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Suppression of Growth by Multiplicative White Noise in a Parametric Resonant System

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Abstract The growth of the amplitude in a Mathieu-like equation with multiplicative white noise is studied. To obtain an approximate analytical expression for the exponent at the extremum on parametric resonance regions, a time-interval width is introduced. To determine the exponents numerically, the stochastic differential equations are solved by a symplectic numerical method. The Mathieu-like equation contains a parameter α determined by the intensity of noise and the strength of the coupling between the variable and noise; without loss of generality, only non-negative α can be considered. The exponent is shown to decrease with α, reach a minimum and increase after that. The minimum exponent is obtained analytically and numerically. As a function of α, the minimum at $\alpha \neq 0$, occurs on the parametric resonance regions of $\alpha = 0$. This minimum indicates suppression of growth by multiplicative white noise.

Keywords Suppression of growth · Exponent · Multiplicative White Noise · Parametric Resonance

1 Introduction

In the past few decades, many researchers have investigated the roles of noise, which led to the discovery of remarkable phenomena. Examples are stochastic resonance [1–4], phase transition induced by multiplicative noise [5], etc [6–10]. A basic system influenced by multiplicative noise is an oscillator with variable mass. In systems of this kind, noise has been shown to amplify the amplitude [11–15].

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Another growth mechanism is parametric resonance [16]. The effects of additive white noise upon a harmonic oscillator with a periodic coefficient has been investigated [17]. The mean-square displacement of an oscillator driven by a periodic coefficient has also been studied in the presence of additive white noise [18, 19]. Reference [20] has studied the parametric resonance induced by multiplicative colored noise. On the experimental front, certain physical systems described by equations with a periodic coefficient and a multiplicative noise term have been studied [21].

A differential equation with a periodic coefficient and a multiplicative noise term appears in the description of a few systems. Multiplicative noise may amplify or suppress the amplitude, as does additive noise. The magnitude of the amplitude being directly related to the stability of the system and to such physical quantities as the energy and number of particle, there is ample motivation to study the effects of multiplicative white noise in a parametric resonant system.

The general purpose of this paper is to study whether the amplification in a parametric resonant system is suppressed by multiplicative white noise. A more specific goal is to determine the ratio between the exponent at the minimum in a noisy system to the exponent in a noiseless system, when the suppression occurs. To these ends, a stochastic differential equation is analyzed by introducing a time-interval width in a parametric resonant system. The equation contains a parameter α determined by the noise intensity and the strength of the coupling between the variable and the noise. Only non-negative α is considered, without loss of generality.

The analysis shows that, regarded as a function of α, the exponent has a minimum. To estimate the minimum exponent, an approximate expression is obtained for the exponent on the parametric resonance regions for $\alpha = 0$.

The stochastic differential equations are also solved numerically by a symplectic method, to avoid the growth of numerical error, and the exponent is extracted from the average of the trajectories. The resulting behavior of the exponent as a function of α is displayed.

I find the exponent to have only one minimum at $\alpha \neq 0$ on the parametric resonance regions of $\alpha = 0$ and that the relative variation to be of the order of 10 %. The existence of the minimum indicates that multiplicative white noise suppresses the growth. The results offer insight into systems with periodically varying parameters and multiplicative noise and show that for appropriate noise intensity and coupling strength, multiplicative noise suppresses growth in a parametric resonant system.

2 The Exponent on Parametric Resonance Regions

2.1 Approximate Expression for the Exponent

Equations with a periodic coefficient and a multiplicative white noise term are of interest in certain branches of physics. For example, surface waves are described by a Mathieu-like equation with multiplicative white noise [21]. A Mathieu equation with multiplicative noise appears in the description of an electronic circuit with sufficiently small resistance [21]. Another example of a Mathieu-like equation arises in the study of the early universe [22, 23]. A similar illustration is the equation for the motion of finite mode in ϕ^4 theory, in which the condensate moves sinusoidally. The equation of motion contains a noise term, and when dissipation is negligible the system is described by a Mathieu-like equation [24].

One is typically interested in the equation

$$\ddot{\phi} + [1 + \beta \cos(\gamma z) + \alpha r(z)]\,\phi = 0, \tag{1}$$

where the dot represents the derivative with respect to z, and $r(z)$ has the following properties:

$$\langle r(z)\rangle = 0, \qquad \langle r(z) r(z')\rangle = \delta(z - z'), \tag{2}$$

where $\langle \cdots \rangle$ denotes statistical averaging. In (1), without loss of generality, α can be restricted not to be negative The starting point in this study is (1) with (2).

Equation 1 can be rewritten with the variable p_ϕ, defined by the equality $p_\phi = d\phi/dz$:

$$d\phi = p_\phi dz, \tag{3a}$$

$$dp_\phi = -[1 + \beta \cos(\gamma z)]\,\phi dz - \alpha \phi \circ dW. \tag{3b}$$

Here, the symbol $\circ$ denotes the Stratonovich, and $W(z)$ is defined by the equation $W(z) = \int_{z_0}^{z} ds\; r(s)$, a Wiener process, where z_0 is the value of z at an initial time. Section 3 describes the numerical solution of (3a) and (3b).

When $\alpha = 0$, (1) is just a Mathieu equation, which has resonance bands. With the definition $2u = \gamma z$, the Mathieu equation corresponding to (1) is given by the expression

$$\frac{d^2\phi}{du^2} + (a - 2q\cos(2u))\,\phi = 0, \tag{4}$$

where $a = 4/\gamma^2$ and $-2q = 4\beta/\gamma^2$. The bands are then distinguished by the positive integer n given by the relation $n^2 = 4/\gamma^2$. In other words, in each resonance band for $\alpha = 0$, γ is close to $2/n$.

Here, I attempt to estimate the amplitude growth rate in time. The rate is obtained from the exponent, which is given by $\limsup\, z^{-1} \ln[|\langle\phi(z)\rangle| / |\phi_0|]$, where ϕ_0 is the initial value. To solve (1) approximately in the $\alpha = 0$ resonance regions, I refer to the solution of the Mathieu equation. With $\alpha = 0$, the equation reads

$$\ddot{\Phi} + [1 + \beta\cos(\gamma z)]\,\Phi = 0. \tag{5}$$

I then write ϕ as the product of Φ by a new variable ψ, i. e., $\phi = \Phi\psi$, where ψ satisfies the equation

$$\ddot{\psi} + 2\left(\dot{\Phi}/\Phi\right)\dot{\psi} + \alpha r(z)\psi = 0. \tag{6}$$

The exponent $s \equiv s(\beta, \gamma)$ of Φ has been the subject of many detailed studies in the literature. To estimate the exponent of ϕ, I will, therefore, obtain an approximate expression for the exponent of ψ.

The time dependence of Φ is obtained from (5). In a resonance band, one approximate method is to write the following expression for Φ, under the assumption that $\ddot{P}_n \sim 0$ [16, 25–27]:

$$\Phi = \sum_{n=1}\left[P_n(z)e^{in\gamma z/2} + P_n^*(z)e^{-in\gamma z/2}\right] + R(z). \tag{7}$$

In the mth resonance band, the growth of the function $P_m(z)$ is dominant, and Φ is therefore approximately given by the equalities

$$\Phi \sim e^{s_m z} F_m(z), \tag{8a}$$

$$F_m(z) := Ce^{im\gamma z/2} + C^* e^{-im\gamma z/2}, \tag{8b}$$

where C is a complex constant and s_m is the exponent.

It is conjectured that, in the mth resonance band, the exponent s_m is close to the exponent s, it follows from (8a) and (8b) that

$$\dot{\Phi}/\Phi \sim s_m + \dot{F}_m/F_m. \tag{9}$$

The exponent of ϕ must be estimated from the solutions of (6) and (9). Unfortunately, it is not easy to handle (6). I will therefore substitute the time average of $\dot{\Phi}/\Phi$ for $(\dot{\Phi}/\Phi)$ on the right-hand side of (6). Since the average of $\dot{\Phi}/\Phi$ in one period of $F_m(z)$ is equal to s_m, under this approximation, in the mth resonance band, ψ obeys the equation

$$\ddot{\psi} + 2s_m\dot{\psi} + \alpha r(z)\psi = 0. \tag{10}$$

To estimate the exponent, I now write ψ in the following form:

$$\psi = \psi_0 \exp\left(\int_{z_0}^{z} dz'\sigma(z')\right). \tag{11}$$

Substituting (11) into (10), I obtain the equation for σ:

$$\dot{\sigma} + \sigma^2 + 2s\sigma + \alpha r(z) = 0, \tag{12}$$

where the subscript m of s_m has been omitted.

In Section 2.2, an estimate of the statistical average $\langle\sigma\rangle$ approximately yields the exponent of ϕ.

2.2 Exponent at the Extremum on Parametric Resonance Regions

To estimate the minimum exponent of ϕ from the expression (12) for σ, I assume that $r(z)$ is constant in the quite small time interval under consideration. Since $r(z)$ varies randomly, the statistical average is taken, with respect to it. The exponent of ϕ is therefore estimated from the exponents s and $\langle\sigma\rangle$. To show that the exponent has an extremum, the exponent is differentiated with respect to α.

First, consider the solution for constant $r(z)$. The solution of (12) is categorized by the quantity $\mathcal{D} \equiv 4s^2 - 4\alpha r$. I have that

$$\int_{z_0}^{z} dz'\sigma(z') = \begin{cases} \left(-s + \frac{\sqrt{\mathcal{D}}}{2}\right)(z - z_0) + \ln\left|\frac{1-Ce^{-\sqrt{\mathcal{D}}z}}{1-Ce^{-\sqrt{\mathcal{D}}z_0}}\right| & \mathcal{D} > 0 \\ -s\,(z - z_0) + \ln\left|\frac{z+C'}{z_0+C'}\right| & \mathcal{D} = 0 \\ -s\,(z - z_0) + \ln\left|\frac{\cos\left(\frac{\sqrt{-\mathcal{D}}}{2}z_0+C''\right)}{\cos\left(\frac{\sqrt{-\mathcal{D}}}{2}z+C''\right)}\right| & \mathcal{D} < 0 \end{cases}, \tag{13}$$

where C, C', and C'' are constants related to $\sigma(z_0)$. The logarithmic terms on the right-hand side of (13) do not contribute to the growth substantially.

Let now $r(z)$ be time dependent. The region $[z_0, z]$ is then divided into small time intervals Δz. Moreover, the region of width Δz is subdivided into N quite small regions labeled j ($j = 1, 2, \ldots, N$) in which r is constant. I define the $\Delta W_j \equiv r_j \Delta z/N$, where r_j is the value of r in the jth region. ΔW_j is a Wiener process, the distribution function of which is given by the expression

$$P(\Delta W_j) = \frac{1}{\sqrt{2\pi(\Delta z)/N}} \exp\left(-\frac{(\Delta W_j)^2}{2(\Delta z)/N}\right). \tag{14}$$

The sum $\Delta W \equiv \sum_{j=1}^{N} \Delta W_j$ then obeys the distribution function $P(\Delta W)$, given by the equality

$$P(\Delta W) = \frac{1}{\sqrt{2\pi(\Delta z)}} \exp\left(-\frac{(\Delta W)^2}{2(\Delta z)}\right). \tag{15}$$

The values of ΔW in the regions within the time interval Δz are therefore distributed with probability $P(\Delta W)$.

The statistical average of a variable $\mathcal{O}$ is given by $\langle\mathcal{O}\rangle = \int_{-\infty}^{\infty} d(\Delta W) P(\Delta W)\, \mathcal{O}$. Let $\mathcal{G}$ denote the exponent of ϕ in time units of z. Equations (8a) and (13) yield the following estimate for the exponent:

$$\mathcal{G} = \int_{-\infty}^{\infty} d(\Delta W)\ P(\Delta W)\Theta(\mathcal{D})\,\frac{\sqrt{\mathcal{D}}}{2}, \tag{16}$$

where $\Theta(x)$ is the step function ($\Theta(x > 0) = 1$ and $\Theta(x < 0) = 0$), and $\mathcal{D} = 4s^2 - 4\alpha(\Delta W)/\Delta z$.

Evaluation of the integral yields the following expression for $\mathcal{G}$:

$$\mathcal{G} = \frac{s}{2^{3/2}\kappa} \exp\left(-\frac{\kappa^4}{4}\right) D_{-3/2}\left(-\kappa^2\right), \tag{17}$$

where $\kappa \equiv (\Delta z)^{1/4} s/\alpha^{1/2}$ and D_ν is the parabolic cylinder function [28, 29].

The ratio $\mathcal{G}/s$ depends only on κ. Only through κ, therefore, does $\mathcal{G}/s$ depend on Δz. For given Δz, one can always adjust α to reach a certain κ. Following this procedure, I can read the α dependence of $\mathcal{G}$ from (17). As α grows, κ is reduced. The exponent decreases to a minimum and grows steadily beyond that. This behavior is robust because all factors associated with Δz are lumped into the variable κ, which controls the ratio $\mathcal{G}/s$. The numerical calculations will confirm this conclusion.

The extremum of the exponent $\mathcal{G}$ as a function of α is determined by the condition $d\mathcal{G}/d\alpha = 0$, from which I find that

$$D_{1/2}\left(-\kappa^2\right) = 0. \tag{18}$$

For positive ν, the parabolic cylinder function $D_\nu(x)$ has $[\nu+1]$ zeros [30], where $[\nu+1]$ is the maximum integer not exceeding $(\nu + 1)$. The equation $D_{1/2}(x) = 0$ has therefore a single solution, which I denote x_{sol}. Since x_{sol} is negative, it follows from (18) that α is positive at the extremum of $\mathcal{G}$. This confirms that $\mathcal{G}$ has a single extremum at positive α. At the extremum, the exponent $\mathcal{G}$ is given by the equality

$$\mathcal{G}_{\mathrm{min}} = \frac{s}{2^{3/2}}\frac{1}{[-x_{\mathrm{sol}}]^{1/2}} \exp\left(-\frac{1}{4}(x_{\mathrm{sol}})^2\right) D_{-3/2}(x_{\mathrm{sol}}), \tag{19}$$

which shows that $\mathcal{G}_{\mathrm{min}}$ is smaller than s.

This shows that, for a range of α, multiplicative white noise reduces the exponent $\mathcal{G}$. Also important, $\mathcal{G}_{\mathrm{min}}$ is independent of Δz, (19) being therefore sufficient to determine whether suppression by noise occurs. For very small s, however, (19) may be invalid, because the expression for Φ in (8a) and (8b) becomes unreliable.

Equation 19 is expected to offer a good estimate of the ratio $\mathcal{G}_{\mathrm{min}}/s$, which will be calculated numerically in the following section, as explained by the following argument.

The true exponent $\mathcal{G}_{\rm true}$ at the minimum can be written in the form $\mathcal{G}_{\rm true} = (\mathcal{G}_{\rm min}/s)s + \delta$, where δ is the deviation from the true exponent. It follows that $\mathcal{G}_{\rm true}/s = (\mathcal{G}_{\rm min}/s) + \delta/s$. For large s, the ratio δ/s is small, when δ is weakly dependent on s. $\mathcal{G}_{\rm min}/s$ is therefore expected to express the ratio between the minimum exponent in a noisy system and the exponent in a noiseless system, approximately.

3 Numerical Calculation of the Exponents by a Symplectic Method

I now attempt to solve (3a) and (3b) numerically. The goal is to obtain the amplitude of ϕ when white noise acts multiplicatively. The amplitude must therefore be accurately calculated, at least when a periodic coefficient and a white noise term are absent. If certain conditions are satisfied, the system has the symplectic structure even when there is noise [31]. This considered, I use the first-order symplectic method in Ref. [32] to solve the stochastic differential equations with multiplicative white noise.

The equations are solved numerically from $z = 0$ to $z = 500$, in z-time steps of 0.05, with the initial conditions $\phi(0) = 1$ and $\dot{\phi}(0) = 0$.

Given a noise sequence, one $\phi(z)$ trajectory can be calculated. I calculate many trajectories and take their average to obtain the mean value of the trajectories of the variable $\phi_i^{(j)}(z)$, where the subscript i indicates the batch and the superscript (j) indicates the trajectory in a certain batch i. My study comprised of 20 batches with 500 trajectories each. The mean value $\mathcal{M}_i(z)$ of the trajectories in batch i and the mean value $\bar{\phi}(z)$ over 20 batches were obtained from the equalities

$$\bar{\phi}(z) = \frac{1}{20}\sum_{i=1}^{20}\mathcal{M}_i(z), \quad \mathcal{M}_i(z) = \frac{1}{500}\sum_{j=1}^{500}\phi_i^{(j)}(z). \quad (20)$$

It is possible to perform interval estimation by using $\bar{\phi}$ and $\mathcal{M}_i$. For $\alpha = 0$, there is no need to calculate many trajectories; only one trajectory was calculated numerically.

The exponent is estimated from the average $\bar{\phi}(z)$ in the range of $200 < z < 500$ to decrease the effects of the initial conditions. The estimation was performed as follows: (1) the sets $(z_k, \ln\bar{\phi}(z_k))$ were determined, where z_k is the time at which $\bar{\phi}(z_k)$ is a positive local maximum; (2) the sets were fitted with a linear function. The coefficient of the time z was adopted as the exponent.

Brief justification of this procedure seems appropriate. One way to estimate the parameters would be to directly fit the average $\bar{\phi}(z_k)$. This implicitly assumes that the dispersion of the data distributions at times z and z' $(z' \neq z)$ are (approximately) the same. In the present case, however, the dispersion becomes wider with time z because a Wiener process is involved. The effects of non-equivalent dispersions are decreased by taking the logarithm of the data. For this reason, the transformed data $\ln\bar{\phi}(z_k)$ are fitted with a linear function.

The exponents resulting from the aforementioned procedure are generally different from the Lyapunov exponents obtained from the mean value of the logarithm of $\phi_i^{(j)}$. I calculate $\ln\bar{\phi}(z_k)$ in this study to focus on the enhancement of ϕ.

Figure 1 a displays the map of exponents of the Mathieu (5) on the γ–β plane. To draw this figure, the exponents were calculated over the plane in step sizes $\Delta\gamma = \Delta\beta = 0.02$. The color of a square is determined from the arithmetic mean of the exponents at four corners which are located at (γ, β), $(\gamma + \Delta\gamma, \beta)$, $(\gamma, \beta + \Delta\beta)$ and $(\gamma + \Delta\gamma, \beta + \Delta\beta)$.

The resonance band around $\gamma = 2$ corresponds to the first resonance band of (4). The nth resonance band is found around $\gamma = 2/n$, where n is a positive integer.

The remaining panels in the figure show the exponents for various α on the γ–β plane. Figure 1 b is the map for $\alpha = 0.5$, Fig. 1c for $\alpha = 1.0$, Fig. 1d for $\alpha = 1.5$, Fig. 1e for $\alpha = 2.0$, and Fig. 1f for $\alpha = 2.5$. The exponents in Fig. 1b, c, d, e, and f have been numerically computed in steps $\Delta\beta = \Delta\gamma = 0.05$. The color of a square is determined as in Fig. 1a.

As Fig. 1b, c, d, e, and f show, noise destroys the band structure, and the exponent grows with α over large regions of the γ–β plane. On the resonance bands, however, these figures indicate that the exponent does not grow monotonically with α. Moreover, the β dependence of the exponent in Fig. 1f is weak in comparison with the dependences on panels Fig. 1a, b, and c. This shows that, for large α, the exponents of the equation with the periodic coefficient $(\beta \neq 0)$ are close to those without the periodic coefficient $(\beta = 0)$.

To give more detailed attention to the α dependence of the exponent on the first and the second resonance bands, Fig. 2 displays the exponent as a function of α for fixed γ and β. Panel (a) shows the exponents for $\gamma = 2$ and $\beta = 2$, on the first resonance band, and panel (b), for $\gamma = 0.9$ and $\beta = 2$, on the second resonance band. The crosses represent the data obtained from the numerical solution of (3a) and (3b). The suppression by noise is clearly seen, and each panel shows a single local minimum, which confirms the result in Section 2.2. As α grows, the exponent is initially reduced, reaches the minimum, and then grows rapidly.

This behavior is interpreted as follows. For small α, the mechanism of parametric resonance controls the growth of the amplitude. As α grows, noise destroys this mechanism. The exponent therefore decreases with α. By contrast, for large α, the amplitude is amplified by noise, as many

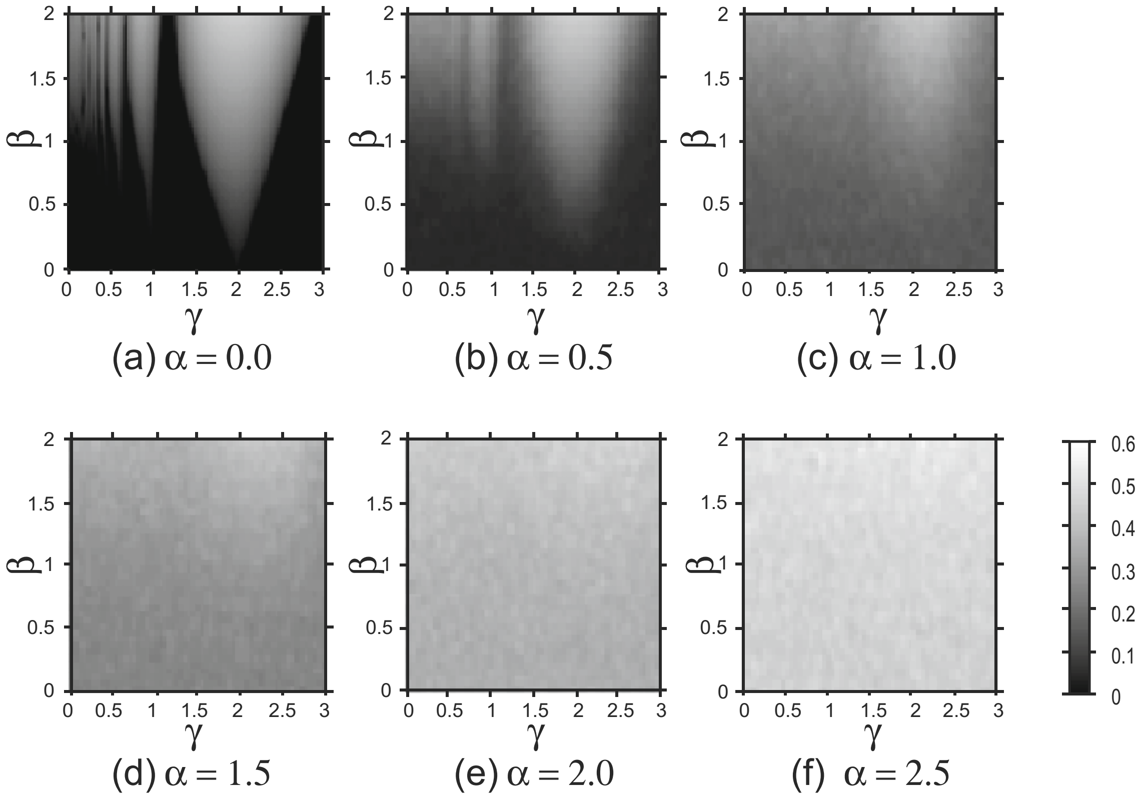

Fig. 1 Exponents on the γ–β plane for various α. The exponents are calculated by numerically solving the stochastic differential equations via the symplectic method. The parameters are **a** $\alpha = 0.0$, **b** $\alpha = 0.5$, **c** $\alpha = 1.0$, **d** $\alpha = 1.5$, **e** $\alpha = 2.0$, **f** $\alpha = 2.5$

reports in the literature have shown. In summary, the exponent decreases with α, reaches the minimum, and increases after that. For other γ, β pairs on the resonance bands, the exponents behave similarly.

Finally, I have estimate the minimum of the exponent as a function of α for various β and γ. Let $s_{\min}$ denotes the numerical estimate for the minimum exponent. Clearly, $s_{\min}$ depends on β and γ. I calculated the exponent for α in the [0, 2] range in steps of 0.01, and γ in the range [0.7, 2.7] in steps of 0.05. The minimum $s_{\min}$ was obtained from the exponents computed at various α at fixed β and γ.

From the minimum, I have calculated the ratio $s_{\min}/s$, because the exponent s at $\alpha = 0$ is also a function of β and γ. The crosses in Fig. 3 show $s_{\min}/s$ for $s \geq 0.3$ as a function of γ, compared with the horizontal dashed line representing the ratio obtained from (19), $\mathcal{G}_{\min}/s = 0.893$.

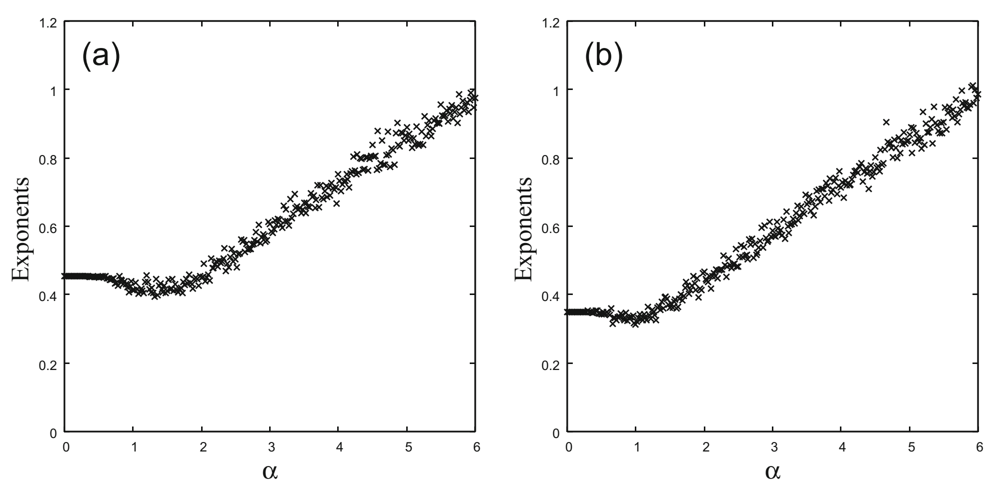

Fig. 2 Exponents in the resonance bands. The crosses represent the numerical data resulting from the solution of (3a) and (3b). **a** $\beta = \gamma = 2$. **b** $\beta = 2$ and $\gamma = 0.9$

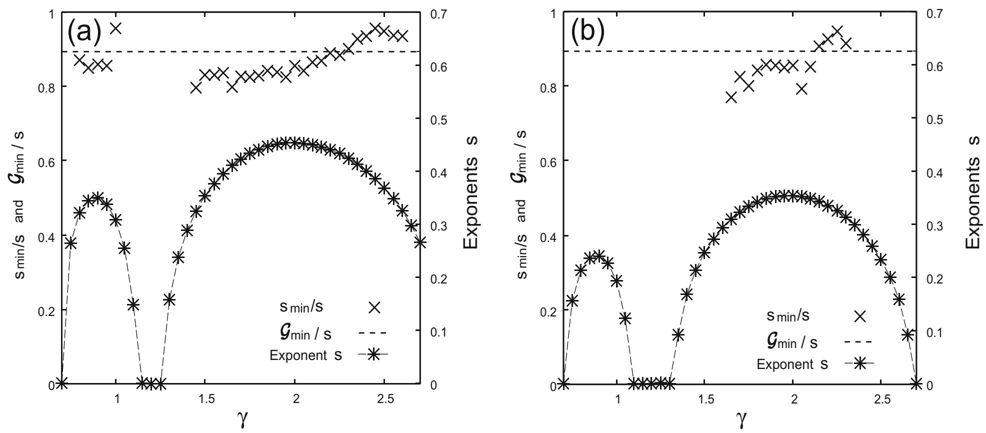

Fig. 3 Ratio $s_{\min}/s$ for various γ's. In panel **a** $\beta = 2.0$, and in panel **b** $\beta = 1.5$. The parameter γ is in the range [0.7, 2.7]. The crosses represent $s_{\min}/s$ for $s \geq 0.3$, and the asterisks represent s. The dashed line represents the ratio $\mathcal{G}_{\min}/s$, approximately equal to 0.893

The parameter β is set to 2.0 in Fig. 3a and 1.5 in Fig. 3b. The asterisks represent s.

As seen in Figs. 1 and 2, noise affects the exponents. It is therefore likely that the ratio $s_{\min}/s$ fluctuates and that the ratios $s_{\min}/s$ around the maximum of s are below $\mathcal{G}_{\min}/s$. I have therefore also calculated the simple moving average of the exponents, and attempted to find their minimum. To this end, I have taken the average $s_{\min}^{\mathrm{SMA}n}$ of n adjoining exponents. For example, the minimum of the averages of three adjoining exponents is denoted $s_{\min}^{\mathrm{SMA3}}$.

Figure 4 displays $s_{\min}^{\mathrm{SMA3}}/s$ for $s \geq 0.3$ and the exponents s. The parameter β is set to 2.0 in Fig. 4a and 1.5 in Fig. 4b. The symbols in Fig. 4a and b follow the convention in Fig. 3a and b. Figures 3 and 4 show that $\mathcal{G}_{\min}/s$ is close to the numerical estimates around the peaks of s in the resonance regions. These coincidences indicate that the exponent at the minimum $\mathcal{G}_{\min}$ in the theoretical estimation is close to the exponent at the minimum $s_{\min}$ in the numerical estimation for large exponent s in the noiseless system.

4 Discussion and Conclusion

I have studied the growth of the amplitude in the Mathieu-like equation with multiplicative white noise. To obtain an approximate expression for the exponent at the extremum, I have introduced the width of time interval on the parametric resonance regions, where parametric resonance occurs in the absence of noise. I have also solved the stochastic differential equations numerically by the symplectic numerical method to calculate the exponents. The intensity of noise and the strength of the coupling between noise and the variable are reflected in the parameter α, which was restricted not to be negative in the present study, without loss of generality. The behavior of the exponents as a function of α was shown.

Multiplicative white noise was shown to substantially affect the growth, the band structure of the Mathieu equation being destroyed by noise. The resonance structure survives for small α, but is lost for larger values of α.

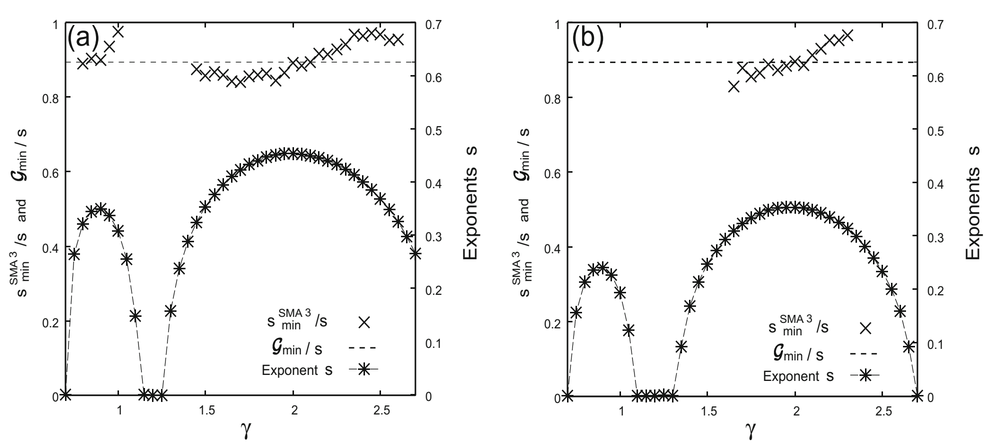

Fig. 4 Ratio $s_{\min}^{\mathrm{SMA3}}/s$ for various γ's. In panel **a** $\beta = 2.0$, and in panel **b** $\beta = 1.5$. The parameter γ lies in the range [0.7, 2.7]. The crosses represents $s_{\min}/s$ for $s \geq 0.3$, the asterisks represent s, and the dashed line indicates $\mathcal{G}_{\min}/s$, approximately equal to 0.893

In a previous paper [33], I have investigated the growth in a stochastic differential equation without periodic coefficient and found that the exponent is a monotonic increasing function of α. Here, by contrast, the exponent has one minimum as a function of α, on the $\alpha = 0$ parametric resonance region. This shows that growth can be suppressed by multiplicative white noise, for appropriate values of α. (17) roughly explains the behavior of the exponent as a function of α, showing that the exponent first decreases with α, reaches a minimum and increases after that.

As briefly discussed in Section 3, the α dependence of the exponent can be explained as follows. Parametric resonance is caused by the periodic coefficient. This periodicity is destroyed by noise, which causes suppression of the exponent associated with parametric resonance. Noise tends to amplify the exponent, but the amplification is small for weak noise. The the exponent is therefore suppressed by weak noise. For large α, (1) is approximately of the form $\ddot{\phi} + \alpha r \phi = 0$, where α is non-negative, and r is white noise with the properties in (2). The solution of the equation is sinusoidal for $r > 0$, and the amplitude of the solution, not amplified. In contrast, the solution is exponential for $r < 0$, and the amplitude either increases or decreases, exponentially: $\phi = A\exp(\mu t) + B\exp(-\mu t)$. The coefficient A on the right-hand side of this equality is generally non-zero. It follows that ϕ grows exponentially when $r < 0$, the growth being strong for large α. In other words, ϕ grows exponentially for large α. All considered, this reasoning shows that the exponent must initially decrease with α, reach a minimum, and then grow.

Intuitively, one expects the exponent to have one minimum as a function of α. It might, however, have more minima due to noise. The theoretical expression (18) indicates that only one minimum exists, a finding supported by the numerical data. The results in Sections 2.2 and 3 therefore show that, regarded as a function of α, the exponent has only one minimum The minimum of the exponent was estimated from the numerical calculations. More specifically, I calculated the ratio $s_{\rm min}/s$, i. e., the minimum value divided by the $\alpha = 0$ exponent s. This numerical results for this ratio are in rough agreement with the ratios obtained analytically around the peaks of s on the resonance regions. The comparisons in Figs. 3 and 4 thus support the analytical derivation, in Section 2.2, of an approximate expression for the ratio $\mathcal{G}_{\rm min}/s$ for large s. The minimum value of the exponent is approximately proportional to the exponent s. As the figures and (19) show, the relative variation is small, of the order of 10%. In contrast with the exponent, the amplitude is affected by changes in s.

For large s, one expects $\mathcal{G}_{\rm min}$ to be close to the exponent $\mathcal{G}_{\rm true}$, because the relative deviations from the correct value tend to become small as s grows. That is, the ratio $|\mathcal{G}_{\rm true} - \mathcal{G}_{\rm min}|/s$ is expected to be small for large s. The ratio $\mathcal{G}_{\rm min}/s$ is therefore approximately equal to the ratio between the exponents in the noisy and noiseless systems.

The noise-induced reduction in the Lyapunov exponent has been determined in a system constituted by an inverted Duffing oscillator with noise [34]. In the absence of noise, the growth mechanisms in the present case and in the case of the inverted Duffing oscillator are distinct. Nonetheless, the mechanism of the suppression is surely the same. In both cases, noise of appropriate intensity suppresses growth. In both cases, as the intensity grows, the exponent goes through a minimum before rising.

Given that white noise decreases the exponent, it is possible that colored noise will strongly suppress growth. The conditions under which parametric resonance occurs may favor stabilization by colored noise, as it has been found in the inverted Duffing-oscillator system.

The analytical expression for the exponent involves the artificial parameter Δz. The α value that minimizes $\mathcal{G}$ is therefore dependent on Δz, while the minimum $\mathcal{G}$ is independent of Δz. This contrast poses a problem that I would like to solve, in a future study.

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