as a scalar field, to gravitation in a cosmological context [5, 6].

These findings are important because their singularity-free features turn regular black holes into strong candidates in representing realistic final stages of collapsing regular configurations, e.g., of a common star. It therefore seems unfortunate that, notwithstanding the successful computation of quasinormal modes of black holes or neutron stars, the same procedure has not been extended to regular black holes under gravitational perturbations.

Motivated by these considerations, the present paper calculates the quasinormal modes of the Bardeen black hole, under gravitational perturbations. The quasinormal modes are the solutions of Einstein's equations in the presence of a nonlinear electromagnetic field, an entirely new paradigm when compared to the procedure in Ref. [1] for the Schwarzschild black hole, since the latter is a vacuum solution. The first step, however, is to construct axial

S. C. Ulhoa (✉)
Instituto de Física, Universidade de Brasília, 70910-900 Brasília, DF, Brazil
e-mail: sc.ulhoa@gmail.com

S. C. Ulhoa
Faculdade Gama, Universidade de Brasília, 72444-240Setor Leste (Gama), Brasília, DF, Brazil

perturbations, which are simpler than polar perturbations. A ϕ-independent gauge will be chosen.

The paper is organized as follows. Section 2 summarizes the general theory of black hole stability for a spherical symmetric background metric under gravitational perturbations. The master equation and the effective potential are obtained for the Bardeen solution. Section 3 computes the complex frequencies for this solution, to third order in the WKB approximation. The results are collected in Tables 1, 2, and 3. Finally, the last section presents the conclusions.

2 Gravitational Perturbations of a Regular Black Hole

In this section, we develop a treatment of a gravitational perturbation of regular black holes. A generic spherically symmetric background metric $\eta_{\mu\nu}$ is defined by the line element

$$ds^2 = -f(r)\ dt^2 + f^{-1}(r)\ dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\ , \quad (1)$$

so that the metric tensor can be written as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$.

The perturbations $h_{\mu\nu}$ can be decomposed as follows:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r,t) \\ 0 & 0 & 0 & h_1(r,t) \\ 0 & 0 & 0 & 0 \\ h_0(r,t) & h_1(r,t) & 0 & 0 \end{pmatrix} h(\theta). \quad (2)$$

Equation (2) is similar to the axial decomposition in the Regge-Wheeler gauge [1]. A different gauge is used here, however, since we would like to find the form of $h(\theta)$ by means of the field equations rather than by imposing a spherical-harmonics expansion.

The perturbed Einstein equations are given by the equality

$$\delta G_{\mu\nu} = k\delta T_{\mu\nu} \quad (3)$$

where k=8π in units such that G=c=1 and

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\left(h_{\mu\nu}R + \eta_{\mu\nu}\delta R\right)\ , \quad (4)$$

$$\delta R = \eta^{\mu\alpha}\eta^{\nu\gamma}h_{\alpha\gamma}R_{\mu\nu}\ , \quad (5)$$

$$\delta R_{\mu\nu} = -\nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} + \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha}\ , \quad (6)$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\nu}\left(\partial_\gamma h_{\beta\nu} + \partial_\beta h_{\gamma\nu} - \partial_\nu h_{\beta\gamma}\right)\ . \quad (7)$$

Combined with Eq. (2), the above equations yield the expression

$$\delta G_{23} = \frac{1}{2}\left[-\frac{1}{f(r)}\frac{\partial h_0}{\partial t} + \frac{\partial[f(r)h_1]}{\partial r}\right]\left[\frac{dh(\theta)}{d\theta} - 2\frac{cos\theta}{sin\theta}\ h(\theta)\right]$$

and

$$\delta G_{13} = \frac{1}{2}\left\{\frac{1}{f(r)}\left[\frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial t\partial r} + \frac{2}{r}\frac{\partial h_0}{\partial t}\right] - \left(\frac{2}{r^2} - \frac{2}{r}\frac{df}{dr} - \frac{d^2 f}{dr^2}\right)h_1\right\}h(\theta) - \frac{h_1}{2r^2}\left(\frac{d^2h(\theta)}{d\theta^2} - \frac{cos\theta}{sin\theta}\frac{dh(\theta)}{d\theta}\right)\ .$$

Therefore, given a background metric that solves the unperturbed Einstein equation, it is possible to find the gravitational perturbations by the above equation, provided only that the perturbed energy-momentum tensor be determined.

2.1 Master Equation for Bardeen Solution

A great deal of literature has been dedicated to the Bardeen black hole. The geodesic structure of test particles has been

Table 1 Quasinormal modes of the Bardeen space-time for α=0

l	n	ω_{nl}	l	n	ω_{nl}
1	0	0.1171192993–0.08879106527i	4	0	0.8090978140–0.09417105983i
	1	0.05501430875–0.2873057745i		1	0.7964989017–0.2843663754i
2	0	0.3731620888–0.08921749033i		2	0.7736360348–0.4789739946i
	1	0.3460174754–0.2749155289i		3	0.7433125214–0.6783003279i
	2	0.3029353684–0.4710642944i	5	0	1.012252026–0.09487331582i
3	0	0.5992651163–0.09272839457i		1	1.002148953–0.2858304756i
	1	0.5823546522–0.2814060077i		2	0.9832631880–0.4798984691i
	2	0.553199534–0.4766840022i		3	0.9574780375–0.6777975349i
	3	0.5157471801–0.6774290938i		4	0.9263580230–0.8791948073i

Table 2 Quasinormal modes of the Bardeen space-time for α=0.3

l	n	ω_{nl}	l	n	ω_{nl}
0	0	0.05982992424–0.3650702720i	3	0	0.62677100570.09160068738i
				1	0.6128131830–0.2776876367i
1	0	0.1884164237–0.2186679687i		2	0.5891039911–0.4697416379i
	1	0.6363972564–0.7536092924i		3	0.5593512286–0.6667476229i
	2	1.329439135–1.493839029i	4	0	0.8355064868–0.09308120018i
2	0	0.4066400107–0.08797961013i		1	0.8244154310–0.2809325272i
	1	0.3918842187–0.2699024702i		2	0.8043475445–0.4728380035i
	2	0.3720498716–0.4598638795i		3	0.7779038150–0.6691359619i
	3	0.3501499693–0.6529380130i		4	0.7467267888–0.8688420648i

studied by Ref. [7], while the gravitational lensing features of the Bardeen solution have been described in [8]. The quasinormal modes of the space-time for scalar field perturbations were calculated in Ref. [9], and a good review can be found in Ref. [10].

Let us consider a nonlinear electromagnetic energy-momentum tensor given by the equality [4]

$$T_\mu^\nu = 2\left(L_F \ F_{\mu\lambda} F^{\nu\lambda} - \delta_\mu^\nu \ L\right) \ , \tag{8}$$

where L_F=$\delta L/\delta F$, with $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and

$$L = \frac{3m}{|\alpha|^3}\left(\frac{\sqrt{2\alpha^2 F}}{1+\sqrt{2\alpha^2 F}}\right)^{5/2} . \tag{9}$$

The parameter α represents a magnetic monopole and m, its mass. If we consider

$$F_{\mu\nu} = 2\delta^2_{[\mu}\delta^3_{\nu]}\alpha sin\theta \ , \tag{10}$$

we then find that F=$\alpha^2/2r^4$.

By coupling this nonlinear-electrodynamics model to the Einstein equations, it is therefore possible to find an exact solution [3, 4], known as the Bardeen black hole, which can be brought to the form of Eq. (1), with

$$f(r) = 1 - \frac{2mr^2}{\left(r^2+\alpha^2\right)^{3/2}} \ . \tag{11}$$

This solution, which was initially only approximately obtained, was the first regular black hole to be identified. We can picture this system as a self-gravitating magnetic monopole of charge α and mass m. An interesting feature of the Bardeen solution is that it can have zero, one, or two horizons of events depending on the choice of the magnetic charge α. Although it is always a regular black hole at r=0 for $\alpha \neq 0$, it describes a regular space-time only when the following inequality holds:

$$\alpha^2 \leq \frac{16}{27} \ m^2 . \tag{12}$$

Clearly, for α=0, the solution reduces to the well-known Schwarzschild metric, which does not represent a regular black hole.

Here, we intend to obtain the master equation for gravitational perturbations of the Bardeen space-time. Therefore, the nonvanishing components of the perturbed energy-momentum tensor have the form

$$\begin{aligned} \delta T_{03} &= \ -2kLh_0(r,t)h(\theta), \\ \delta T_{13} &= \ -2kLh_1(r,t)h(\theta) \ . \end{aligned}$$

Table 3 Quasinormal modes of the Bardeen space-time for $\alpha = \frac{4}{\sqrt{27}}$

l	n	ω_{nl}	l	n	ω_{nl}
0	0	0.6218228066–0.07896506498i	3	0	0.9315004445–0.07399275203i
				1	0.9189649894–0.2227244676i
1	0	0.6767694809–0.07815597820i		2	0.8943294277–0.3736175009i
	1	0.6619780155–0.2360238599i		3	0.8584153696–0.5278493353i
	2	0.6340480798–0.3978834282i	4	0	1.109517971–0.07301443485i
2	0	0.7828924561–0.07622286290i		1	1.099525117–0.2193579532i
	1	0.7688856410–0.2299687824i		2	1.079605666–0.3666520402i
	2	0.7420488856–0.3872459558i		3	1.049912024–0.5155433322i
	3	0.7042736801–0.5493249063i		4	1.010709699–0.6666705011

The perturbed Einstein equations then lead to the equalities

$$\frac{1}{2}\left[-\frac{1}{f(r)}\frac{\partial h_0}{\partial t}+\frac{\partial[f(r)h_1]}{\partial r}\right]=0\ ,\tag{13}$$

$$\frac{1}{f(r)}\left[\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial^2 h_0}{\partial t\partial r}+\frac{2}{r}\frac{\partial h_0}{\partial t}\right]+$$

$$+\left[\frac{(\gamma-2)}{r^2}+\frac{2}{r}\frac{df}{dr}+\frac{d^2f}{dr^2}+2kL\right]h_1=0\ ,\tag{14}$$

$$\frac{d^2h}{d\theta^2}-\frac{cos\theta}{sin\theta}\frac{dh}{d\theta}+\gamma\ h=0\ .\tag{15}$$

for the functions $h(\theta)$, $h_0(r,t)$, and $h_1(r,t)$.

From Eq. (15), it follows that $\gamma=l(l+1)$ and $h(\theta)=P_l(cos\theta)$, i.e., the Legendre Polynomials.

If we use the definition $\psi=\left(\frac{1}{r}\right)f(r)h_1(r,t)$, then Eqs. (13) and (14) can be combined into a single equation, which reads

$$\frac{\partial^2\psi}{\partial t^2}-f^2\frac{\partial^2\psi}{\partial r^2}-f\frac{df}{dr}\frac{\partial\psi}{\partial r}$$
$$f\left(\frac{l(l+1)+2(f-1)}{r^2}+\frac{1}{r}\frac{df}{dr}+\right.\tag{16}$$
$$\left.+\frac{d^2f}{dr^2}+2kL\right)\psi=0\ .$$

In order to bring this equation to a more familiar form, we change the variable r to the *tortoise* coordinate, defined by the equality $dx=dr/f(r)$, which leads to the equation

$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+V(x)\ \psi=0\ ,\tag{17}$$

with the effective potential given by the expression

$$V=f\left[\frac{l(l+1)+2(f-1)}{r^2}+\frac{1}{r}\frac{df}{dr}+\frac{d^2f}{dr^2}+2kL\right]\ .$$

The temporal dependence is assumed to be given by the expression $\psi=e^{-i\omega t}\phi$, so that

$$\left[\frac{\partial^2}{\partial x^2}+\omega^2-V(x)\right]\phi(x)=0\ ,\tag{18}$$

which is the master equation for gravitational perturbations of the Bardeen solution as the background metric.

Once Eq. (18) is solved, one can construct the perturbed metric and all the space-time features will be known. The frequencies ω represent modes that are dissipative in time, so that after the perturbation, the whole black hole will oscillate and go back to a stable configuration. The frequencies of the oscillatory modes are called quasinormal frequencies.

3 Quasinormal Modes in WKB Approximation

The WKB method is a semi-analytic technique to solve Schöedinger-like equations, such as Eq. (18). Once a general solution $\phi(x)$ is found, two conditions are imposed to specify the desired function, with ongoing waves at infinity and ingoing waves at the event horizon. In the WKB approximation, the frequency is written in the form [11]

$$\frac{i\ (\omega^2-V_0)}{\sqrt{-2V_0''}}-\sum_{i=2}^{k}\Lambda_i=n+\frac{1}{2}\ ,\tag{19}$$

where V_0 and V_0'' are the effective potential and its second derivative respectively, taken at the maximum of V.

The second term on the left-hand side of the above equation represents the higher-order WKB corrections, from the second to the kth order [12]). An attractive feature of the WKB method, for low n ad l, is the progressive improvement of its results as successively higher orders of approximation are included. The procedure is therefore very accurate, even when compared with numerical methods [11]. We will work with the third-order WKB approximation [13], defined by the equality.

$$\omega_{n,l}^2=\left[V_0+\left(-2V_0''\right)^{1/2}\Lambda\right]$$
$$-i\left(n+\frac{1}{2}\right)\left(-2V_0''\right)^{1/2}(1+\Omega),\tag{20}$$

where

$$\Lambda=\frac{1}{(-2V_0'')^{1/2}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4}+\beta^2\right)-\frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2(7+60\beta^2)\right],$$

and

$$\Omega=\frac{1}{(-2V_0'')}\left[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4(77+188\beta^2)-\frac{1}{384}\left(\frac{(V_0''')^2V_0^{(4)}}{(V_0'')^3}\right)(51+100\beta^2)+\right.$$
$$\left.+\frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2(67+68\beta^2)+\frac{1}{288}\left(\frac{V_0'''V_0^{(5)}}{(V_0'')^2}\right)(19+28\beta^2)-\frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)(5+4\beta^2)\right],$$

Here, $\beta = n + \frac{1}{2}$, $V_0^{(n)} = \frac{d^n V}{dx^n}|_{x=x_0}$, where x_0 is the solution of the equation $\frac{dV}{dx}(x_0) = 0$, i.e., a maximum effective potential. The variable x is the well-known tortoise coordinate.

From Eq. (20) one can calculate the quasinormal frequencies, which are presented in Tables 1, 2, and 3 for Bardeen space-time. All results, which refer to quasinormal modes for gravitational perturbations, were computed in units of m, so that ω_{nl}=m $\omega_{n,l}$, where ω_{nl} is dimensionless.

To comment on these results, we first recall that the Bardeen space-time for α=0 is exactly the Schwarzschild space-time. As it is well known, the ground state of the quasinormal modes for the Schwarzschild metric is given by $\{l,n\}=\{2,0\}$. Modes below this state hence have no physical meaning, although we have displayed them on Table 1. Inspection of the tables shows that the imaginary part of the frequency, i.e., the damping term, varies little with α. By contrast, Table 2 shows that the real part of the frequency follows a peculiar behavior: for l=1, it grows with n. The general trend is for the imaginary part to grow while the real part decreases. Therefore, by comparison with the ground state of the quasinormal modes of Schwarzschild space-time, it seems natural to also take the mode $\{l,n\}=\{2,0\}$ on Tables 2 and 3 as the ground state.

4 Conclusion

We have calculated the quasinormal frequencies associated with gravitational perturbations on a regular black hole. Our results were obtained for the Bardeen solution of the Einstein equations that represent a regular black hole. To obtain the quasinormal modes, we have used the third-order WKB approximation, which for low n and ℓ has been shown to yield good results in comparisons with accurate numerical techniques. The function $h(\theta)$ in our solution is a Legendre Polynomial, a result that may seem surprising, since the θ dependence in the Regge-Wheeler gauge is given by the expression $h(\theta)=sin\theta\partial_\theta P_l(cos\theta)$. In the Regge-Wheeler gauge, however, the product $l(l+1)$ appears in the radial equation, while the same quantity appearing here is due to the eigenvalue of Legendre equation. Our results have been summarized in Tables 1, 2, and 3. Although the quasinormal modes for the Schwarzschild solution had already been obtained—see e.g. Table III of Ref. [13]—we have also shown them, for comparison.

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