



Brazilian Journal of Physics
ISSN: 0103-9733
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Sociedade Brasileira de Física
Brasil

Koide, Tomoi; Ramos, Rudnei O.; Vicente, Gustavo S.
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Brazilian Journal of Physics, vol. 45, núm. 1, 2015, pp. 102-111
Sociedade Brasileira de Física
São Paulo, Brasil

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Bivelocity Picture in the Nonrelativistic Limit of Relativistic Hydrodynamics

Tomoi Koide · Rudnei O. Ramos · Gustavo S. Vicente

Received: 12 October 2014 / Published online: 17 December 2014
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Abstract We discuss the nonrelativistic limit of the relativistic Navier-Fourier-Stokes (NFS) theory. The next-to-leading order relativistic corrections to the NFS theory for the Landau-Lifshitz fluid are obtained. While the lowest order truncation of the velocity expansion leads to the usual NFS equations of nonrelativistic fluids, we show that when the next-to-leading order relativistic corrections are included, the equations can be expressed concurrently with two different fluid velocities. One of the fluid velocities is parallel to the conserved charge current (which follows the Eckart definition) and the other one is parallel to the energy current (which follows the Landau-Lifshitz definition). We compare this next-to-leading order relativistic hydrodynamics with bivelocity hydrodynamics, which is one of the generalizations of the NFS theory and is formulated in such a way to include the usual mass velocity and also a new velocity, called the volume velocity. We find that the volume velocity can be identified with the velocity obtained in the Landau-Lifshitz definition. Then, the structure of bivelocity hydrodynamics, which is derived using various nontrivial assumptions, is reproduced in the NFS theory including the next-to-leading order relativistic corrections.

Keywords Nonrelativistic hydrodynamics · Relativistic corrections · Bivelocity hydrodynamics

1 Introduction

Macroscopic matter, such as fluids, consists of extraordinary large number of microscopic particles and the dynamics is determined by solving highly coupled equations. However, it is also known that the long-wavelength and low-frequency behaviors are approximately described by a coarse-grained dynamics called the Navier-Fourier-Stokes (NFS) theory. This coarse-grained dynamics is much more tractable than the original microscopic dynamics and has been applied to various nonrelativistic collective phenomena successfully. Thus, it is quite natural to expect that this approach is also useful for the application to relativistic phenomena. As a matter of fact, relativistic hydrodynamics has been used to study relativistic collective behaviors in astrophysics, cosmology, and nuclear physics.

Despite the widespread use of relativistic hydrodynamic models, their theoretical properties are still not fully understood because of the difficulties inherent in the relativistic kinematics. For example, it is well known that first-order dissipative relativistic theories have problems concerning causality and generic stability, and in general, they do not have a well-posed initial value formulation (see, for example, Hiscock and Lindblom [1]). Other proposals for relativistic hydrodynamics led to dissipative relativistic theories satisfying causality¹ such as the second-order theory by Israel and Stewart [9, 10].

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¹However, the complexity of the generic stability of these theories led some authors to reconsider first-order theories in order to establish this property properly [2–8].

In this work, we investigate another aspect of relativistic hydrodynamics, that is, the nonuniqueness of the definitions of fluid velocities. For example, the conserved energy and charge densities in relativistic systems are given by the sum and subtraction of the particle and anti-particle contributions, respectively. Thus, the flows of energy and charge are, in general, not parallel to each other and we observe two different definitions for the fluid velocities. In one case, we can define the fluid velocity to be parallel to the charge current, and the other possibility is when the fluid velocity is chosen to be parallel to the energy current. The former case was introduced by Eckart [11] and the local rest frame associated with this velocity is called the Eckart frame. By definition, there are no spatial components of the charge current in the Eckart frame. It should be noted, however, that the Eckart definition is not applicable, for example, in relativistic heavy-ion collisions at vanishing baryon chemical potentials and in early universe cosmology, where the flow of a conserved charge is usually not considered. The other possibility for defining the fluid velocity, and also of common usage, was proposed by Landau-Lifshitz [12]. In this case, the fluid velocity is chosen to be parallel to the energy current, i.e., there are no spatial components of the energy current in this rest frame, called the Landau-Lifshitz frame.

As is well known, in the nonrelativistic NFS theory, the Eckart and Landau-Lifshitz rest frames are equivalent and conserved charge and energy of fluids are transported by a unique fluid velocity (the mass velocity). However, when relativistic corrections are considered, from the argument above, it is natural to expect that deviations from the NSF theory should depend on the choice of fluid velocity because the definitions of the local rest frame are changed. The next-to-leading order (NLO) corrections to the standard NSF fluid equations were first considered by Chandrasekhar [13, 14] for ideal fluids. That study was later extended by Greenberg [15] for nonideal fluids. However, only the Eckart definition of fluid velocity was considered and the role of the two fluid velocities in the modified hydrodynamics was not discussed. Moreover, the two different local rest frames required in the definitions of nonrelativistic limit of hydrodynamic variables, like the energy density and the conserved charge density, were not distinguished. We extend those earlier works and apply to the case of Landau-Lifshitz fluids, determining the NLO relativistic corrections to the NSF theory for the Landau-Lifshitz definition of fluid velocity in this paper.

Then, it is interesting to contrast the obtained NLO relativistic equations with a recent proposal of the modification of the NFS theory by Brenner [16–26] to clarify the role of the two fluid velocities. Since the velocity of a tracer particle of nonrelativistic fluids is not necessarily parallel to the

mass velocity, it is claimed in his bivelocity formulation that the existence of these two velocities should be included in a consistent formulation of the nonrelativistic hydrodynamics. This additional fluid velocity is called volume velocity. So far, there are various studies following this scenario (for a list of related works, although far from complete, see, e.g., Refs. [27–37]), but it is still controversial whether this bivelocity scenario is realized or not.

As mentioned, as far as the presence of the two definitions of velocities is concerned, the bivelocity argument is similar to what is familiar in the community of relativistic hydrodynamics. Then, it is interesting to discuss the structure of the NLO relativistic corrections from the point of view of bivelocity hydrodynamics because, as will be discussed in this paper, the corrections stem from the fact that the nonrelativistic energy density is defined in the Landau-Lifshitz frame, while the nonrelativistic conserved charge density is in the Eckart frame. In other words, the NLO relativistic corrections are directly affected by the difference of the two fluid velocities. Therefore, the detailed analysis of the NLO corrections is useful even to inspect the consistency of the structure of bivelocity hydrodynamics.

In the present work, we also study the formulation of bivelocity hydrodynamics by comparing it to relativistic hydrodynamics.² We start by considering the nonrelativistic limit of relativistic hydrodynamics in the Landau-Lifshitz frame, and we show that the standard NFS theory is reproduced in the leading order approximation. Moreover, it is found that the derived hydrodynamics at the NLO can be cast into a form of bivelocity hydrodynamics which is generalized so as to permit to include the effect of relativistic corrections.

The rest of this paper is organized as follows. In Section 2, we briefly summarize relativistic hydrodynamics and discuss the different possibilities of the choice of fluid velocities in the context of both Landau-Lifshitz and Eckart theories. In Section 3, we express the nonrelativistic limit of the various hydrodynamic variables in the relativistic theory in terms of the corresponding nonrelativistic ones. The leading order truncation of the velocity expansion is implemented and we derive the NFS theory. In Section 4, we discuss the NLO corrections. Our results, including the NLO corrections, are then contrasted with bivelocity hydrodynamics in Section 5. Section 6 is devoted to the concluding remarks.

²It is well known that higher-order kinetic corrections to the NFS theory leads to the Burnett and super-Burnett equations [38]. The relation between these kinetic corrections and the bivelocity picture was discussed in Refs. [24, 25].

2 Relativistic Hydrodynamics

2.1 Ideal Fluid

In relativistic hydrodynamics, the energy-momentum tensor and the conserved charge current are expressed in terms of hydrodynamic variables describing the macroscopic motion of many-body systems.³ In the case of an ideal fluid, two proper scalar densities (ε and P) and one four-vector field (the four-velocity u^μ) are used to express the energy-momentum tensor,

$$T_0^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - g^{\mu\nu}P, \quad (1)$$

where the Lorentz four-velocity field u^μ is expressed as

$$u^\mu = \gamma(1, \mathbf{v}/c), \quad (2)$$

and $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c^2}$ is the usual Lorentz factor, with the spatial velocity \mathbf{v} and the speed of light c . The four-velocity field is normalized such that $u^\mu u_\mu = 1$ and $u^\mu = (1, 0, 0, 0)$ in the rest frame. We use $g^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$ as the flat space-time metric. Besides the energy-momentum tensor (1), a charge current is also defined and can be expressed in terms of one proper scalar density (n) and the four-velocity field as

$$N_0^\mu = nu^\mu, \quad (3)$$

or, equivalently, $n = N_0^\mu u_\mu$, with the four-velocity normalization given above.

It should be noted that the proper scalar densities ε , P and n coincide, respectively, with the energy density, the pressure and the charge densities *only* in the local rest frame because of the effect of the Lorentz contraction.

One can see that the introduced four-velocity field u^μ for an ideal fluid satisfies the following equation,

$$T_0^{\mu\nu} u_\nu = \varepsilon u^\mu, \quad (4)$$

where εu^μ is interpreted as the energy current. This equation means that u^μ is parallel to the energy current. On the other hand, from (3), one can see that this velocity is also parallel to N_0^μ . Therefore, we conclude that there is no deviation between the energy current and the charge current in an ideal fluid and, hence, there is no ambiguity for the definition of the fluid velocity. However, this situation changes when the effects of dissipation are taken into account, which is the case of nonideal fluids.

2.2 Nonideal Fluids

By using $T_0^{\mu\nu}$ and N_0^μ , which were introduced above, the general energy-momentum tensor and conserved charge current, in the presence of dissipative effects, are changed, respectively, to

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}, \quad (5)$$

$$N^\mu = N_0^\mu + \Delta N^\mu, \quad (6)$$

where

$$\Delta T^{\mu\nu} = -(g^{\mu\nu} - u^\mu u^\nu)\Pi + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu}, \quad (7)$$

$$\Delta N^\mu = v^\mu, \quad (8)$$

and where Π is the bulk viscous scalar pressure, $\pi^{\mu\nu}$ is the shear viscous tensor, h^μ is the heat current, and v^μ is the diffusion current. These quantities satisfy the following orthogonal conditions,

$$u^\mu h_\mu = 0, \quad (9)$$

$$u^\mu v_\mu = 0, \quad (10)$$

$$u_\mu \pi^{\mu\nu} = 0. \quad (11)$$

In addition, the shear viscous tensor is traceless, $\pi^\mu_\mu = 0$. These dissipative quantities will be explicitly defined below, in Section 3.

The four new variables, Π , h_μ , v_μ , and $\pi_{\mu\nu}$, are introduced to represent the dissipative effects. However, we can reduce this number of variables from four to three under an appropriate choice for the four-velocity field in the context of nonideal fluids.

2.2.1 Nonideal Fluids in the Landau-Lifshitz Frame

The Landau-Lifshitz fluid velocity is defined to satisfy the following condition,

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu. \quad (12)$$

Substituting the general expression of the energy momentum tensor (5) into (12), we obtain

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu + h^\mu. \quad (13)$$

Thus, in the Landau-Lifshitz definition of fluid velocity, $h^\mu = 0$.

In short, the energy-momentum tensor and the conserved charge current in the Landau-Lifshitz theory are then given by

$$T^{\mu\nu} = (\varepsilon_L + P_L + \Pi_L)u_L^\mu u_L^\nu - g^{\mu\nu}(P_L + \Pi_L) + \pi_L^{\mu\nu}, \quad (14)$$

$$N^\mu = n_L u_L^\mu + v^\mu. \quad (15)$$

Here, the index L indicates the quantities defined in the Landau-Lifshitz theory. From the latter equation, one can also notice that u_L^μ is not parallel to N_L^μ due to the diffusion

³The time scale of the evolution of non-conserved quantities are considered to be short and these are usually not included as hydrodynamic variables.

current v^μ . This will bring to another possibility of choice for the fluid four-velocity field.

2.2.2 Nonideal Fluids in the Eckart Frame

In the Eckart frame, the velocity is defined to satisfy the condition

$$N^\mu = nu^\mu. \quad (16)$$

Substituting the general expression of the conserved charge current (6) in (16), we obtain

$$v^\mu = 0. \quad (17)$$

Likewise, the energy-momentum tensor and the conserved charge current in the Eckart theory are defined as

$$T^{\mu\nu} = (\varepsilon_E + P_E + \Pi_E)u_E^\mu u_E^\nu - g^{\mu\nu}(P_E + \Pi_E) + h^\mu u_E^\nu + h^\nu u_E^\mu + \pi_E^{\mu\nu}, \quad (18)$$

$$N^\mu = n_E u_E^\mu. \quad (19)$$

Here, the index E is used to indicate the quantities defined in the Eckart frame.

Evaluating $T^{\mu\nu}(u_E)_\nu$ with the help of the orthogonality conditions for h^μ (9) and $\pi^{\mu\nu}$ (11), we obtain

$$T^{\mu\nu}(u_E)_\nu = \varepsilon_E u_E^\mu + h^\mu. \quad (20)$$

From the above equation, one can notice that u_E^μ is not parallel to $T^{\mu\nu}(u_E)_\nu$ due to the heat current h^μ .

2.3 Connecting the Landau-Lifshitz and Eckart Definitions of Fluid Velocity

In order to discuss the nonrelativistic limit of relativistic hydrodynamics, it is necessary to express the relativistic hydrodynamic variables in terms of those defined in the NFS theory. As a first example, following Landau and Lifshitz [12], the energy density is given by the proper scalar density ε_L expressed in terms of the mass density ρ_m and the internal energy per unit mass \hat{u} as

$$\varepsilon_L = \frac{\rho_m}{\gamma_L} (c^2 + \hat{u}). \quad (21)$$

As concerning the expression for the conserved charge density n_L , it is much less trivial. In nonrelativistic case, the corresponding conserved charge density is defined by the number of charged particle per unit volume. In the Eckart frame, it is trivial to show that n_E can be expressed in terms of the nonrelativistic conserved charge density as

$$u_E^\mu N_\mu \equiv n_E = \frac{q}{m} \frac{\rho_m}{\gamma_E}. \quad (22)$$

Note that in the case where the nonrelativistic limit is permitted, the contribution from the anti-particle is negligibly

small. On the other hand, n_L is the proper scalar density associated with u_L^μ and it is clear that

$$n_L = u_L^\mu N_\mu \neq u_E^\mu N_\mu = n_E. \quad (23)$$

That is, to express n_L with ρ_m , we need to know the relation between n_L and n_E .

Because N^μ can be expressed in the two different ways with u_L^μ and u_E^μ , one can easily find that

$$n_L = \sqrt{n_E^2 - v^\mu v_\mu}. \quad (24)$$

Substituting (22) on the right-hand side, we can express n_L in terms of ρ_m .

In short, to introduce nonrelativistic hydrodynamic variables \hat{u} and ρ_m simultaneously, we need to consider, e.g., both the Eckart and the Landau rest frames concomitantly. As was mentioned in the introduction, this is the reason why the difference of the two fluid velocities affects the NLO relativistic correction terms. It should be mentioned that this issue has not been discussed in Refs. [13–15].

3 Fluid Equations in the Nonrelativistic Limit

Let us now discuss the nonrelativistic limit for the hydrodynamic equations. As the first step to obtain the nonrelativistic limit of the hydrodynamic variables, it is necessary to specify the irreversible variables.

3.1 Nonrelativistic Limit of Hydrodynamic Variables

We obtain the relativistic covariant expression of the NFS theory when the linear irreversible thermodynamics (LIT) is applied to determine the irreversible currents, satisfying the positivity of the entropy production rate. However, as was pointed out in Ref. [39, 40], such a theory is inconsistent with the relativistic kinematics in the sense that the stability of the relativistic fluid changes depending on the choice of reference frames. There are several proposals to calculate these currents, but there is still no established model (see, for example, Ref. [41] for references and discussions regarding this issue). However, in the present argument, our intention is to discuss the behavior of relativistic hydrodynamics in the nonrelativistic limit and, therefore, the inconsistency mentioned above will not be of relevance. Thus, using the following results obtained in LIT [12, 42], the linear expressions for the irreversible variables can be expressed as

$$v^\mu = \frac{\kappa}{c} \left(\frac{n_L T_L}{\varepsilon_L + P_L} \right)^2 \Delta_L^{\mu\nu} \partial_\nu \frac{\mu_{rel}^L}{T_L}, \quad (25)$$

$$\pi_L^{\mu\nu} = 2c\eta \Delta_L^{\mu\nu\alpha\beta} \partial_\alpha (u_L)_\beta, \quad (26)$$

$$\Pi_L = -c\zeta \partial_\mu u_L^\mu, \quad (27)$$

where $\Delta_L^{\mu\nu}$ and $\Delta_L^{\mu\nu\alpha\beta}$ are projection operators defined as

$$\Delta_L^{\mu\nu} = g^{\mu\nu} - u_L^\mu u_L^\nu, \quad (28)$$

$$\Delta_L^{\mu\nu\alpha\beta} = \frac{1}{2} \left(\Delta_L^{\mu\alpha} \Delta_L^{\nu\beta} + \Delta_L^{\mu\beta} \Delta_L^{\nu\alpha} \right) - \frac{1}{3} \Delta_L^{\mu\nu} \Delta_L^{\alpha\beta}. \quad (29)$$

Here, T_L and μ_{rel}^L are the temperature and the chemical potential, which are obtained by employing the local equilibrium in the Landau-Lifshitz frame. Let us recall that the chemical potentials in relativistic and nonrelativistic systems, μ_{rel}^L and μ_{nrel}^L , respectively, are related through $\mu_{rel}^L = mc^2 + \mu_{nrel}^L$. The other coefficients, κ , η and ζ , appearing in (25), (26), and (27), represent the coefficients of the thermal conductivity, the shear viscosity, and the bulk viscosity, respectively.

In order to obtain the nonrelativistic limit of relativistic hydrodynamics, we need to perform an expansion of the hydrodynamic variables in powers of v_L/c , which is a velocity expansion. For the linear irreversible variables, we find that the leading order contributions are

$$v^i \propto \mathcal{O}(v_L^3/c^3), \quad (30)$$

$$\pi_L^{ij} \propto \mathcal{O}(v_L^0/c^0), \quad (31)$$

$$\Pi_L \propto \mathcal{O}(v_L^0/c^0). \quad (32)$$

Likewise, for the other components, we have that (see also the argument in Ref. [12])

$$v^0 \propto \mathcal{O}(v_L^4/c^4), \quad (33)$$

$$\pi_L^{00} \propto \mathcal{O}(v_L^2/c^2), \quad (34)$$

$$\pi_L^{0i} = \pi_L^{i0} \propto \mathcal{O}(v_L/c). \quad (35)$$

It can be noted that only the purely spatial components of the irreversible variables, (30) and (31), are important at the leading order in a velocity expansion, whereas the other components contribute only at higher orders.

By using the velocity expansion, one can obtain the expression of n_L from (22) and (24) as

$$n_L = \frac{q}{m} \rho_m - \frac{q}{2m} \rho_m \frac{\mathbf{v}_L^2}{c^2} - \frac{1}{c^4} \frac{q}{8m} \rho_m \mathbf{v}_L^4 + \mathcal{O}(v_L^6/c^6). \quad (36)$$

Substituting (36) into (24), the fundamental relation between the two fluid velocities in the nonrelativistic limit is derived as

$$v_E^i - v_L^i = \frac{mc}{q\rho_m} v^i + \mathcal{O}(v_L^4/c^4), \quad (37)$$

where the diffusion current v^i can be obtained from (25). Since $v^i \propto \mathcal{O}(v_L^3/c^3)$, the two fluid velocities differ only at second order in the relativistic corrections, i.e., $v_E^i - v_L^i \propto \mathcal{O}(v_L^2/c^2)$.

The proper scalar energy density ε_L , when expanded in v_L/c , gives

$$\begin{aligned} \varepsilon_L = & \rho_m \left(c^2 + \hat{u} \right) + \frac{1}{2} \rho_m \mathbf{v}_L^2 \\ & - \frac{1}{c^2} \left[\frac{1}{2} \rho_m \mathbf{v}_L^2 \hat{u} + \frac{1}{8} \rho_m \left(\mathbf{v}_L^2 \right)^2 \right] + \mathcal{O}(v_L^4/c^4). \end{aligned} \quad (38)$$

Note that the above equation, defined in the Landau-Lifshitz frame and derived from (21), gives only part of the relativistic corrections to the energy defined from T^{00} .

3.2 Leading Order Truncation and Navier-Fourier-Stokes Theory

The conservation of energy, momentum, and charge are expressed by the equations of continuity of the energy-momentum tensor and the conserved charge current,

$$\partial_\mu T_L^{\mu\nu} = 0, \quad (39)$$

$$\partial_\mu N_L^\mu = 0. \quad (40)$$

By using (14) and (15), we can obtain the relativistic hydrodynamic model of Landau and Lifshitz, the Landau-Lifshitz theory. The nonrelativistic limit of this theory can be obtained from the substitution of the relativistic hydrodynamic variables by the leading order expressions for these variables that we have obtained in the previous section. In the present work, we adapt the following orders for the hydrodynamic variables \hat{u} , P_L and μ_{nrel}^L ,

$$\frac{\hat{u}}{c^2}, \frac{P_L}{c^2}, \frac{\mu_{nrel}^L}{c^2} \propto \mathcal{O}(v_L^2/c^2). \quad (41)$$

Then, as we will see soon later, the NSF theory is reproduced in the leading order approximation.

As it was shown in (37), the difference of the two fluid velocities, \mathbf{v}_E and \mathbf{v}_L , appears only at order $\mathcal{O}(v_L^2/c^2)$. By truncating at $\mathcal{O}(v_L^0/c^0)$ of the velocity expansion, we simply have that

$$\mathbf{v}_E = \mathbf{v}_L = \mathbf{v}. \quad (42)$$

This equality shows that the velocities \mathbf{v}_L and \mathbf{v}_E define the same rest frame at the leading order in the v_L/c expansion. Thus, the energy and mass flows are both parallel to the fluid velocity, which is the case of the usual nonrelativistic hydrodynamics.

The hydrodynamic variables P_L , Π_L , and π_L^{ij} occurring in the energy-momentum tensor $T_L^{\mu\nu}$ are obtained by employing the local equilibrium in the Landau-Lifshitz frame. It should be noted, however, that there is a unique rest frame because of (42) and these hydrodynamic variables do not have any frame dependences in the leading order truncation. Therefore, we can verify that the nonrelativistic

limit of relativistic hydrodynamics reproduces the NFS theory,

$$\partial_t \rho_m + \nabla \cdot (\rho_m \mathbf{v}_L) = 0, \quad (43)$$

$$\rho_m (\partial_t + \mathbf{v}_L \cdot \nabla) v_L^i = - \sum_{j=1}^3 \nabla_j \mathcal{P}_{L0}^{ij}, \quad (44)$$

$$\rho_m (\partial_t + \mathbf{v}_L \cdot \nabla) \hat{u} = - \nabla \cdot \mathbf{q} - \sum_{j,k=1}^3 \mathcal{P}_{L0}^{jk} \nabla_j v_L^k, \quad (45)$$

where the stress tensor \mathcal{P}_{L0}^{ij} is given by

$$\mathcal{P}_{L0}^{ij} = \delta^{ij} (P_L + \Pi_L) + \pi_L^{ij}, \quad (46)$$

with the viscosities Π_L and π_L^{ij} , which are also used in the next Section, are defined by their usual leading order expressions [12]

$$\Pi_L = -\zeta \nabla \cdot \mathbf{v}_L, \quad (47)$$

$$\pi_L^{ij} = -\eta \left(\nabla_i v_{Lj} + \nabla_j v_{Li} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v}_L \right), \quad (48)$$

while the heat current vector \mathbf{q} is defined by

$$\mathbf{q} = -\kappa \nabla T_L. \quad (49)$$

4 Next-to-Leading Order Relativistic Corrections

To determine the difference between the two fluid velocities, we calculate the NLO corrections for the NFS theory. By keeping the relativistic corrections terms up to $\mathcal{O}(v_L^2/c^2)$ in the fluid equations (43), (44), and (45), we obtain the mass, the momentum, and the energy equations, respectively, as

$$\partial_t \rho_m + \nabla \cdot (\rho_m \mathbf{v}_L) = -\frac{mc}{e} \nabla \cdot \mathbf{v} + \mathcal{O}(v_L^4/c^4), \quad (50)$$

$$\begin{aligned} \rho_m (\partial_t + \mathbf{v}_L \cdot \nabla) v_L^i = & - \sum_{j=1}^3 \left[\delta_{ij} - \frac{\delta_{ij}}{c^2} \left(\frac{\mathbf{v}_L^2 + 2\hat{u}}{2} + \frac{P_L + \Pi_L}{\rho_m} \right) - \frac{1}{c^2} \frac{\pi_L^{ij}}{\rho_m} \right] \\ & \times \nabla_j (P_L + \Pi_L) \\ & - \sum_{j,k=1}^3 \left[\delta_{ij} - \frac{\delta_{ij}}{c^2} \left(\frac{\mathbf{v}_L^2 + 2\hat{u}}{2} + \frac{P_L + \Pi_L}{\rho_m} \right) \right. \\ & \left. - \frac{1}{c^2} \left(v_L^i v_L^j + \frac{\pi_L^{ij}}{\rho_m} \right) \right] \nabla_k \pi_L^{jk} \\ & + \frac{1}{c^2} \sum_{j,k=1}^3 \left(v_L^i \pi_L^{jk} + v_L^k \pi_L^{ij} \right) \nabla_k v_L^j - \frac{1}{c^2} v_L^i \partial_t (P_L + \Pi_L) \\ & - \frac{1}{c^2} \sum_{j=1}^3 v_L^j \partial_t \pi_L^{ij} + \mathcal{O}(v_L^4/c^4), \end{aligned} \quad (51)$$

$$\begin{aligned} \rho_m (\partial_t + \mathbf{v}_L \cdot \nabla) \hat{u} = & - \sum_{i,j=1}^3 \left\{ \left(1 + \frac{1}{2} \frac{\mathbf{v}_L^2}{c^2} \right) \left[\delta_{ij} (P_L + \Pi_L) + \pi_L^{ij} \right] \right. \\ & \left. - \frac{1}{c^2} \sum_{k=1}^3 v_L^i v_L^k \pi_L^{jk} \right\} \nabla_i v_L^j \\ & + \frac{1}{c^2} \sum_{i=1}^3 \left(v_L^i \frac{P_L + \Pi_L}{\rho_m} + \sum_{k=1}^3 v_L^k \frac{\pi_L^{ik}}{\rho_m} \right) \\ & \times \left[\nabla_i (P_L + \Pi_L) + \sum_{j=1}^3 \nabla_j \pi_L^{ij} \right] \\ & + \frac{mc^3}{q} \sum_{i=1}^3 \nabla_i \left(v^i - \sum_{j=1}^3 \frac{v_L^i v_L^j}{c} \frac{v^j}{c} \right) \\ & + \frac{mc}{q} \hat{u} \nabla \cdot \mathbf{v} + \mathcal{O}(v_L^4/c^4). \end{aligned} \quad (52)$$

The irreversible variables, which are also expanded up to the next-to-leading order $\mathcal{O}(v_L^2/c^2)$, are given by

$$\begin{aligned} v^i = & \kappa \frac{q}{mc^3} \left[\left(\nabla_i - \frac{\hat{u}}{c^2} \nabla_i + \frac{v_L^i}{c^2} D_L \right) T_L \right. \\ & \left. - \frac{T_L}{\rho_m c^2} \nabla_i P_L \right], \end{aligned} \quad (53)$$

$$\begin{aligned} \pi_L^{ij} = & -\eta \left\{ \partial_j v_{Li} + \partial_i v_{Lj} + \frac{1}{c^2} \left[(v_{Li} \partial_j + v_{Lj} \partial_i) \left(\frac{\mathbf{v}_L^2}{2} \right) \right. \right. \\ & \left. \left. + \frac{\mathbf{v}_L^2}{2} (\partial_j v_{Li} + \partial_i v_{Lj}) - v_{Lj} D_L v_{Li} - v_{Li} D_L v_{Lj} \right] \right\} \\ & + \frac{2}{3} \eta \left\{ \nabla \cdot \mathbf{v}_L \delta_{ij} + \frac{1}{c^2} \left[\left(\frac{\mathbf{v}_L^2}{2} \delta_{ij} + v_{Li} v_{Lj} \right) \nabla \cdot \mathbf{v}_L \right. \right. \\ & \left. \left. + D_L \left(\frac{\mathbf{v}_L^2}{2} \right) \delta_{ij} \right] \right\}, \end{aligned} \quad (54)$$

$$\Pi_L = -\zeta \left\{ \nabla \cdot \mathbf{v}_L + \frac{1}{c^2} \left[D_L \left(\frac{\mathbf{v}_L^2}{2} \right) + \frac{\mathbf{v}_L^2}{2} \nabla \cdot \mathbf{v}_L \right] \right\}. \quad (55)$$

One can notice that in order to satisfy the NLO energy and momentum conservation equations, the relativistic correction terms appearing in the definitions of energy and momentum should be considered. The expressions for the nonrelativistic energy density and momentum current, $\rho_m \hat{e} \equiv \rho_m \left(\frac{1}{2} \mathbf{v}^2 + \hat{u} \right)$ and $\rho_m \mathbf{m} \equiv \rho_m \mathbf{v}$, respectively, which are conserved in the NFS theory, are no longer conserved in the NLO equations. However, the appropriate expressions in the relativistic context are exactly those obtained from the components of the relativistic energy-momentum tensor, $\rho_m \hat{e}_L = T^{00}$ and $\rho_m \mathbf{m}_L^i = T^{0i}/c$, respectively. Thus, from the energy-momentum tensor in the Landau-Lifshitz frame,

(14), the expressions for the energy per unit mass and for the momentum per unit mass, that account up to the $\mathcal{O}(v_L^2/c^2)$ relativistic corrections, are given, respectively, by

$$\hat{e}_L = c^2 + \left(\frac{v_L^2}{2} + \hat{u} \right) + \frac{1}{c^2} \left[\left(\frac{3}{8} v_L^2 + \frac{1}{2} \hat{u} \right) \mathbf{v}_L^2 + \sum_{i,j=1}^3 \frac{(P_L + \Pi_L) \delta^{ij} + \pi_L^{ij}}{\rho_m} v_L^i v_L^j \right], \quad (56)$$

and

$$m_L^i = v_L^i + \frac{1}{c^2} \left[\left(\frac{1}{2} v_L^2 + \hat{u} \right) v_L^i + \sum_{j=1}^3 \frac{(P_L + \Pi_L) \delta^{ij} + \pi_L^{ij}}{\rho_m} v_L^j \right]. \quad (57)$$

From these expressions, (50), (51), and (52) can be cast into much simpler forms,

$$\partial_t \rho_m + \nabla \cdot (\rho_m \mathbf{v}_E) = 0, \quad (58)$$

$$\rho_m (\partial_t + \mathbf{v}_E \cdot \nabla) m_L^i = -(\nabla \cdot \mathcal{P}_L)^i, \quad (59)$$

$$\rho_m (\partial_t + \mathbf{v}_E \cdot \nabla) \hat{e}_L = -\nabla \cdot \mathbf{j}_L - \sum_{i=1}^3 \partial_i (\mathcal{P}_L \cdot \mathbf{v}_L)^i, \quad (60)$$

where we have used (37) to take \mathbf{v}_E into account. Here, $(\nabla \cdot \mathcal{P}_L)^i = \sum_{j=1}^3 \partial_j \mathcal{P}_L^{ij}$ and $(\mathcal{P}_L \cdot \mathbf{v}_L)^i = \sum_{j=1}^3 \mathcal{P}_L^{ij} v_L^j$. The spatial components for the stress tensor \mathcal{P}_L and for the heat current vector \mathbf{j}_L are, respectively, given by

$$\mathcal{P}_L^{ij} = \mathcal{P}_{L0}^{ij} - \frac{1}{c^2} \sum_{k=1}^3 \pi_L^{ik} v_L^k v_L^j - \frac{mc}{q} m_L^i v_L^j, \quad (61)$$

and

$$\begin{aligned} j_L^i &= -\frac{mc}{q} \left(\hat{e}_L v^i - \sum_{j=1}^3 m_L^i v^j v_L^j \right) \\ &= -\kappa \left[\left(\nabla_i + \frac{v_L^2}{c^2} \nabla_i + \frac{v_L^i}{c^2} \partial_t \right) T_L - \frac{T_L}{\rho_m c^2} \nabla_i P_L \right], \end{aligned} \quad (62)$$

where, in the calculation of j_L^i , we have used the explicit expression for v^i given by (53). One can easily check that the relativistic quantities $\rho_m \hat{e}_L$ and $\rho_m \mathbf{m}_L$ are conserved densities in the next-to-leading order relativistic hydrodynamics, as well as $\rho_m \hat{e}$ and $\rho_m \mathbf{m}$ are in the NFS theory.

The hydrodynamic equations given by (58), (59), and (60) represent the main result of this work. Most importantly, we note that the two fluid velocities \mathbf{v}_E and \mathbf{v}_L appear in the above equations. This shows that it is possible to work concurrently with two different velocities in the NLO hydrodynamics. Of course, one of the velocities is eliminated by substituting the relation given by (37), but the above expressions are essential to compare with bivelocity hydrodynamics in the next section (see also the comment

below (65)). We also notice that the last term on the right-hand side of the energy equation, (60), associated with the work done by the stress tensor, is implemented following \mathbf{v}_L . On the other hand, in bivelocity hydrodynamics, this term follows \mathbf{v}_V . The discussion concerning this term plays an important role in the comparison.

Before closing this section, as a final remark concerning the stress tensor defined by (61), we note that it is asymmetric. But this is only because we wrote (59) in a way it can be compared to the bivelocity hydrodynamics in the next section. In fact, the momentum equation in the NLO, (59), can be expressed with a symmetric tensor as

$$\partial (\rho_m m_L^i) = - \sum_j \partial_j \tilde{\mathcal{P}}_L^{ij}, \quad (63)$$

where, when we move the term depending on the Eckart fluid velocity in (59) to the right-hand-side of that equation, and upon using also (58), we obtain that

$$\begin{aligned} \tilde{\mathcal{P}}_L^{ij} &= \mathcal{P}_{L0}^{ij} \\ &+ \frac{1}{c^2} \left(\rho_m c^2 + \frac{\rho_m}{2} v_L^2 + \hat{u} + P_L + \Pi_L \right) v_L^i v_L^j, \end{aligned} \quad (64)$$

which is symmetric.

5 Comparison with Bivelocity Hydrodynamics

As it was shown in the previous section, the NLO relativistic corrections to the NFS theory leads to a new hydrodynamic model that is described by the two different fluid velocities, \mathbf{v}_E and \mathbf{v}_L . In this section, we compare this NLO equations with bivelocity hydrodynamics.

5.1 Velocity in Landau-Lifshitz Frame and Volume Velocity

Bivelocity hydrodynamics is constructed with the mass velocity \mathbf{v}_M and the volume velocity \mathbf{v}_V . The mass velocity is defined to be parallel to the mass flow as usual. The origin of the volume velocity is attributed to the fact that the flow of the constituent particles of the fluid (velocity of the tracer particles) is not necessarily parallel to the mass velocity [16–26].

The velocity \mathbf{v}_E in relativistic hydrodynamics is defined to be parallel to the conserved charge current and, therefore, it is quite natural to ask if it can be identified with the mass velocity. However, it is not trivial to know in principle whether \mathbf{v}_L corresponds to the volume velocity because the physical meaning of these two velocities seem to be different. Thus, we need to investigate the explicit relation between \mathbf{v}_L and \mathbf{v}_V .

The relation between \mathbf{v}_M and \mathbf{v}_V is known in the context of bivelocity hydrodynamics and is given by [16–21]

$$\begin{aligned} \mathbf{v}_M - \mathbf{v}_V &= -C_v \frac{\eta}{\rho_m} \nabla \ln \rho_m \\ &= -C_v \frac{\eta}{\rho_m^2} \left\{ \left(\frac{\partial \rho_m}{\partial T} \right)_P \nabla T + \left(\frac{\partial \rho_m}{\partial P} \right)_T \nabla P \right\}, \end{aligned} \quad (65)$$

where the coefficient C_v is a free parameter that can be obtained once a particular application or theory is given. For example, some results for the coefficient C_v can be found in Table I of Ref. [24].

It is clear from (65) that the volume velocity is expressed in terms of the mass velocity and, thus, these are not independent. This is simply the notation introduced in bivelocity hydrodynamics, and for the sake of the comparison with this theory, we will also express our results of the previous section by using the two velocities. In this manner, although the existence of the two fluid velocities is assumed in bivelocity hydrodynamics, they do not represent independent variables.

On the other hand, as it was shown in (37), the difference between the two fluid velocities in relativistic hydrodynamics is given by

$$\begin{aligned} \mathbf{v}_E - \mathbf{v}_L &= \frac{mc}{q\rho_m} v^i \\ &= \kappa \frac{1}{c^2 \rho_m} \left[\left(\nabla_i - \frac{\hat{u}}{c^2} \nabla_i + \frac{v_L^i}{c^2} D_L \right) T_L \right. \\ &\quad \left. - \frac{T_L}{\rho_m c^2} \nabla_i P_L \right], \end{aligned} \quad (66)$$

where $D_L = \partial_t + \mathbf{v}_L \cdot \nabla$ and we have used the explicit expression of v^i , (53). Here, we have expanded v^i using the thermodynamic relation,

$$\nabla_i \frac{\mu_{rel}^L}{T_L} = - \left(\frac{\varepsilon_L + P_L}{n_L T_L^2} \right) \nabla_i T_L + \left(\frac{1}{n_L T_L} \right) \nabla_i P_L. \quad (67)$$

It can be verified that $D_L T_L$ is approximately given by $\nabla^2 T_L$ when the contribution from the energy dissipation is sufficiently small in the energy equation. Then, the third term in the right-hand side of (66) is a higher-order contribution of the spatial derivative and can be neglected. The difference between \mathbf{v}_E and \mathbf{v}_L is, then, determined by the gradients of temperature and pressure similarly to the case of \mathbf{v}_M and \mathbf{v}_V . Therefore, it can be concluded that the volume velocity in bivelocity hydrodynamics \mathbf{v}_V is related to the velocity in the Landau-Lifshitz frame \mathbf{v}_L .

5.2 Equations in Bivelocity Hydrodynamics

Let us investigate further whether the NLO equations given by (58), (59), and (60) can have the same structure as the fluid equations in bivelocity hydrodynamics.

In the following, we use $\mathbf{v}_V = \mathbf{v}_L$ and $\mathbf{v}_M = \mathbf{v}_E$ to express the equations in bivelocity hydrodynamics to avoid confusion in the comparison.

The bivelocity hydrodynamics model is characterized by the following set of equations [24, 25],

$$\partial_t \rho_m + \nabla \cdot (\rho_m \mathbf{v}_E) = 0, \quad (68)$$

$$\rho_m (\partial_t + \mathbf{v}_E \cdot \nabla) m_{bi}^i = -(\nabla \cdot P_{L0})^i, \quad (69)$$

$$\rho_m (\partial_t + \mathbf{v}_E \cdot \nabla) \hat{e}_{bi} = -\nabla \cdot \mathbf{j}_u - \nabla \cdot (P_{L0} \mathbf{v}_L), \quad (70)$$

where

$$\mathbf{m}_{bi} = \mathbf{v}_E, \quad (71)$$

$$\hat{e}_{bi} = \frac{\mathbf{v}_E^2}{2} + \hat{u}. \quad (72)$$

On the other hand, the heat current vector in bivelocity hydrodynamics, by using LIT, is given by

$$\begin{aligned} \mathbf{j}_u &= - \left[\kappa + \frac{C_v \eta}{\rho_m^2} \left(\frac{\partial \rho_m}{\partial T} \right)_P \right] \nabla T \\ &\quad + \frac{C_v \eta}{\rho_m^2} \left[T \left(\frac{\partial \rho_m}{\partial T} \right)_P - P \left(\frac{\partial \rho_m}{\partial P} \right)_T \right] \nabla P. \end{aligned} \quad (73)$$

One can note that the heat current is given by the linear combination of the two thermodynamic forces: one is for the pure heat conduction, ∇T , and the other is induced by the existence of the volume velocity, ∇P . Then, because of the Curie principle, the most general expression is given by their linear combination.

5.3 Comparison Between the Two Approaches

By comparing the NLO equations (58), (59), and (60) with those of the bivelocity hydrodynamics, (68), (69), and (70), one can find that the structures of the two theories are similar. In fact, the various assumptions used in the derivation of bivelocity hydrodynamics are naturally reproduced in the NLO equations.

In both theories, the equations are expressed with the material (substance) derivative for the mass velocity \mathbf{v}_E . That is, the evolution of the hydrodynamic variables are defined in terms of the fluid element, which moves with the mass velocity. However, the work done by the stress tensor, which appears in the second terms on the right-hand side of the energy equation of each theory, and the forms of the bulk and shear viscosities are given in terms of the volume velocity \mathbf{v}_L , but not \mathbf{v}_E .

It is also verified that in both theories, the heat currents are induced even by the pressure gradient. In bivelocity hydrodynamics, this behavior is because LIT leads to the pressure gradient as the thermodynamic force associated with the volume velocity [24, 25]. On the other hand, in the NLO equations, the thermodynamic force associated with the diffusion current v^μ is given by the gradient $\nabla(\mu/T)$

and the pressure gradient is induced by the chemical potential dependence included in this term. These behaviors are assumed in the derivation of bivelocity hydrodynamics, while the very same are automatically reproduced in the NLO equations. The consistency that we found in the comparison above can be considered as an indication of support for the validity of the application of LIT for the construction of bivelocity hydrodynamics.

There are still qualitative differences that we cannot ignore: (1) the energy and momentum variables definitions in bivelocity hydrodynamics are given in terms of the mass velocity, which is argued to be the universal behavior [24], whereas in the NLO equations, these variables are defined in terms of the volume velocity and (2) the symmetric stress tensor in bivelocity hydrodynamics \mathcal{P}_{L0}^{ij} is replaced by an asymmetric one in the NLO equations \mathcal{P}_L^{ij} . However, these problems are essentially connected to and can be explained as an effect of the relativistic corrections, which are not considered in the original formulation of bivelocity hydrodynamics.

In the framework of bivelocity hydrodynamics, in fact, the definitions of energy and momentum are not trivial because of the existence of the two different fluid velocities. Then, in Ref. [24], it is assumed that the nonrelativistic forms of the energy and momentum are still held even in the formulation of bivelocity hydrodynamics. In other words, \mathbf{m}_{bi} and \hat{e}_{bi} are, respectively, given by the linear and quadratic functions of a certain velocity. Under this assumption, Brenner succeeded in showing that this velocity is given by the mass velocity. In short, (71) and (72) are derived. It is however obvious that this argument is not applicable to the NLO equations because the definitions of momentum and energy are modified due to the relativistic corrections. Therefore, we can still consider the momentum and energy per unit mass as functions of the volume velocity in the present case.

The modification of the velocity dependence in the energy and momentum of bivelocity hydrodynamics naturally leads to the introduction of an asymmetric stress tensor. It is difficult to predict the velocity dependence when the relativistic corrections are allowed to be included in the framework of bivelocity hydrodynamics. However, as a simplest example, let us consider the case where the momentum is simply $\mathbf{m}_{bi} = \mathbf{v}_L$. Then, the momentum equation (69) is expressed as

$$\partial_t (\rho_m \mathbf{v}_L^i) = - \sum_{j=1}^3 \nabla_j (\mathcal{P}_{L0}^{ij} + \rho_m \mathbf{v}_E^j \mathbf{v}_L^i). \quad (74)$$

Note that the second term on the right-hand side in (74) has two velocities: one comes from the definition of \mathbf{m}_{bi} and the other from the material derivative. As a conse-

quence, the second-rank tensor on the right-hand side in (74), $(\mathcal{P}_{L0}^{ij} + \rho_m \mathbf{v}_E^j \mathbf{v}_L^i)$, is *not* symmetric for the exchange of the indexes i and j , and hence, the angular momentum density defined by $\rho_m (\mathbf{r} \times \mathbf{m}_b^i) = \mathbf{r} \times \rho_m \mathbf{v}_L^i$ is not a conserved density. Therefore, to satisfy the angular momentum conservation in this case, the stress tensor \mathcal{P}_{L0}^{ij} should contain an asymmetric part that cancels the last term on the right-hand side, $\rho_m \mathbf{v}_E^j \mathbf{v}_L^i$. This conclusion is still the same even if we consider a more complex velocity dependence. In fact, one can easily confirm that the angular momentum density $\rho_m (\mathbf{r} \times \mathbf{m}_L^i)$ is conserved in the NLO (63).

In summary, the formulation of bivelocity hydrodynamics is based on various assumptions. Most of these assumptions (material derivatives, work done by the stress tensor, the forms of the viscosities, the thermodynamic force of the volume velocity) are automatically reproduced in the NLO equations. On the other hand, the stress tensor in bivelocity hydrodynamics is given by the symmetric form, while the corresponding stress tensor in the NLO equations is asymmetric. That is, the structure of the NLO equations do not reproduce bivelocity hydrodynamics completely. However, this is because bivelocity hydrodynamics is not constructed in such a way to include the relativistic corrections. When we generalize the argument to include the relativistic corrections, an asymmetric stress tensor emerges even in bivelocity hydrodynamics in order to satisfy the angular momentum conservation. That is, the nonrelativistic hydrodynamics with the NLO corrections is qualitatively equivalent to this generalized version of bivelocity hydrodynamics.

6 Concluding Remarks

In this work, we have discussed the nonrelativistic limit of relativistic hydrodynamics. In relativistic hydrodynamics, it is possible to define two fluid velocities: one is parallel to the energy current and the other can be defined to be parallel to the conserved charge current. The difference between these velocities disappears in the nonrelativistic limit and the NFS theory is reproduced.

From the results we have obtained, we do not observe any bivelocity effect in the nonrelativistic limit, but it does not necessarily mean that bivelocity hydrodynamics does not exist in the nonrelativistic regime. It is because the relativistic hydrodynamics used here is obtained by employing the linear irreversible thermodynamics, and hence, the possible nonlinear effects in the irreversible currents are not considered. If such effects are taken into account, the correction terms may appear even in this regime. As a matter of fact, there are arguments that the bivelocity effect can be induced from such nonlinearities (see, for example, Refs. [27–37]). How these nonlinearities also manifest in the context of

relativistic hydrodynamics is an interesting subject to be explored in a future work.

Afterwards, we have explicitly obtained the NLO relativistic corrections to the NFS theory for the case of the Landau-Lifshitz definition of fluid velocity. Previous studies of the NLO relativistic corrections to the NSF theory were available only for a Eckart fluid [15]. We believe that our results represent an important contribution to the study of cases where the Eckart scenario does not apply.

The derived NLO hydrodynamics can be expressed concurrently in terms of both fluid velocities, where one of them is a Eckart fluid velocity and the other the Landau-Lifshitz fluid velocity. Comparing this result with bivelocity hydrodynamics, we found that the Landau-Lifshitz velocity can be identified with the volume velocity in bivelocity hydrodynamics.

Using this identification, we have confirmed that most of the assumptions used in bivelocity hydrodynamics (material derivatives, work done by the stress tensor, the forms of the viscosities, the thermodynamic force of the volume velocity) are automatically reproduced in the NLO equations. The difference comes from the stress tensor; the stress tensor in the bivelocity hydrodynamics is symmetric, while the one in the NLO equations is asymmetric. However, this difference can be explained by the different origins of the volume velocity; in bivelocity hydrodynamics, the volume velocity is induced as a consequence of the definition of the diffusive flux of volume \mathbf{j}_v (which is absent in the NFS theory), while in the NLO equations, it appears as an effect of the relativistic corrections. Then, by discussing the symmetry properties of the stress tensors in connection to the angular momentum conservation in both theories, we have found that the argument of bivelocity hydrodynamics can be extended so as to include the relativistic corrections, and then the stress tensor is permitted to be asymmetric even in bivelocity hydrodynamics to satisfy the angular momentum conservation. In short, in the sense discussed above, the hydrodynamics including the NLO corrections is qualitatively equivalent to bivelocity hydrodynamics.

In the original idea of bivelocity hydrodynamics, the origin of the volume velocity is identified with the flow of the constituent particles of the fluid that is not parallel to the mass velocity, and this deviation is enhanced for the compressible fluid. However, as we have shown in the present work, a similar situation can be expected as the result of relativistic effects and that is possible to be observed even for incompressible fluids. The study we have performed in this work points out, thus, that analogous effects expected from the bivelocity picture can be obtained by observing the behavior of, for example, high energy fluids in cosmology and relativistic heavy-ion collisions. These are in fact areas of research that our results may have immediate applications and that are worth of future investigation.

Acknowledgments T. K. acknowledge discussions with G. S. Denicol, X. Huang, and L. S. García-Colin in the initial stages of the present study and financial support from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). R.O.R. is partially supported by research grants from CNPq and Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ). G. S. V. is supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES).

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