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CONDENSED MATTER

Correlation Identities, Correlation Inequalities, and Upper Bounds on Critical Temperature of Spin Systems

J. P. Santos · F. C. Sá Barreto

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Abstract A review on spin correlation functions identities and on rigorous correlation functions inequalities is presented for various spin models. The spin correlation identities are exactly obtained from finite cluster of spins for different models, and the rigorous spin correlation inequalities are presented for the discrete and continuous classical spin models and also for quantum spin models. Through these correlation identities and rigorous inequalities, the upper bounds on critical temperatures are obtained by the decay of the correlation function.

Keywords Correlation function identities · Correlation function inequalities · Upper bounds on critical temperatures

1 Introduction

The knowledge of the correlation function is important to determine the physical properties, dynamic and thermodynamic, of the models described by a Hamiltonian H. In the literature exists results on identities for the spin correlation functions of some models and also rigorous inequalities for those correlation functions. One of the objectives of this paper is to present a review of the results already obtained for various models of the correlation identities and the rigorous correlation inequalities. Also, the results will be presented, not yet published, for the exact correlation function of some spin models. The correlation identities are exact equations relating correlation functions of lower order, say two-spin correlation functions, to higher order correlation functions. So, as such, the identities are not useful to give a direct information on the spin correlation functions. On the other hand, correlation inequalities are obtained by rigorous decoupling of higher order correlation functions into lower order functions. Upper bound on Tc, the critical temperature, is defined to be that temperature at which the two-point function no longer falls exponentially. The upper bounds are obtained from an exact spin-spin correlation function identity and on an iterative procedure ignited by rigorous spin-spin correlation inequalities. The method itself is inspired on the applications of the Simon-Lieb inequality [1], and it is based upon the correlation inequalities for the models. In words, the method may be described as follows: starting from an identity for the spin-spin correlation, an iterative procedure leads to the exponential decay of correlations, equivalently, to a finite correlation length, if a certain condition on the temperature is satisfied. Then we find, according to our estimates, the lowest temperature up to which the correlation length is finite thus obtaining an upper bound on the critical temperature. Therefore, the objectives of this paper are to (1) describe the correlation identities, those obtained previously and the new ones presented in this paper; (2) describe the correlation inequalities presented in the literature; (3) combine the identities and inequalities with the use of the decay of the correlation functions [1]; and to obtain rigorous upper bounds for the critical temperature. So, we present a review on correlation functions and a general procedure to obtain rigorous upper bounds on the critical temperatures of different spin models. In Sections 2 and 3, we describe the correlation identities and the correlation inequalities, respectively. In Section 4,

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we describe the general theorems for the decay of the correlation functions, and in Section 5, we apply the results described in Sections 2 and 3 to obtain the upper bounds on the critical temperature of some models.

In Section 2, we present the deduction of the exact equation of the multispin correlation functions for an n-site cluster of different spin models. This correlation identity relating correlation functions of lower order to higher order correlation functions is a generalization of Callen's identity [2] for the spin-1/2 Ising model [3] which was derived from the one-site cluster. From those identities, one obtains the mean-field approximation results on the critical temperature by discarding correlations, i.e., by decoupling the multispin correlation functions appearing in the r.h.s of the identity into a product of single-spin average. The differential operator technique [4] and the van der Waerden relations are shown and applied to the general expression of the spin correlation functions. This procedure will incorporate in the formalism the autocorrelation function. Therefore, by applying the mean field-type approximation to those equations, the results on the critical temperature will improve over those obtained by the mean field approximation directly. In the sequence, we present the correlation identities obtained from different size clusters for various models: the spin-1/2 Ising model on a Kagome lattice for a five-site cluster; the spin-1/2 Ising model on a pyrochlore lattice for a four-site cluster; the spin-1/2 Ashkin-Teller model [5] for a one-site cluster; the spin-1/2 Ising model with four-spin interactions for a one-site cluster [6]; the Blume-Capel (BC) model [7, 8] for a one-site cluster [9, 10]; the Blume-Emery-Griffiths (BEG)[11] model for a one-site cluster [12, 13]; the spin-1 Ashkin-Teller Model for a one-site cluster; the Potts model [14] for an one-site cluster [15]; the Z_2 Lattice Gauge model for a one-site cluster [16]; and the transverse Ising model (TIM) for a one-site cluster [17, 18]. In Section 3, we present the rigorous inequalities for the correlation functions. First, we present the Griffiths inequalities [19, 20], the generalization of Griffiths inequalities obtained by Kelly and Sherman [21] and Ginibre [22] and Newman's inequality [23, 24]. Following these results, we present the correlation inequalities for the continuous spins [25]. It is a generalization of the Griffiths inequalities, obtained by Ginibre [26], Ellis and Monroe [27], Percus [28], Lebowitz [29], and Griffiths et al. [30]. We conclude this section showing the correlations inequalities for some quantum models with the results obtained by Hurst and Sherman [31] , Suzuki [32], Monroe [33, 34], and Contucci and Lebowitz [35, 36]. In Section 4, we present the results obtained by Simon [1] for the decay of the correlation function. In Section 5, we obtain rigorous upper bounds on the critical temperature T_c, resulting from the application of the correlation identities, the rigorous correlation inequalities, and the decay of the correlation function.

We establish an inequality of the following type $\langle S_i S_r\rangle \leq \left[\sum_j a_j(T)\right]\langle S_j S_r\rangle$ for the correlation function which, when iterated under the condition $\sum_j a_j(T) < 1$, implies the existence of $\overline{T}_c$ and exponential decay for $T > \overline{T}_c$. The upper bounds are determined for the following models: the spin-1/2 Ising model [37] in rectangular lattices, the spin-1/2 Ising model on a Kagome lattice, the spin-1/2 Ising model on a pyrochlore lattice, the spin-1/2 Ashkin-Teller model, the spin-1/2 Ising model with four-spin interactions [6], the Blume-Capel (BC) model [10] in rectangular lattices, the Blume-Capel model in a Kagome lattice, the Z_2 Lattice Gauge model [16], and the transverse Ising model [17, 18]. The concluding remarks are presented in Section 6.

2 Correlation Functions Identities

2.1 The Multispin Correlation Function Identity and Callen's Identity

In order to motivate the deduction of the multispin correlation function identity, let us present Callen's identity [2] which was obtained for the one-spin cluster of the spin-1/2 Ising model. The thermal average of the spin variable S_1 located at site 1, i.e. ,$\langle S_1\rangle$, is given by

$$\langle S_1\rangle = \frac{1}{Z} Tr S_1\, e^{-\beta H}, \tag{1}$$

where $H = -\sum_{<ij>} J_{ij} S_i S_j$, $Z = Tr\, e^{-\beta H}$, J_{ij} is the coupling interaction between neighbors i and j, $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant and T is the temperature.

Let $H = H_1 + H'$, where $H_1 = S_1 \sum_{|j|=1} J_{1j} S_j \equiv S_1 h_1$, $h_1 = \sum_{|j|=1} J_{1j} S_j$, and j denotes the nearest neighbors of the site 1. So,

$$\langle S_1\rangle = \frac{1}{Z} Tr' \left(\frac{Tr_{(1)} S_1\, e^{-\beta H_1}}{Tr_{(1)}\, e^{-\beta H_1}}\right)\left(Tr_{(1)}\, e^{-\beta H_1}\right) e^{-\beta H'} \tag{2}$$

or,

$$\langle S_1\rangle = \frac{1}{Z} Tr \left(\frac{Tr_{(1)} S_1\, e^{-\beta H_1}}{Tr_{(1)}\, e^{-\beta H_1}}\right) e^{-\beta H}, \tag{3}$$

where $Tr = Tr' Tr_i$.

So,

$$\langle S_1\rangle = \left\langle \frac{\sum_{S_1=-1}^{1} S_1\, e^{-\beta H_1}}{\sum_{S_1=-1}^{1} e^{-\beta H_1}} \right\rangle = \left\langle \frac{e^{\beta h_1} - e^{-\beta h_1}}{e^{\beta h_1} + e^{-\beta h_1}} \right\rangle, \tag{4}$$

or,

$$\langle S_1 \rangle = \left\langle \tanh\left(\beta \sum_{|j|=1} J_{1j} S_j\right)\right\rangle, \tag{5}$$

which is Callen's identity.

We now present the derivation of the multispin correlation function identity, which is the generalization of Callen's identity, to be used in other cases.

Let H be the Hamiltonian of spin model described by classical variables. The partition function Z and the thermal average of the spin variables are

$$Z = Tr\, e^{-\beta H}, \tag{6}$$

$$\left\langle F(\{S\})S_{\{n\}}\right\rangle = \frac{1}{Z} Tr\left\{F(\{S\})S_{\{n\}}\, e^{-\beta H}\right\}. \tag{7}$$

The thermal average $\langle \ldots \rangle$ is defined in a cluster $\{n\}$, $n = 1, 2, \ldots, N$, where $S_{\{n\}}$ is a spin of the cluster $\{n\}$ and $F(\{S\})$ is a function of the variables of spins, except the spins of the cluster $\{n\}$.

In order to obtain the exact relation for the correlation function defined in (7), the Hamiltonian is separated into two parts

$$H = H_{\{n\}} + H', \tag{8}$$

where $H_{\{n\}}$ is the Hamiltonian of the spins of the cluster $\{n\}$ and H' corresponds to the Hamiltonian of the rest of the lattice. The spin variables are classical, so they commute, i.e., $[S_i, S_j] = 0$ for all i and j. Consequently, $[H_{\{n\}}, H'] = 0$ and $e^{-\beta H} = e^{-\beta H_{\{n\}}} e^{-\beta H'}$. From (7) and (8), we get

$$\left\langle F(\{S\})S_{\{n\}}\right\rangle = \frac{1}{Z} Tr' tr_{\{n\}} F(\{S\}) S_{\{n\}} e^{-\beta H_{\{n\}}} e^{-\beta H'}, \tag{9}$$

where $Tr' tr_{\{n\}} = Tr$. We now multiply and divide expression (9) by $tr_{\{n\}} e^{-\beta H_{\{n\}}}$ to obtain

$$\left\langle F(\{S\})S_{\{n\}}\right\rangle = \frac{1}{Z} Tr F(\{S\}) \left(\frac{tr_{\{n\}} S_{\{n\}} e^{-\beta H_{\{n\}}}}{tr_{\{n\}} e^{-\beta H_{\{n\}}}}\right) e^{-\beta H} \tag{10}$$

Finally, we obtain,

$$\left\langle F(\{S\})S_{\{n\}}\right\rangle = \left\langle F(\{S\}) \left(\frac{tr_{\{n\}} S_{\{n\}} e^{-\beta H_{\{n\}}}}{tr_{\{n\}} e^{-\beta H_{\{n\}}}}\right)\right\rangle. \tag{11}$$

The identity (11) can be used for classical spin models and for clusters of n sites. It is also used in lattices which, in order to guarantee its symmetry, require an n-site cluster which contains such symmetry, not present in lower size clusters, such as the $2d$ Kagome and the $3d$ Pyrochlore lattices. Note that by taking in (11), $F(\{S\})$=1 and n=1, one obtains Callen's identity (3).

2.2 The Multispin Correlation Function Identity and the Differential Operator Technique

In 1979, Honmura and Kaneyoshi [4] introduced the differential operator technique. This technique combined with the van der Waerden relations applied to (5) results in an identity for the correlation function which incorporates the autocorrelation of the spins [38]. The differential operator $\nabla_r \equiv \partial/\partial x_r$ $(r = 0, 1, 2 \ldots, n)$ is defined by

$$e^{\alpha \nabla_r} f(x_0, \ldots, x_r, .., x_n) = f(x_0, \ldots, x_r + \alpha, .., x_n), \tag{12}$$

where $f(x_0, \ldots, x_r, \ldots, x_n)$ is any analytic function. This can obtain by expanding the exponential operator in Taylor series, i.e.,

$$e^{\alpha \nabla_r} = 1 + \alpha \nabla_r + \frac{\alpha^2}{2!}\nabla_r^2 + \frac{\alpha^3}{3!}\nabla_r^3 + \ldots, \tag{13}$$

and applying it to the function $f(x_0, \ldots, x_r, \ldots, x_n)$, we get

$$f(x_0, \ldots, x_r + \alpha, \ldots, x_n) = [1 + \alpha \nabla_r + \frac{\alpha^2}{2!}\nabla_r^2 + \ldots] f(\ldots, x_r, \ldots). \tag{14}$$

Let us use the differential operator in Callen's identity, i.e., (5). We have

$$\langle S_1 \rangle = \left\langle \exp(\sum_{|j|=1} \beta J_{1j} S_j \nabla_x)\right\rangle . \tanh(x)\,|_{x=0}\,, \tag{15}$$

where $\nabla_x = \frac{\partial}{\partial x}$.

As the spins commute, i.e., $[S_i, S_j] = 0$, we have

$$\langle S_1 \rangle = \left\langle \prod_{|j|=1} e^{\beta J_{1j} S_j \nabla_x}\right\rangle . \tanh(x)\,|_{x=0}\,, \tag{16}$$

where

$$\begin{aligned} e^{\beta J_{1j} S_j \nabla_x} &= \sum_{n=0}^{\infty} \frac{(\beta J_{1j} \nabla_x)^n}{n!} (S_j)^n \\ &= \sum_{n=0}^{\infty} \frac{(\beta J_{1j} \nabla_x)^{2n}}{(2n)!} (S_j)^{2n} \\ &\quad + \sum_{n=0}^{\infty} \frac{(\beta J_{1j} \nabla_x)^{2n+1}}{(2n+1)!} (S_j)^{2n+1}. \end{aligned} \tag{17}$$

Using van der Waerden relations $(S_j)^{2n}$ =1 e $(S_j)^{2n+1} = S_j$, we get

$$e^{\beta J_{1j} S_j \nabla_x} = \cosh(\beta J_{1j} \nabla_x) + S_j \sinh(\beta J_{1j} \nabla_x). \tag{18}$$

Finally, we obtain the identity for single-spin average (one-spin correlation function) of the spin-1/2 Ising model of the one-site cluster $n = 1$,

$$\langle S_1\rangle = \left\langle \prod_j [\cosh(\beta J_{1j}\nabla_x) + S_j \sinh(\beta J_{1j}\nabla_x)] \right\rangle \tanh(x)\Big|_{x=0} . \quad (19)$$

Equation (19) is exact and contains the autocorrelation of the spins due to the application of van der Waerden relations. Therefore, because of the incorporation of the autocorrelations in (19), if one applies the mean field-type approximation (neglecting correlations in the r.h.s) to those equations, one gets for the critical temperature a result which improves over the standard mean field result, which is obtained by neglecting correlations in the r.h.s of (16). An approximate relation to the expression (17) was proposed by Kaneyoshi and others [39], to treat models with spins different of the spin-1/2 and spin-1.

2.3 Spin-1/2 Ising Model. One-Site Cluster

According to McCoy and Wu [40], the Ising model is the most studied among the models of statistical mechanics. In one-dimensional lattice, the model was resolved by Ising in 1925 [3]. In 1944, Onsager [41] solved the Ising model on the square lattice at zero external field, finding exact expressions for the partition function, the internal energy, the specific heat, and the magnetization. In tridimensional lattices, there are no exact results.

The Hamiltonian for the spin-1/2 Ising model, presented in a previous Section 2.1, is

$$H = -\sum_{<ij>} J_{ij} S_i S_j , \quad (20)$$

where $S_i = \{-1, 1\}$, J_{ij} is a coupling interaction between variables S_i and S_j and $< ij >$ denote a pair of nearest-neighbor spins.

From the preceding section, the thermal average of the spin variable at the site i is given by

$$\langle S_i\rangle = \langle \tanh\ \beta E_i\rangle , \quad (21)$$

where $E_i = \sum_j J_{ij} S_j$. Using the differential operator, defined in (12) and van der Waerden relation ($S_i^2 = 1$), we obtain an exact identity of correlation function $\langle F(\{S\})S_i\rangle$ for the Ising model

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\}) \prod_j \left[\cosh(\beta J_{ij}\nabla_x) + S_j \sinh(\beta J_{ij}\nabla_x)\right] \right\rangle f(x)\,|_{x=0} , \quad (22)$$

where $F(\{S\})$ is a function of the variables of spins, except for the spin S_i and

$$f(x) = \tanh(x) = -f(-x) \quad (23)$$

2.4 Spin-1/2 Ising Model on a Kagome Lattice. Five-Site Cluster

In 1953, using Onsager's method, Kanô and Naya solved the Ising model on the Kagome lattice [42]. Statistical mechanics on a Kagome lattice has attracted a lot of attention recently. For example, in crystal engineering, colloidal Kagome lattices are used as a substrate to form superstructures from designed building blocks of organic molecules [43].

Consider a cluster of five sites which contain the symmetries of the Kagome lattice (see Fig. 1).

From (11), we deduced the identity formula for $< F(\{S\})S_i >$, whose details of the calculation will be presented elsewhere, and it is given by,

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\}) \frac{\sum_{\{S_0,S_1,\ldots,S_4\}} S_i e^{-\beta H_{\{5\}}}}{\sum_{\{S_0,S_1,\ldots,S_4\}} e^{-\beta H_{\{5\}}}} \right\rangle , \quad (24)$$

where $F(\{S\})$ is a function of the spins variables, except the spins of the cluster $\{n{=}5\}$ and $H_{\{5\}}$ is given by

$$H_{\{5\}} = -J S_0 \left(\sum_{i=1}^{4} S_i\right) - J\,(S_1S_2 + S_3S_4) - \sum_{i=1}^{4} S_i \left(\sum_{j=1}^{2} J_{ij} S_{ij}\right) \quad (25)$$

Using the differential operator defined in (12) and van der Waerden relation ($S_i^2 = 1$), we obtain the spin correlation function identity for the Ising model on a Kagome lattice,

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\}) \prod_{i=1}^{4} \prod_{j=1}^{2} [\cosh(\beta J_{ij}\nabla_j) + S_{ij} \sinh(\beta J_{ij}\nabla_j)] \right\rangle f(a_1, a_2, a_3, a_4)\,|_{(0,0,0,0)} , \quad (26)$$

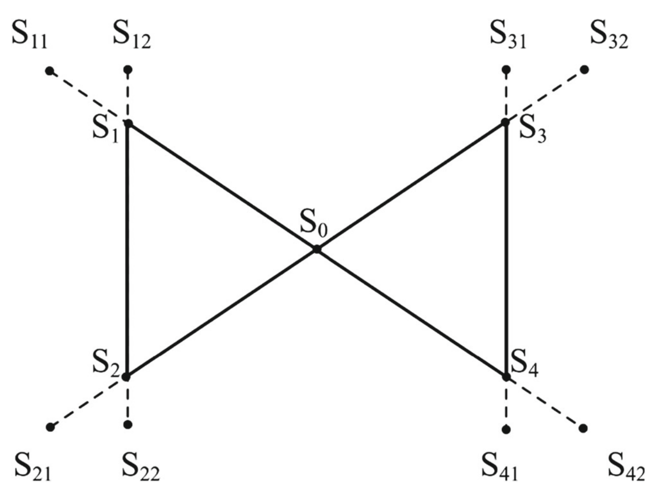

Fig. 1 Butterfly type structure for the Kagome lattice

where $f(a_1, a_2, a_3, a_4)$ is given by,

$$f(a_1,a_2,a_3,a_4) = \Big[e^{2J}\sinh(4J)\sinh(a_1+a_2+a_3+a_4)+\sinh(2J)[\sinh(a_1+a_2+a_3-a_4) +\sinh(a_1+a_2-a_3+a_4)]+\sinh(a_1-a_2+a_3+a_4)+\sinh(-a_1+a_2 +a_3+a_4)\Big]/\Big[e^{2J}\cosh(4J)\cosh(a_1+a_2+a_3+a_4)+\cosh(2J)[\cosh(a_1 +a_2+a_3-a_4)+\cosh(a_1+a_2-a_3+a_4)+\cosh(a_1-a_2+a_3+a_4) +\cosh(-a_1+a_2+a_3+a_4)]+\Big[e^{2J}\cosh(a_1+a_2-a_3-a_4) +e^{-2J}[\cosh(-a_1+a_2-a_3+a_4)+\cosh(-a_1+a_2+a_3-a_4)]\Big]\Big], \tag{27}$$

and $a_j = \beta\sum_{j=1}^{2} J_{ij}S_{ij}$, $i = 1, ..., 4$.

Beyond the single-site cluster theory, the complexity of the algebra increases and we used an algebraic computation software for the determination of the expansion coefficients on the right hand side of (26). This comment also applies in next Section 2.5.

2.5 Spin-1/2 Ising Model on a Pyrochlore Lattice. Four-Site Cluster

The pyrochlore lattice has attracted a lot of attention recently, as for instance in soft condensed matter, the effects of superfrustation and the possibility of theoretically realizing a spin ice, a possible new state of matter [44, 45].

Consider a cluster of four sites which contains all the symmetries of the pyrochlore lattice (see Fig. 2). From (11), we deduced the identity formula for $< F(\{S\})S_i >$, whose details of the calculation will be presented elsewhere, and it is given by,

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\})\frac{\sum_{\{S_1,..,S_4\}} S_i e^{-\beta H_{\{4\}}}}{\sum_{\{S_1,..,S_4\}} e^{-\beta H_{\{4\}}}}\right\rangle, \tag{28}$$

where $F(\{S\})$ is a function of the spin variables, except the spins of the cluster $\{n = 4\}$ and $H_{\{4\}}$ is given by,

$$H_{\{4\}} = -J\,(S_1S_2+S_1S_3+S_1S_4+S_2S_3+S_2S_4+S_3S_4) - \left(\sum_{i=1}^{4} S_i \sum_{j=1}^{3} J_{ij}S_{ij}\right) \tag{29}$$

Using the differential operator defined in (12) and van der Waerden relation ($S_i^2 = 1$), we obtain the correlation identity,

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\})\prod_{i=1}^{4}\prod_{j=1}^{3}[\cosh(\beta J_{ij}\nabla_j) +S_{ij}\sinh(\beta J_{ij}\nabla_j)]\right\rangle f(a_1,a_2,a_3,a_4)\,|_{(0,0,0,0)}\,, \tag{30}$$

where $f(a_1, a_2, a_3, a_4)$ is given by,

$$f(a_1,a_2,a_3,a_4) = [\cosh(6J-a_1-a_2-a_3-a_4)+\sinh(6J-a_1-a_2-a_3-a_4)+\sinh(a_1-a_2 -a_3-a_4)-\sinh(a_1-a_2+a_3+a_4)-\sinh(a_1+a_2-a_3+a_4)-\sinh(a_1+a_2 +a_3-a_4)-\cosh(6J+a_1+a_2+a_3+a_4)-\sinh(6J+a_1+a_2+a_3+a_4)] /[-2\cosh(a_1-a_2+a_3+a_4)-2\cosh(a_1+a_2-a_3+a_4)-\cosh(6J-a_1-a_2 -a_3-a_4)-\sinh(6J-a_1-a_2-a_3-a_4)-2\cosh(a_1-a_2-a_3-a_4) -\cosh(2J+a_1+a_2-a_3-a_4)+\sinh(2J+a_1+a_2-a_3-a_4)-\cosh(2J+a_1 -a_2+a_3-a_4)+\sinh(2J+a_1-a_2+a_3-a_4)+\sinh(2J+a_1-a_2-a_3+a_4) -\cosh(2J-a_1+a_2+a_3-a_4)+\sinh(2J-a_1+a_2+a_3-a_4)-\cosh(2J-a_1 -a_2+a_3+a_4)+\sinh(2J-a_1-a_2+a_3+a_4)-\cosh(6J+a_1+a_2+a_3+a_4) -\sinh(6J+a_1+a_2+a_3+a_4)-\cosh(2J-a_1+a_2-a_3+a_4)+\sinh(2J-a_1 +a_2-a_3+a_4)-\cosh(2J+a_1-a_2-a_3+a_4)-2\cosh(a_1+a_2+a_3-a_4)], \tag{31}$$

and $a_j = \beta\sum_{j=1}^{3} J_{ij}S_{ij}$, $i = 1, \ldots, 4$.

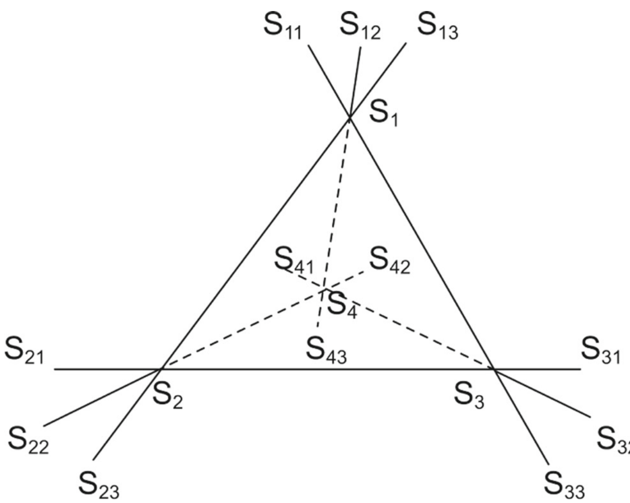

Fig. 2 Tetrahedron type structure for the pyrochlore lattice

2.6 Spin-1/2 Ashkin-Teller Model. One-Site Cluster

The Ashkin-Teller model [5] is a generalization of the Ising model, and it is used to describe four-component systems. It may be considered as a two superposed Ising models which are described by the spin variables S_i and σ_i sitting on each site of a hypercubic lattice. Within each Ising model, there is a two-spin nearest-neighbor interaction J_{ij} and the different Ising models are coupled by a four-spin interaction K_{ij}. The Hamiltonian can be written as

$$H = -\sum_{<ij>} J_{ij}(S_i S_j + \sigma_i \sigma_j) - \sum_{<ij>} K_{ij} S_i S_j \sigma_i \sigma_j, \quad (32)$$

where $< ij >$ denote a pair of nearest-neighbor spins.

Now, we present the Ashkin-Teller model spin correlation identity. Those identities are formally generalization of Callen's identity for the Ising model [2]. The multispin correlation function $\langle F(\{\tau\})\tau_i\rangle$ of the Ashkin-Teller model was deduced, the details of the calculation will be presented elsewhere, and it is given by

$$\langle F(\{\tau\})\tau_i\rangle = \left\langle F(\{\tau\}) f_{(\tau)}(E_i^{(S)}, E_i^{(\sigma)}, E_i^{(\sigma S)})\right\rangle, \quad (33)$$

where $E_i^{(\tau)} = \sum_j L_{ij}^{(\tau)} \tau_j$ with $L_{ij}^{(\tau)}$ given by $L_{ij}^{(S)} = L_{ij}^{(\sigma)} = J_{ij}$ and $L_{ij}^{(\sigma S)} = K_{ij}$, and $\tau \equiv \{S, \sigma, S\sigma\}$. The variable τ_i is restricted to $\tau_i = \{-1, 1\}$ and $F(\tau)$ is any function of τ different from τ_i.

Using the differential operator defined in (12), we obtain,

$$\langle F(\{\tau\})\tau_i\rangle = \left\langle F(\{\tau\}) \prod_j \prod_{\{\tau\}} e^{\beta L_{ij}^{(\tau)} \tau_j \nabla_\tau} \right\rangle . f_{(\tau)}(x, y, z)\Big|_{(0,0,0)}, \quad (34)$$

where $\nabla_S = \frac{\partial}{\partial x}$, $\nabla_\sigma = \frac{\partial}{\partial y}$ and $\nabla_{\sigma S} = \frac{\partial}{\partial z}$. Using the van der Waerden relation, $(\tau_i)^2 = 1$, we get the exact relation,

$$\langle F(\{\tau\})\tau_i\rangle = \Big\langle F(\{\tau\}) \prod_j \prod_{\{\tau\}} [\cosh(\beta L_{ij}^{(\tau)} \nabla_\tau) + \tau_j \sinh(\beta L_{ij}^{(\tau)} \nabla_\tau)] \Big\rangle f_{(\tau)}(x, y, z)\big|_{(0,0,0)}, \quad (35)$$

where j are neighbors of i and the functions $f_{(\tau)}(x, y, z)$ are given by,

$$f_{(S)}(x, y, z) = \frac{\tanh(x) + \tanh(y)\tanh(z)}{1 + \tanh(x)\tanh(y)\tanh(z)}, \quad (36)$$

$$f_{(\sigma)}(x, y, z) = \frac{\tanh(y) + \tanh(x)\tanh(z)}{1 + \tanh(x)\tanh(y)\tanh(z)} \quad (37)$$

and

$$f_{(\sigma S)}(x, y, z) = \frac{\tanh(z) + \tanh(x)\tanh(y)}{1 + \tanh(x)\tanh(y)\tanh(z)}. \quad (38)$$

2.7 Spin-1/2 Ising Model with Four-Spin Interactions. One-Site Cluster

The Ising model with four-spin interaction shows a bilinear interaction between pairs of nearest neighbor and another term with the interaction of four spins. This model has exact solution connected with the eight vertex model [46–48]. Models with multispin interactions have been used to describe various physical systems such as binary alloys [49], classical fluids [50], rare gases [51], among others.

The Hamiltonian of the spin-1/2 Ising system with four-spin interactions is given by

$$H = -\sum_{ij} J_{ij} S_i S_j - \sum_{ijkl} K_{ijkl} S_i S_j S_k S_l, \quad (39)$$

where $S_i = \pm 1$, $J_{ij} > 0$ is the coupling between pairs of nearest neighbors and K_{ijkl} represents the four-spin interaction coupling. The spin correlation identity for this model [6] is given by

$$\langle F(\{S\})S_i\rangle = \left\langle F(\{S\}) \tanh[\beta(E_i + E_q)]\right\rangle, \quad (40)$$

where $E_i = \sum_j J_{ij} S_j$ and $E_q = \sum_{jkl} K_{ijkl} S_j S_k S_l$. $F(\{S\})$ is a function of the variables of spins, except the spin S_i. Using the differential operator (12) and van der Waerden relation, we get

$$\langle F(\{S\})S_i\rangle = \Big\langle F(\{S\}) \prod_j [\cosh(\beta J_{ij} \nabla_x) + S_j \sinh(\beta J_{ij} \nabla_x)] \times \prod_{jkl} [\cosh(\beta K_{ijkl} \nabla_x) + S_j S_k S_l \sinh(\beta K_{ijkl} \nabla_x)] \Big\rangle . f(x)\Big|_{x=0}, \quad (41)$$

where $\nabla_x \equiv \frac{\partial}{\partial x}$, j, k, l are neighbors of i and the function f is given by

$$f(x) = \tanh x = -f(-x). \quad (42)$$

2.8 Spin-1 Blume-Capel Model. One-Site Cluster

The Blume-Capel model was introduced by Blume [7] and independently by Capel [8] as a generalization of the Ising model. This is a model of spin $S = 1$, which originally was used to explain some thermodynamic properties and phase transition in the $He^3 - He^4$ mixture.

The Hamiltonian for the Blume-Capel model is given by

$$H = -\sum_{<ij>} J_{ij} S_i S_j - D \sum_i S_i^2, \quad (43)$$

where J_{ij} is the coupling between the nearest neighbors of the lattice, D is the single ion anisotropy and S_i is restricted by $S_i = \{-1, 0, 1\}$.

We present the correlation function identities for the Blume-Capel model which was obtained by Siqueira and Fittipaldi [9]. They are given by

$$\langle F(\{S\}) S_i \rangle = \left\langle F(\{S\}) \frac{2 \sinh \beta E_i}{2 \cosh \beta E_i + e^{-\beta D}} \right\rangle, \quad (44)$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \left\langle G(\{S\}) \frac{2 \cosh \beta E_i}{2 \cosh \beta E_i + e^{-\beta D}} \right\rangle, \quad (45)$$

with $E_i = \sum_j J_{ij} S_j$. Here, $F(\{S\})$ is different from S_i and $G(\{S\})$ is different from S_i^2.

Using the differential operator (12), we obtain

$$\langle F(\{S\}) S_i \rangle = \left\langle F(\{S\}) [\exp \sum_j \beta J_{ij} S_j \nabla_x] \right\rangle f(x) \Big|_{x=0} \quad (46)$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \left\langle G(\{S\}) [\exp \sum_j \beta J_{ij} S_j \nabla_x] \right\rangle g(x) \Big|_{x=0}, \quad (47)$$

with $\nabla_x = \partial/\partial x$. The functions $f(x)$ and $g(x)$ are given by

$$f(x) = \frac{2 \exp(\beta D) \sinh x}{1 + 2 \exp(\beta D) \cosh x} = -f(-x), \quad (48)$$

and

$$g(x) = \frac{2 \exp(\beta D) \cosh x}{1 + 2 \exp(\beta D) \cosh x} = g(-x). \quad (49)$$

Using the van der Waerden relation for the spin S=1, $S^{2n+1} = S$ and $S^{2n} = S^2$, we get the correlation function identities

$$\langle F(\{S\}) S_i \rangle = \Big\langle F(\{S\}) \prod_j [1 + S_j \sinh(\beta J_{ij} \nabla_x) + S_j^2 (\cosh(\beta J_{ij} \nabla_x) - 1)] \Big\rangle f(x)|_{x=0} \quad (50)$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \Big\langle G(\{S\}) \prod_j [1 + S_j \sinh(\beta J \nabla_x) + S_j^2 (\cosh(\beta J_{ij} \nabla_x) - 1)] \Big\rangle g(x)|_{x=0}. \quad (51)$$

2.9 Spin-1 Blume-Capel Model in a Kagome Lattice. Five-Site Cluster

In this section, we present the correlation function identity for the Blume-Capel model in a Kagome lattice. A spin-1 model with triquadratic interactions, single-ion anisotropy and magnetic field [52], was studied in a Kagome lattice. Using the Hamiltonian (43) and considering a cluster of five sites which retain the main aspects of the Kagome lattice (see Fig. 1), we deduced from (11) the identity formula for $< F(\{S\}) S_i >$ and $< G(\{S\}) S_i^2 >$, whose details of the derivation will be presented elsewhere and they are given by

$$\langle F(\{S\}) S_i \rangle = \left\langle F(\{S\}) \frac{\sum_{\{S_0, S_1, .., S_4\}} S_i e^{-\beta H_{\{5\}}}}{\sum_{\{S_0, S_1, ..., S_4\}} e^{-\beta H_{\{5\}}}} \right\rangle \quad (52)$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \left\langle G(\{S\}) \frac{\sum_{\{S_0, S_1, .., S_4\}} S_i^2 e^{-\beta H_{\{5\}}}}{\sum_{\{S_0, S_1, ..., S_4\}} e^{-\beta H_{\{5\}}}} \right\rangle, \quad (53)$$

where $F(\{S\})$ is a function of the variables of spins except for the spins $S_{\{n\}}$ of the cluster $\{n = 5\}$ and similarly $G(\{S\})$is a function of the variables of the spins except for the spins $S_{\{n\}}^2$ of the cluster $\{n = 5\}$. Moreover, $H_{\{5\}}$ is given by

$$H_{\{5\}} = -J S_0 \left(\sum_{i=1}^{4} S_i \right) - J (S_1 S_2 + S_3 S_4) - \sum_{i=1}^{4} S_j \left(\sum_{j=1}^{2} J_{ij} S_{ij} \right) - D \left(S_0^2 + \sum_{i=1}^{4} S_i^2 \right), \quad (54)$$

Using the van der Waerden relation for the spin $S = 1$, $S^{2n+1} = S$ and $S^{2n} = S^2$, we get the spin correlation identity

$$\langle F(\{S\}) S_i \rangle = \Big\langle F(\{S\}) \prod_{i=1}^{4} \prod_{j=1}^{2} [1 + S_{ij} \sinh(\beta J_{ij} \nabla_j) + S_{ij}^2 (\cosh(\beta J_{ij} \nabla_j) - 1)] \Big\rangle f(a_1, a_2, a_3, a_4)|_{(0,0,0,0)} \quad (55)$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \Big\langle G(\{S\}) \prod_{i=1}^{4} \prod_{j=1}^{2} [1 + S_{ij} \sinh(\beta J_{ij} \nabla_j) + S_{ij}^2 (\cosh(\beta J_{ij} \nabla_j) - 1)] \Big\rangle g(a_1, a_2, a_3, a_4)|_{(0,0,0,0)}, \quad (56)$$

where $f(a_1, a_2, a_3, a_4)$ and $g(a_1, a_2, a_3, a_4)$ are obtained from an explicit summation

$$f(a_1, a_2, a_3, a_4) = \left\langle F(\{S\}) \frac{\sum_{\{S_1,..,S_4\}} S_i e^{-\beta H_{\{5\}}}}{\sum_{\{S_1,..,S_4\}} e^{-\beta H_{\{5\}}}} \right\rangle \tag{57}$$

and

$$g(a_1, a_2, a_3, a_4) = \left\langle G(\{S\}) \frac{\sum_{\{S_1,..,S_4\}} S_i^2 e^{-\beta H_{\{5\}}}}{\sum_{\{S_1,..,S_4\}} e^{-\beta H_{\{5\}}}} \right\rangle, \tag{58}$$

with $a_j = \beta \sum_{j=1}^{2} J_{ij} S_{ij}, i = 1, \ldots, 4$.

As has been said in Section 2.4, beyond the single-site cluster theory, the complexity of the algebra increases and in the present case it is not possible to write down explicit expressions for the coefficients. We made use of an algebraic computation software for the determination of the expansion coefficients on the right hand side of (57) and (58) (see [53]). This comment also applies in Section 2.13.

2.10 Spin-1 Blume-Emery-Griffiths Model. One-Site Cluster

The Blume-Emery-Griffiths (BEG) model [11] is a generalization of the Blume-Capel model. The BEG model has been applied to magnetic system and also in the description of various physical systems such as $He^3{-}He^4$ mixture, ternary alloys, and fluids. It is a spin-1 model containing bilinear interactions with coupling J_{ij}, biquadratic interactions with coupling K_{ij}, and a single spin anisotropy D.

The Hamiltonian of the BEG model is defined by

$$H = -\sum_{<ij>} J_{ij} S_i S_j - \sum_{<ij>} K_{ij} S_i^2 S_j^2 - D \sum_i S_i^2, \tag{59}$$

where each S_i is restricted by $S_i = \{-1, 0, 1\}$.

Analogous to the deduction of the identity for correlation function of the Blume-Capel model, the identity for the Blume-Emery-Griffiths model was obtained [13] and is given by

$$\langle F(\{S\}) S_i \rangle = \left\langle F(\{S\}) \frac{2e^{\beta(E_i^{(S^2)} - D)} \sinh(\beta E_i^{(S)})}{1 + 2e^{\beta(E_i^{(S^2)} - D)} \cosh(\beta E_i^{(S)})} \right\rangle \tag{60}$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \left\langle G(\{S\}) \frac{2e^{\beta(E_i^{(S^2)} - D)} \cosh(\beta E_i^{(S)})}{1 + 2e^{\beta(E_i^{(S^2)} - D)} \cosh(\beta E_i^{(S)})} \right\rangle, \tag{61}$$

where $E_i^{(S)} = \sum_j J_{ij} S_j$ and $E_i^{(S^2)} = \sum_j K_{ij} S_j^2$. Here, $F(\{S\})$ is a function of the spin variables except for the spin S_i and $G(\{S\})$is a function of the spin variables except for the spins S_i^2.

Using the van der Waerden relation for the spin $S = 1$, $S^{2n+1} = S$ and $S^{2n} = S^2$, we get the identities for the correlation functions

$$\langle F(\{S\}) S_i \rangle = \Big\langle F(\{S\}) \prod_j [1 + S_j e^{(\beta K_{ij} \nabla_y)} \sinh(\beta J_{ij} \nabla_x) + S_j^2 (e^{(\beta K_{ij} \nabla_y)} \cosh(\beta J_{ij} \nabla_x) - 1)] \Big\rangle f(x, y)|_{(0,0)} \tag{62}$$

and

$$\left\langle G(\{S\}) S_i^2 \right\rangle = \Big\langle G(\{S\}) \prod_j [1 + S_j e^{(\beta K_{ij} \nabla_y)} \sinh(\beta J_{ij} \nabla_x) + S_j^2 (e^{(\beta K_{ij} \nabla_y)} \cosh(\beta J_{ij} \nabla_x) - 1)] \Big\rangle g(x, y)|_{(0,0)}, \tag{63}$$

where $\nabla_x = \frac{\partial}{\partial x}$ and $\nabla_y = \frac{\partial}{\partial y}$ are the two differential operators and the functions $f(x, y)$ and $g(x, y)$ are defined by

$$f(x, y) = \frac{2e^{(y - \beta D)} \sinh x}{1 + 2e^{(y - \beta D)} \cosh x} = -f(-x, y) \tag{64}$$

and

$$g(x, y) = \frac{2e^{(y - \beta D)} \cosh x}{1 + 2e^{(y - \beta D)} \cosh x} = g(-x, y). \tag{65}$$

2.11 Spin-1 Ashkin-Teller Model. One-Site Cluster

In order to study the effect of the anisotropy in the Ashkin-Teller model, we consider a lattice whose sites are occupied by two spin-1 variables, S_i and σ_i. The model contains a bilinear coupling J_{ij} between the variables S_i and between the variables σ_i, a quadratic coupling K_{ij} between the variables $S_i \sigma_i$ and in addition there is the single ion anisotropy D_1 for the two variables S_i and σ_i and the anisotropy D_2 for the variables $S_i \sigma_i$. Thus, the Hamiltonian is given by

$$H = -\sum_{<ij>} J_{ij} (S_i S_j + \sigma_i \sigma_j) - \sum_{<ij>} K_{ij} S_i S_j \sigma_i \sigma_j + D_1 \sum_i \left(S_i^2 + \sigma_i^2\right) + D_2 \sum_i (S_i \sigma_i)^2, \tag{66}$$

where $< ij >$ denote a pair of nearest-neighbor spins. With the same notations used for the spin-1/2 Ashkin-Teller model, we have obtained the expressions for the thermal averages $\langle F(\{\tau\}) \tau_i \rangle$ and $\left\langle G(\{\tau\}) \tau_i^2 \right\rangle$, whose details of the derivation will be presented elsewhere, and it is given by

$$\langle F(\{\tau\}) \tau_i \rangle = \left\langle F(\{\tau\}) f_{(\tau)}(E_i^{(S)}, E_i^{(\sigma)}, E_i^{(\sigma S)}) \right\rangle \tag{67}$$

and

$$\left\langle G(\{\tau\}) \tau_i^2 \right\rangle = \left\langle G(\{\tau\}) g_{(\tau)}(E_i^{(S)}, E_i^{(\sigma)}, E_i^{(\sigma S)}) \right\rangle, \tag{68}$$

where $E_i^{(\tau)}$, $L^{(\tau)}$, and τ have been defined in Section 2.7 as $\tau \equiv \{S, \sigma, S\sigma\}$, with each τ_i restricted by $\tau_i = \{-1, 0, 1\}$. Also, $F(\{\tau\})$ is a function of the variables of spins except for the spin τ_i and $G(\{\tau\})$ is a function of the variables of the spins except for the spins τ_i^2. Introducing the differential operator (12), we obtain

$$\langle F(\{\tau\})\tau_i\rangle = \left\langle F(\{\tau\})\prod_j\prod_{\{\tau\}} e^{\beta L_{ij}^{(\tau)}\tau_j\nabla_\tau}\right\rangle f_{(\tau)}(x,y,z)\big|_{(0,0,0)} \quad (69)$$

and

$$\left\langle G(\{\tau\})\tau_i^2\right\rangle = \left\langle G(\{\tau\}))\prod_j\prod_{\{\tau\}} e^{\beta L_{ij}^{(\tau)}\tau_j\nabla_\tau}\right\rangle g_{(\tau)}(x,y,z)\big|_{(0,0,0)}, \quad (70)$$

where $\nabla_S = \frac{\partial}{\partial x}$, $\nabla_\sigma = \frac{\partial}{\partial y}$ and $\nabla_{\sigma S} = \frac{\partial}{\partial z}$. Using the van der Waerden relation for spin $S = 1$, $\tau^{2n+1} = \tau$ and $\tau^{2n} = \tau^2$, we get

$$\langle F(\{\tau\})\tau_i\rangle = \Big\langle F(\{\tau\})\prod_j\prod_{\{\tau\}}[1+\tau_j\sinh(\beta L_{ij}^{(\tau)}\nabla_\tau) + \tau_j^2(\cosh(\beta L_{ij}^{(\tau)}\nabla_\tau)-1)]\Big\rangle . f_{(\tau)}(x,y,z)\big|_{(0,0,0)} \quad (71)$$

and

$$\left\langle G(\{\tau\})\tau_i^2\right\rangle = \Big\langle G(\{\tau\})\prod_j\prod_{\{\tau\}}[1+\tau_j\sinh(\beta L_{ij}^{(\tau)}\nabla_\tau) + \tau_j^2(\cosh(\beta L_{ij}^{(\tau)}\nabla_\tau)-1)]\Big\rangle . g_{(\tau)}(x,y,z)\big|_{(0,0,0)}. \quad (72)$$

The functions $f_{(\tau)}(x,y,z)$ and $g_{(\tau)}(x,y,z)$ are given by

$$f_{(S)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_x\mathcal{S}_y\mathcal{S}_z + \mathcal{S}_x\mathcal{C}_y\mathcal{C}_z] + 2\mathcal{S}_x}{\Lambda}, \quad (73)$$

$$f_{(\sigma)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_y\mathcal{S}_x\mathcal{S}_z + \mathcal{S}_y\mathcal{C}_x\mathcal{C}_z] + 2\mathcal{S}_y}{\Lambda}, \quad (74)$$

$$f_{(\sigma S)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_z\mathcal{S}_x\mathcal{S}_y + \mathcal{S}_z\mathcal{C}_x\mathcal{C}_y]}{\Lambda}, \quad (75)$$

$$g_{(S)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_x\mathcal{C}_y\mathcal{C}_z + \mathcal{S}_x\mathcal{S}_y\mathcal{S}_z] + 2\mathcal{C}_x}{\Lambda}, \quad (76)$$

$$g_{(\sigma)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_x\mathcal{C}_y\mathcal{C}_z + \mathcal{S}_x\mathcal{S}_y\mathcal{S}_z] + 2\mathcal{C}_y}{\Lambda} \quad (77)$$

and

$$g_{(\sigma S)}(x,y,z) = \frac{4e^{\beta(D_1+D_2)}[\mathcal{C}_x\mathcal{C}_y\mathcal{C}_z + \mathcal{S}_x\mathcal{S}_y\mathcal{S}_z]}{\Lambda}, \quad (78)$$

with

$$\Lambda = 4e^{\beta(D_1+D_2)}[\mathcal{C}_x\mathcal{C}_y\mathcal{C}_z + \mathcal{S}_x\mathcal{S}_y\mathcal{S}_z] + 2[\mathcal{C}_x+\mathcal{C}_y] + e^{-\beta D_1}, \quad (79)$$

where $\mathcal{C}_x$, $\mathcal{C}_y$, $\mathcal{C}_z$, $\mathcal{S}_x$, $\mathcal{S}_y$, and $\mathcal{S}_z$ are defined by

$$\mathcal{C}_x = \cosh(x) \quad \mathcal{C}_y = \cosh(y) \quad \mathcal{C}_z = \cosh(z)$$
$$\mathcal{S}_x = \sinh(x) \quad \mathcal{S}_y = \sinh(y) \quad \mathcal{S}_z = \sinh(z). \quad (80)$$

The same comment made at the end of Section 2.9 applies here. The algebra to obtain the coefficients is too long and prohibits the explicitation of the coefficients.

2.12 Potts Model. One-Site Cluster

The q-state Potts model [14] is a generalization of the Ising model to more than two states and has been subject of intense research interest. Historically, a four-state version of the model was first studied by Ashkin and Teller [5]. But the model of general q-states bears its current name after it was proposed by Domb [54] four decades ago to his then research student Potts as a thesis topic [55] .

The Hamiltonian of the q-state Potts model is

$$H = -q\sum_{i,j} J_{ij}\delta_{S_i,S_j} - q\sum_i h_i\delta_{S_i,0}, \quad (81)$$

where, $S_i = 0, 1, 2....., q-1$, δ_{S_i,S_j} denotes the Kronecker δ function and $h_i \geq 0$.

A generalization of Callen's identity was obtained for the q-state Potts model [15] for arbitrary functions $f(S_K)$ Therefore, the correlation function identity, involving a function $F(\{S\})$ different from $f(S_k)$, is

$$\langle F(\{S\})f(S_k)\rangle = \left\langle \frac{F(\{S\})\sum_{S_k=0}^{q-1} f(S_k)\mathrm{e}^{-\beta H}}{\sum_{S_k=0}^{q-1}\mathrm{e}^{-\beta H}}\right\rangle, \quad (82)$$

where $-\beta H = \sum_j K_{k,j}\delta_{S_k,S_j} + L_k\delta_{S_k,0}$, with $K_{k,j} = \beta q J_{k,j}$ and $L_k = \beta q h_k$. By choosing $f(S_k) = \delta_{S_k,0}$ and using (12), we rewrite (82) as follows

$$\left\langle F(\{S\})\delta_{S_k,0}\right\rangle = \left\langle F(\{S\})\prod_{n=0}^{q-1}\mathrm{e}^{\left(\sum_j K_{k,j}\delta_{n,S_j}\nabla_n\right)}\right\rangle f(\{x_n\})\Big|_{\{x_n\}=0} \quad (83)$$

or

$$\left\langle F(\{S\})\delta_{S_k,0}\right\rangle = \left\langle F(\{S\})\prod_j\sum_{n=0}^{q-1}\mathrm{e}^{(K_{k,j}\nabla_n)}\delta_{n,S_j}\right\rangle f(\{x_n\})\Big|_{\{x_n\}=0} \quad (84)$$

where $\nabla_n = \frac{\partial}{\partial x_n}$, $(n = 0, 1, 2....., q-1)$, and

$$f(\{x_n\}) = \frac{\mathrm{e}^{(L_k+x_0)}}{\mathrm{e}^{(L_k+x_0)} + \sum_{n=1}^{q-1}\mathrm{e}^{x_n}} \quad (85)$$

The order parameter $m \equiv \langle m_k\rangle \in [0, 1]$, associated with the q-state Potts model, is defined by

$$m_k \equiv \frac{q\delta_{(S_k,0)} - 1}{q-1}. \quad (86)$$

2.13 Z_2 Lattice Gauge Model. One-Site Cluster

Correlation identities were obtained [16] for Z_2 lattice gauge theory where the bonds of the plaquettes are decorated by generalized two-state Ising variables. The hamiltonian for the lattice gauge model is given by

$$H = -\beta J \sum_{P \subset \Lambda} \chi_P, \tag{87}$$

where i denotes the bonds of the lattice, P denotes the unit squares (plaquettes) of Λ, and $\chi_P = S_1 S_2 S_3 S_4$, with $S_i = \pm 1$ for all i. Let $S_D = S_{i1} \ldots S_{iD}$ denote a product of distinct bond variables in the lattice. We have

$$\langle S_D \rangle = \frac{Tr\left(S_D e^{-H}\right)}{Tr(e^{-H})}. \tag{88}$$

For a fixed bond, b occurring in S_D give a numerical ordering 1,2,... to the $2(d-1)$ plaquettes that have one bond in common with b. The following identity is obtained [16]

$$\langle S_D \rangle = \left\langle S_D^{(b)} \prod_{j,k,l} \left[\cosh(\beta J \nabla_x) + S_j S_k S_l \ \sinh(\beta J \nabla_x)\right] \right\rangle f(x) \bigg|_{x=0}, \tag{89}$$

where $\nabla_x \equiv \frac{\partial}{\partial x}$, j, k, l are bonds neighbors of bond b. The function f is given by

$$f(x) = \tanh x = -f(-x) \tag{90}$$

and $S_D^{(b)}$ is S_D with the bond b deleted.

2.14 Transverse Ising Model. One-Site Cluster

This model is a quantum model and is has been applied to describe the phase transitions and the properties of hydrogen bonded ferroelectrics [56, 57] and magnetic ordered materials [58]. In one dimension, it has no phase transition at finite temperatures. However, in one dimension, at zero temperature, it is ordered up to a critical value of the transverse field. This model was solved exactly in one dimension [32, 59, 60] and in upper dimensions exists approximations to regions of low temperature and high temperature.

The transverse Ising model is described by a two-state model with bilinear coupling between the longitudinal components of spin and transverse field coupled to the transverse component of the spin. The Hamiltonian is given by

$$H = -\sum_{<ij>} J_{ij} S_i^z S_j^z - \Omega \sum_i S_i^x, \tag{91}$$

where J_{ij} is the bilinear coupling, Ω is the transverse field, and $< ij >$ represents the nearest neighbors of the lattice. S^z is the longitudinal component of the spin and S^x is the transverse component of the spin, which are spin-1/2 Pauli operators.

The identity of the correlation function for the Ising model in a transverse field [17, 61] is given by

$$\langle F(\{S^\alpha\}) S_i^z \rangle = \left\langle F(\{S^\alpha\}) \frac{Tr_{(i)} S_i^z e^{-\beta H_i}}{Tr_{(i)} e^{-\beta H_i}} \right\rangle - \left\langle F(\{S^\alpha\}) \left[\frac{Tr_{(i)} S_i^z e^{-\beta H_i}}{Tr_{(i)} e^{-\beta H_i}} - S_i^z \right] \Delta \right\rangle, \tag{92}$$

where H_i is the hamiltonian associated with the lattice site i, H' the hamiltonian of the rest of the lattice, and $\Delta = 1 - e^{-\beta H_i} e^{-\beta H'} e^{\beta(H_i + H')}$. Equation (92) is exact, but because of the non-commutativity of the Pauli matrices $[H_i, H] \neq 0$ and $[H_i, H'] \neq 0$. Therefore, we have $\Delta \neq 0$ and (92) is untractable. Let us use an approximation in the second term the (92), i.e.,

$$\left\langle F(\{S^\alpha\}) \left[\frac{Tr_{(i)} S_i^z e^{-\beta H_i}}{Tr_{(i)} e^{-\beta H_i}} - S_i^z \right] \Delta \right\rangle \approx \left\langle F(\{S^\alpha\}) \left[\frac{Tr_{(i)} S_i^z e^{-\beta H_i}}{Tr_{(i)} e^{-\beta H_i}} - S_i^z \right] \right\rangle \langle \Delta \rangle . \tag{93}$$

Diagonalizing and taking the partial trace over the site i, we obtain

$$\langle F(\{\sigma^\alpha\}) \sigma_i^z \rangle \approx \left\langle F(\{\sigma^\alpha\}) \frac{\sum_j J_{ij} \sigma_j^z}{\sqrt{(2\Omega)^2 + (\sum_j J_{ij} \sigma_j^z)^2}} \times \tanh\left(\frac{\beta}{4} \sqrt{(2\Omega)^2 + (\sum_j J_{ij} \sigma_j^z)^2} \right) \right\rangle, \tag{94}$$

where $\sigma_i^z = 2 S_i^z$ and $F(\{\sigma^\alpha\})$ is a function of the variables σ except σ_i^z. Using the differential operator (12), we obtain

$$\langle F(\{\sigma^\alpha\}) \sigma_i^z \rangle \approx \left\langle F(\{\sigma^\alpha\}) \prod_j [\cosh(J_{ij} \nabla_x) + \sigma_j^z \sinh(J_{ij} \nabla_x)] \right\rangle \times f(x)|_{x=0}, \tag{95}$$

with the function $f(x)$ defined by

$$f(x) = \frac{x}{\sqrt{(2\Omega)^2 + x^2}} \tanh\left(\frac{\beta}{4} \sqrt{(2\Omega)^2 + x^2} \right) = -f(-x). \tag{96}$$

We also obtain an approximated expression for transverse correlation function given by

$$\langle F(\{\sigma^\alpha\}) \sigma_i^x \rangle \approx \left\langle F(\{\sigma^\alpha\}) \prod_j [\cosh(J_{ij} \nabla_x) + \sigma_j^z \sinh(J_{ij} \nabla_x)] \right\rangle g(x)|_{x=0}, \tag{97}$$

with the function $g(x)$ defined by

$$g(x) = \frac{2\Omega}{\sqrt{(2\Omega)^2 + x^2}} \tanh\left(\frac{\beta}{4} \sqrt{(2\Omega)^2 + x^2} \right) = g(-x). \tag{98}$$

Because of the approximations (see (93)), the expressions (95) and (97) are not exact.

3 Correlation Function Inequalities

In this section, some results of rigorous inequalities of the correlation function are presented. The first results are Griffiths inequalities [19, 20] and the generalization of Griffiths inequalities by Kelly and Sherman [21] and Ginibre [22, 26]. Also, it is presented Newman's gaussian domination inequality [23, 24]. Then, the correlations inequalities are presented for the continuous spin models described by Sylvester [25] and by Ellis and Monroe, Percus, Lebowitz, and Griffiths et al. [27–30, 33]. Correlation inequalities have been proved for spin systems with non-purely ferromagnetic interactions possessing a certain symmetry [62]. These inequalities generalize the inequalities of Griffiths, Ginibre, Lebowitz, Schrader, Messager-Miracle-Sole [63] and Percus. This section concludes with the correlations inequalities of quantum models with results of Hurst and Sherman [31] for the quantum Heisenberg model, Suzuki [32] for the XY model, Monroe [34] for vector spin systems, and Contucci and Lebowitz [35, 36] for quantum spin systems. It is worth mentioning that Griffiths inequalities were also proven for antiferromagnets [64], for the XY model [65], for the Ashkin-Teller model [66], for non-interacting N-vector models [67], for gaussian spin glass [68], and for the Potts model [69–75] among others. Other correlation inequalities were proven for classical and quantum continuous systems [76] and for the Ashkin-Teller model [77].

3.1 Griffiths Inequalities for the Spin 1/2 Ising Model.

Griffiths [19] obtained remarkable inequalities for the correlation functions of spin 1/2 Ising ferromagnetic model. These inequalities were generalized by Kelly and Sherman [21] for systems involving an arbitrary number of spins. Ginibre [22] obtained a simple proof of the second inequality.

Denote N the set $\{1, 2, \ldots, n\}$, where n is the number of sites of the lattice. Consider the space of 2^n spin configurations $(S_1, S_2, \ldots, S_n)$, where each S_i is restricted to values ± 1.

For each pair (i, j) of distinct indices in N, consider the real number,

$$J_{ji} = J_{ij} \geq 0. \tag{99}$$

Griffiths [19] studied a system of Ising spins 1/2, in zero magnetic field, coupled by a purely ferromagnetic interaction with hamiltonian

$$H = -\sum_{<ij>} J_{ij} S_i S_j, \tag{100}$$

where $< ij >$ represents the nearest neighbors of the lattice.

To all lattice and unlimited range of interactions, we have the first theorem of Griffiths,

Theorem 1 (Griffiths I) *For a system described by* (99) *and* (100)*, with binary correlation functions described by* $\langle S_k S_l \rangle$ *and any pair* k, l*, we have*

$$\langle S_k S_l \rangle \geq 0. \tag{101}$$

It can be observed that increasing the ferromagnetic interaction between any pair of spins tends to enhance the tendency of other pairs to line up parallel of spins. This observation is incorporated in the second theorem of Griffiths,

Theorem 2 (Griffiths II) *For a system described by* (99) *and* (100)*, and where* k, l, m, n *denote any four spins (not necessarily all different), the following is true:*

$$\beta^{-1} \frac{\partial \langle S_k S_l \rangle}{\partial J_{mn}} = \langle S_k S_l S_m S_n \rangle - \langle S_k S_l \rangle \langle S_m S_n \rangle \geq 0. \tag{102}$$

This result still holds when J_{kl} or J_{mn} (or both) are negative.

Theorem 3 (Griffiths III) *For a system described by* (99) *and* (100)*, and where* k, l, *and* n *denote any three spins, the following relation holds:*

$$\langle S_k S_n \rangle \geq \langle S_k S_l \rangle \langle S_l S_n \rangle \tag{103}$$

and it is unnecessary to assume that J_{kl} and J_{ln} are nonnegative.

Remark Consequences of these results, in particular the second one, are

- (i) $\langle S_k S_l \rangle$ never decreases if any J_{mn} is increased.
- (ii) If an Ising model with ferromagnetic interactions displays a long-range order, this long-range order increases if more ferromagnetic interactions are added.

3.2 Generalization of Griffiths Inequalities

Kelly and Sherman [21] under the same hypothesis extending (101) and (102), generalized Griffiths results to an arbitrary number of spins.

For each subset R of the index set N, define

$$S^R = \prod_{i \in R} S_i. \tag{104}$$

For each non-empty subset R of N, let the number

$$J_R = \prod_{i,j} J_{ij} \geq 0 \tag{105}$$

be given.

A physical system is defined by coupling interactions J_{ij}, which are real function on N, and with which are associated, respectively, a Hamiltonian, a probability density, a partition function, and correlation functions by the formula

$$H(R) = -\sum_{R\in N} J_R S^R, \tag{106}$$

$$W = Z^{-1}\exp(-\beta H(R)), \tag{107}$$

$$\left\langle S^R\right\rangle = Z^{-1}\sum_{A\in N} S^R \exp[-\beta H(R)]. \tag{108}$$

$$Z = \sum_{R\in N}\exp[-\beta H(R)]. \tag{109}$$

Then, under the same hypothesis as in (101) and (102), they extended Griffiths results:

Theorem 4 (Kelly e Sherman) *In the probability space defined by* (107) *and* (109)*, we have*

$$\left\langle S^R\right\rangle \geq 0, \qquad \forall\ R\subset N \tag{110}$$

$$\left\langle S^R S^D\right\rangle - \left\langle S^R\right\rangle\left\langle S^D\right\rangle \geq 0, \qquad \forall\ R, D\subset N \tag{111}$$

$$\frac{1}{\beta}\frac{\partial\left\langle S^R\right\rangle}{\partial J_D} = \left\langle S^R S^D\right\rangle - \left\langle S^R\right\rangle\left\langle S^D\right\rangle, \ \ \forall\ R, D\subset N. \tag{112}$$

Ginibre [22] obtained a simple proof of the second inequality of Kelly and Sherman.

Another important inequality for the Ising model was obtained by Newman [23, 24]. It states that

$$\langle S_i F(\{S\})\rangle \leq \sum_j \langle S_i S_j\rangle\langle \partial F(\{S\})/\partial S_j\rangle \tag{113}$$

where $F(\{S\})$ are polynomials of the spin variables with positive coefficients (see Theorem (5.1) of [24]).

3.3 Correlation Inequalities for Ising Models with Continuous Spins

Sylvester [25] presents in his work a sequence of theorems and corollaries of correlation inequalities for ferromagnetic Ising models with continuous spins. These models of continuous spins generalize the spin-1/2 classical Ising model, in which the variable spin is not restricted to the values ± 1, but can take any real value ν, with some measure in a single spin. In this section, correlation inequalities of Griffiths [19], Ginibre [22], Ellis and Monroe [27], Percus [28], Lebowitz [29] and Griffiths et al. [30] are presented for the ferromagnetic Ising model with continuous spins.

Considering the same expressions defined in (106–109), the results presented in the work of Sylvester [25] are

Theorem 5 (Griffiths I) *Let $R \in N$ be a family of sites in a finite Ising ferromagnet (N, H, ν), with Hamiltonian given by* (106) *and arbitrary (even) single-spin measure ν. Then,*

$$\left\langle S^R\right\rangle \geq 0. \tag{114}$$

Theorem 6 (Ginibre) *Let $R, D \in N$ be families of sites in a finite Ising ferromagnet (N, H, ν), with Hamiltonian given by* (106) *and arbitrary (even) single-spin measure ν. Then,*

$$\langle q_R t_D\rangle \geq 0, \tag{115}$$

where q, t are defined by

$$t_i = \frac{1}{\sqrt{2}}(S_i + \sigma_i), \quad q_i = \frac{1}{\sqrt{2}}(S_i - \sigma_i), \quad i\in N. \tag{116}$$

where S_i and σ_i are spin variables.

Corollary 1 (Griffiths II) *Let $R, D \in N$ be families of sites in the model of Theorem 6. Then*

$$\frac{\partial\left\langle S^R\right\rangle}{\partial J_D} \equiv \left\langle S^R S^D\right\rangle - \left\langle S^R\right\rangle\left\langle S^D\right\rangle \geq 0. \tag{117}$$

Theorem 7 (Percus) *Let $R \in N$ be a family of sites in a finite Ising model (N, H, ν) with Hamiltonian, given by*

$$H = -\sum_{i\leq j} J_{ij} S_i S_j - \sum_i h_i S_i, \quad J_{ij}\geq 0, \tag{118}$$

where h_i is an external field and S_i are spins with a single measure ν arbitrary. Then

$$\langle q_R\rangle \geq 0, \tag{119}$$

where q is defined by (116).

Corollary 2 *Let i, j be sites in the model of Theorem 7. Then*

$$\frac{\partial\left\langle S_i\right\rangle}{\partial h_j} \equiv \left\langle S_i S_j\right\rangle - \left\langle S_i\right\rangle\left\langle S_j\right\rangle \geq 0. \tag{120}$$

Theorem 8 (Ellis-Monroe) *Let $R, D, A, B \in N$ be sites in a finite Ising ferromagnet (N, H, ν) with Hamiltonian given by*

$$H = -\sum_{i\leq j} J_{ij} S_i S_j - \sum_i h_i S_i, \quad J_{ij}\geq 0 \text{ and } h_i \geq 0. \tag{121}$$

h_i is an external field and S_i are spins with a single measure ν arbitrary. Then

$$\langle \alpha_R \beta_D \gamma_A \delta_B\rangle \geq 0, \tag{122}$$

where α, β, γ, and δ are defined by

$$\alpha_i = \frac{1}{\sqrt{2}}(t_i + t_i'), \quad \beta_i = \frac{1}{\sqrt{2}}(t_i - t_i') \tag{123}$$

$$\gamma_i = \frac{1}{\sqrt{2}}(q_i' + q_i), \quad \delta_i = \frac{1}{\sqrt{2}}(q_i' - q_i) \tag{124}$$

where t and q are defined by (116). t_i' and q_i' are given by

$$t_i' = \frac{1}{\sqrt{2}}(S_i' + \sigma_i'), \quad q_i' = \frac{1}{\sqrt{2}}(S_i' - \sigma_i'), \quad i \in N. \tag{125}$$

where S_i' and σ_i' are spin variables.

Corollary 3 (Lebowitz) *Let $R, D \in N$ be families of sites in the model of Theorem 8. Then*

$$\langle t_R t_D \rangle - \langle t_R \rangle \langle t_D \rangle \geq 0 \tag{126}$$

$$\langle q_R q_D \rangle - \langle q_R \rangle \langle q_D \rangle \geq 0 \tag{127}$$

$$\langle t_R q_D \rangle - \langle t_R \rangle \langle q_D \rangle \leq 0 \tag{128}$$

where t, q are defined by (116).

Corollary 4 (Griffiths-Hurst-Sherman) *Let i, j, k be sites in the model of Theorem 8. Then*

$$\frac{\partial^2 \langle S_i \rangle}{\partial h_j \partial h_k} \equiv \langle S_i S_j S_k \rangle - \langle S_i \rangle \langle S_j S_k \rangle - \langle S_j \rangle \langle S_i S_k \rangle - \langle S_k \rangle \langle S_i S_j \rangle + 2 \langle S_i \rangle \langle S_j \rangle \langle S_k \rangle \leq 0. \tag{129}$$

Corollary 5 . *Let i, j, k, l be sites in the model of Theorem 8. Then*

$$\langle S_i S_j S_k S_l \rangle - \langle S_i S_j \rangle \langle S_k S_l \rangle - \langle S_i S_k \rangle \langle S_j S_l \rangle - \langle S_i S_l \rangle \langle S_j S_k \rangle + 2 \langle S_i \rangle \langle S_j \rangle \langle S_k \rangle \langle S_l \rangle \leq 0. \tag{130}$$

3.4 Correlation Function Inequalities for Quantum Models

In this section, correlation inequalities are presented for the quantum models. Ginibre [26] presented a general framework in which Griffiths inequalities on the correlations of ferromagnetic spin systems are extended to the noncommutative quantum case; its theory includes as special cases the Ising model with arbitrary spins and the plane rotator model. The first inequality appearing in this section is the generalization of Griffiths inequalities for the quantum Heisenberg model by Hurst and Sherman [31]; then one has the inequality of the work of Suzuki [32] for XY model; follows, a set of correlation function inequalities for a vector spin systems shown by Monroe [34] and concludes with the work of Lebowitz and Contucci[35, 36] presenting correlation inequalities in models of quantum spins.

Hurst and Sherman [31] showed that Griffiths II is not true in general for the quantum ferromagnetic Heisenberg model. The Hamiltonian of the Heisenberg model is given by

$$H = -\sum_{<ij>} J_{ij} \vec{S}_i . \vec{S}_j, \tag{131}$$

where $\vec{S}_i$ are spins represented by Pauli matrices. Hurst and Sherman [31] showed the following inequality

$$\frac{\partial \left\langle \vec{S}_i . \vec{S}_j \right\rangle}{\partial J_{ij}} < 0. \tag{132}$$

They also showed that Griffiths I, which states that $\left\langle \vec{S}_i . \vec{S}_j \right\rangle \geq 0$, is true, even when an external magnetic field is present.

Suzuki [32] proved some inequalities for the XY model. Starting from the Hamiltonian

$$H = -\sum_R \left(J_R^z S_R^z + J_R^x S_R^x \right), \tag{133}$$

where R denotes an arbitrary subset of sites in lattice N and S_R^z, S_R^x are Pauli matrices with $J_A^z \geq 0$, $J_A^x \geq 0$ and

$$S_R^z = \prod_{j \in R} S_j^z \quad \text{and} \quad S_R^x = \prod_{j \in R} S_j^x, \tag{134}$$

Suzuki obtained

$$\left\langle S_R^z \right\rangle \geq 0, \tag{135}$$

$$\left\langle S_R^z S_D^z \right\rangle \geq \left\langle S_R^z \right\rangle \left\langle S_D^z \right\rangle, \tag{136}$$

$$\frac{\partial \left\langle S_R^z \right\rangle}{\partial J_D^z} \geq 0 \tag{137}$$

and

$$\frac{\partial \left\langle S_R^z \right\rangle}{\partial J_D^x} \leq 0, \tag{138}$$

with the same definitions for J_R and S_R given by (104) and (105).

This produces a simple extension of Griffiths-Kelly-Sherman inequalities for the quantum system XY model.

Monroe [34] proved a set of correlation function inequalities, including Griffiths-Kelly-Sherman, for a lattice system of N sites where on each site there is a vector spin $\vec{S} = (S^x, S^z)$, with $|\vec{S}| = 1$. The system is assumed to have a Hamiltonian of the form

$$H = -\sum_{<ij>} J_{ij} (S_i^x S_j^x + S_i^z S_j^z) - \sum_i h^x S_i^x - \sum_i h^z S_i^z, \tag{139}$$

where $h^x \geq 0$ and $h^z \geq 0$ are external field. Monroe [34] showed the following inequalities

$$\left\langle \prod_{i \in R} S_i^x \prod_{j \in D} S_i^x \right\rangle - \left\langle \prod_{i \in R} S_i^x \right\rangle \left\langle \prod_{j \in D} S_i^x \right\rangle \geq 0, \tag{140}$$

$$\left\langle \prod_{i\in R} S_i^z \prod_{j\in D} S_j^z \right\rangle - \left\langle \prod_{i\in R} S_i^z \right\rangle \left\langle \prod_{j\in D} S_j^z \right\rangle \geq 0 \tag{141}$$

and

$$\left\langle \prod_{i\in R} S_i^x \prod_{j\in D} S_j^z \right\rangle - \left\langle \prod_{i\in R} S_i^x \right\rangle \left\langle \prod_{j\in D} S_j^z \right\rangle \leq 0, \tag{142}$$

where R and D denote arbitrary subsets of sites in lattice N. He also showed the following inequality

$$\langle S_i^x \rangle \left\langle S_j^z S_k^z \right\rangle - \left\langle S_j^z \right\rangle \langle S_i^x S_k^z \rangle \geq 0, \tag{143}$$

where the pairs (i, j) and (i, k) are nearest neighbors.

In order to show the results obtained by Contucci and Lebowitz [36], consider a finite set of sites in lattice N, with Hamiltonian given by

$$H = -\sum_{<ij>} J_{ij}(\alpha_x S_i^x.S_j^x + \alpha_y S_i^y.S_j^y + \alpha_z S_i^z.S_j^z) - \sum_{<ij>} K_{ij} S_i^z.S_j^z$$
$$-h_1 \sum_i S_i^x - h_2 \sum_i S_i^y, \tag{144}$$

where $S_i^{(x)}, S_i^{(y)}, S_i^{(z)}, \;\; i\in N$ are Pauli matrices.

Theorem 9 *For systems described by the quantum Hamiltonian H (144), the following inequality holds, for all $R \subset N$ and all $\alpha \geq 0$,*

$$Av\ (J_R \langle \Phi_R \rangle_H) \geq 0. \tag{145}$$

where $\Phi_{ij} = \alpha_x S_i^x.S_j^x + \alpha_y S_i^y.S_j^y + \alpha_z S_i^z.S_j^z$ and Av denotes the average over the distributions of the $J's$,

with the same definitions for J_R and Φ_R or S_R^x, S_R^y, S_R^z given by (104) and (105).

4 Decay of the Correlations in Ferromagnets

Simon [1] proved a variety of correlation inequalities which bound intermediate distance correlations from below by long distance correlations. Exponential decay of correlation functions was also studied [78].

Theorem 10 *Let $\langle S_\alpha S_\gamma \rangle$ denote the two-point function of a spin-1/2 nearest neighbor (infinite volume and free boundary condition) Ising ferromagnet at some fixed temperature. Fix α, γ, and B, a set of spins whose removal breaks the lattice in such a way that α and γ lie in distinct components. Then*

$$\langle S_\alpha S_\gamma \rangle \leq \sum_{\delta\in B} \langle S_\alpha S_\delta \rangle \langle S_\delta S_\gamma \rangle. \tag{146}$$

Theorem 11 *Under the hypothesis of Theorem 10, suppose that*

$$\langle S_\alpha S_\gamma \rangle \leq C\, |\alpha - \gamma|^{-\mu}, \tag{147}$$

with $\mu + 1 > d$, where d the dimension of the lattice. Then

$$\langle S_\alpha S_\gamma \rangle \leq C e^{-m|\alpha-\gamma|}, \tag{148}$$

for some $m > 0$.

Theorem 12 *Let $f(\alpha - \gamma)$ be a translation invariant non-negative bounded function of $\alpha, \gamma \in Z^\nu$. Suppose that for some subset B of Z^ν with $B \subset \{\delta / |\delta| \leq R\}$ and all γ with $|\gamma| > R$, we have that*

$$f(\gamma) \leq \sum_{\delta\in B} a(\delta) f(\gamma - \alpha), \tag{149}$$

where $0 \leq a(\delta)$ and

$$\sum_{\delta\in B} a(\delta) \equiv A_0 < 1. \tag{150}$$

where $a(\delta)$ represents the coefficients of the correlation functions.

Theorem 13 *If $H = -\sum J_{ij} S_i S_j$, $J_{ij} \geq 0$, if*

$$\sum \tanh(J_{i0}) < 1 \tag{151}$$

and if $\{i | J_{i0} \neq 0\}$ is bounded, then

$$\langle S_i S_j \rangle \leq C e^{-m|i-j|}, \tag{152}$$

for some $m > 0$.

5 Rigorous Upper Bounds on the Critical Temperature

In this section, we obtain rigorous upper bounds for the critical temperature of some models. We present the results for the Ising model [37], Ising model on a Kagome lattice, Ising model on a pyrochlore lattice, Ashkin-Teller model, Ising model with four-spin interactions [6], Blume-Capel model (BC) [10], Z_2 Lattice Gauge model [16], and transverse Ising model [18].

In order to compare the results for the transition temperatures obtained by the present methods, described in the next subsections, we quote here the exact or numerical values for T_c for the Ising model and the Blume-Capel model in various lattices. Ising model, square lattice T_c = 2.269 [41]; cubic lattice T_c = 4.511 [79]; Kagome lattice T_c = 2.143 [42]; and pyrochlore lattice T_c = 4.00 [80]. Blume-Capel model (single ion anisotropy D = 0), square lattice T_c = 1.688 [81] and cubic lattice T_c = 3.192 [81].

From the identities of the of spin correlation function $< F(\{S\}) S_i >$ obtained in Section 2 (taking $F(\{S\}) = S_r$) and the application the rigorous inequalities, presented in

Section 3, to the higher order correlation functions appearing in the r.h.s of the correlation identities, we obtain an inequality of the form

$$\langle S_i S_r \rangle \leq \sum_j a_j \langle S_j S_r \rangle, \quad 0 \leq a_j \leq 1 \tag{153}$$

which, when iterated, leads to the exponential decay of spin-spin correlations whenever

$$\sum_j a_j(T) < 1. \tag{154}$$

Defining $\overline{T}_c$ so that $\sum_j a_j(\overline{T}_c) = 1$, it comes out that the critical temperature T_c must satisfy the bound $T_c < \overline{T}_c$. The preceding procedure is the application of Simon's theorems [1] to obtain rigorous upper bound on the critical temperature.

5.1 Spin-1/2 Ising Model

For spin-1/2 Ising Model, in the square and cubic lattices, Sá Barreto and Carroll [37] found the following results for the upper bounds on T_c

5.1.1 For the square lattice (d = 2, z = 4)

$$a_j = A_{11} - |A_{12}| \langle S_j S_k \rangle_{1D}. \tag{155}$$

5.1.2 For the cubic lattice (d = 3, z = 6)

$$a_j = A_{13} - |A_{14}| \langle S_j S_k \rangle_{1D} + 5A_{15}. \tag{156}$$

where $\langle S_j S_k \rangle_{1D} = \tanh^2(\beta J)$ which is the spin correlation of the one-dimensional spin 1/2 Ising model separated by two lattice parameter. See coefficients[1] A_{1i} in reference [37]. Evaluating numerically the value of the critical temperature T_c such that $\sum_j a_j \leq 1,\ a_j > 0$, the upper bounds are obtained

$d = 2,\ \ T_c = 3.01399$ and $d = 3\ \ T_c = 5.42315$.

Another application for the spin-1/2 Ising model, in the square and cubic lattice, was presented by Monroe [82]. He used for the correlation functions of two spins separated by two lattices parameter the exact value determined in a cluster of nine spins in the square lattice. For the cubic lattice, he determined the correlation function of two spins in a cluster of eight spins.

He obtained the following upper bounds

$d = 2,\ \ T_c = 2.8752$ and $d = 3\ \ T_c = 5.0607$.

[1]Remarck:the coefficients A_{si}, where s stands for the subsection, appearing along the next five subsections, have a different notation in the cited papers.

5.2 Spin-1/2 Ising Model on a Kagome Lattice

For the Ising model on a Kagome lattice, the upper bounds on T_c were determined and the following result, whose details of the calculation will be presented elsewhere, was obtained

$$\begin{aligned} a_j = {} & A_{21} - |A_{22}| \langle S_{11} S_{21} \rangle \\ & +(-|A_{23}| - |A_{24}| - |A_{25}|) \langle S_{11} S_{12} \rangle \\ & +5(A_{26} + A_{27} + A_{28} + A_{29}) \langle S_{11} S_{12} S_{21} S_{22} \rangle \\ & -|A_{210}| \langle S_{11} S_{12} \rangle^3 . \end{aligned} \tag{157}$$

In (157), the two-spin correlation function and the four-spin correlation function are bound from above by their values obtained on the limit of an infinite magnetic field applied to all sites except those appearing in the four-spin correlations. Evaluating numerically the value of the critical temperature T_c , it was obtained the upper bounds $T_c = 2.5492$.

5.3 Spin-1/2 Ising Model on a Pyrochlore Lattice

For the Ising model on a pyrochlore lattice, the upper bounds for T_c were determined from an identity obtained from a cluster of four sites (see Section 2.6). The following results, whose details of the calculation will be presented elsewhere, were obtained

$$\begin{aligned} a_j = {} & A_{31} - [(|A_{32}| + |A_{33}|) \langle S_{11} S_{12} \rangle + |A_{34}| \langle S_{11} S_{21} \rangle] \\ & +5[(A_{35} + A_{36} + A_{37} + A_{38}) \langle S_{11} S_{12} S_{21} S_{22} \rangle] \\ & -[(|A_{39}| + |A_{310}| + |A_{311}| + |A_{312}|) \langle S_{11} S_{12} \rangle^3] \\ & +9[(A_{313}+A_{314}+A_{315}) \langle S_{11} S_{12} S_{13} S_{21} S_{22} S_{23} S_{31} S_{32} \rangle] \\ & -|A_{316}| \langle S_{11} S_{12} \rangle^5 . \end{aligned} \tag{158}$$

Similarly to the previous section, in (158), the two- , four-, and eight-spin correlation functions are bound from above by their values obtained on the limit of an infinite magnetic field applied to all sites except those appearing in the four- and eight-spin correlations. Evaluating numerically the value of the critical temperature T_c, it was obtained the upper bounds $T_c = 4.6773$.

5.4 Spin-1/2 Ashkin-Teller Model

For the Ashkin-Teller spin-1/2 model, the upper bounds for T_c in the hexagonal, square, and cubic lattices were determined and the details of the calculation will be presented elsewhere. The following results for the coefficients for the hexagonal and square lattices are presented in the next subsections. We will not show the coefficients for the cubic lattice because they are too long. However, the critical temperature obtained for the three lattices will be shown on Table 1.

Table 1 Upper bounds on T_c of the hexagonal, square, and cubic lattices

		$K_4 = 0$	$K_2 = K_4$	$K_4 \to \infty$
Hexagonal	Bounds 1	1.9988	2.5687	3.9984
	Bounds 2	1.9988	2.5529	3.9984
	Bounds 3	1.9592	2.5290	3.9184
Square	Bounds 1	3.0138	3.7119	6.1349
	Bounds 2	3.0138	3.6913	6.0313
	Bounds 3	2.9010	3.6697	5.6818
Cubic	Bounds 1	0.1844	0.1601	0.0857
	Bounds 2	0.1844	0.1597	0.0861
	Bounds 3	0.2028	0.1572	0.1001

5.4.1 For the hexagonal lattice ($d = 2$, $z = 3$)

$$a_{1j} = [A_{41} + (-|A_{42}| - |A_{43}| - |A_{44}|)\langle\sigma_1\sigma_2\rangle - |A_{45}|\langle\sigma_1\sigma_2\rangle^2], \tag{159}$$

$$a_{2j} = [A_{41} + A_{43} + A_{45} + (-|A_{42}| - |A_{44}|)\langle\sigma_1\sigma_2\rangle], \tag{160}$$

$$b_{1j} = [B_{41} + (-|B_{42}| - |B_{43}| - |B_{44}|)\langle\sigma_1\sigma_2\rangle - |B_{45}|\langle\sigma_1 S_1\sigma_2 S_2\rangle], \tag{161}$$

$$b_{2j} = [B_{41} + (-|B_{42}| - |B_{43}| - |B_{44}|)\langle\sigma_1\sigma_2\rangle - |B_{45}|\langle\sigma_1\sigma_2\rangle]. \tag{162}$$

5.4.2 For the square lattice ($d = 2$, $z = 4$)

$$a_{1j} = [A_{46} + 2A_{411} + (-|A_{47}| - |A_{48}| - |A_{49}|)\langle\sigma_1\sigma_2\rangle - (|A_{410}| - |A_{412}|) \times \langle\sigma_1\sigma_2\rangle^2 - |A_{413}|\langle\sigma_1\sigma_2\rangle^3], \tag{163}$$

$$a_{2j} = [A_{46} + A_{48} + A_{410} + 2A_{411} + A_{413} + (-|A_{47}| - |A_{49}| - |A_{412}|), \times \langle\sigma_1\sigma_2\rangle], \tag{164}$$

$$b_{1j} = [B_{46} + (-|B_{47}| - |B_{48}| - |B_{49}| - |B_{410}| - |B_{411}|)\langle\sigma_1\sigma_2\rangle + 2B_{412} - |B_{413}|\langle\sigma_1 S_1\sigma_2 S_2\rangle], \tag{165}$$

$$b_{2j} = [B_{46} + (-|B_{47}| - |B_{48}| - |B_{49}| - |B_{410}| - |B_{411}|)\langle\sigma_1\sigma_2\rangle + 2B_{412} - |B_{413}|\langle\sigma_1 S_1\sigma_2 S_2\rangle]. \tag{166}$$

where a_{kj} and b_{kj} are the coefficients of the inequalities obtained from the (35) for the variables σ and σS, respectively.

The inequalities obtained contain correlation functions of two spins separated by two lattice parameters (i.e., $\langle\sigma_1\sigma_2\rangle$ and $\langle\sigma_1 S_1\sigma_2 S_2\rangle$). Three approximations were used to calculate these two-spin correlation functions.

(i) (Bounds 1): $\langle\tau_k\tau_l\rangle_{1D} = \tanh^2(\beta J)$ which is the one dimensional spin-spin correlation for the spin-1/2 Ising model separate by two lattice parameter.

(ii) (Bounds 2): With $\gamma_\tau = \langle\tau_k\tau_l\rangle$ determined in the one dimensional spin-1/2 Ashkin-Teller model. The function γ_τ is given by

$$\gamma_\tau = \frac{1 - \sqrt{1 - (f_\tau(2\beta J, 2\beta J, 2\beta K))^2}}{f_\tau(2\beta J, 2\beta J, 2\beta K)} \tag{167}$$

where $f_\tau(2\beta J, 2\beta J, 2\beta K)$ are given by (36), (37), and (38).

(iii) (Bounds 3): For two-dimensional lattices, the correlation functions of two spins separated by two lattices parameter were exactly determined in a cluster of 16 spins in the hexagonal lattice and 9 spins in the square lattice. For the cubic lattice, the correlation function of two spins was determined exactly in a cluster of 27 spins located in the cubic lattice sites (see similar treatment in [82]).

In Table 1, the upper bounds on T_c are presented for the Ising model limit ($K_4 = 0$), the four states Potts model limit ($K_2 = K_4$) and ($K_4 \to \infty$)- limit.

5.5 Spin-1/2 Ising Model with Four Spins Interaction

The upper bounds on T_c for this model were obtained for three lattices [6]. In the hexagonal lattice, the four-spin interaction contains the center spin and its neighbors and the upper bound is obtained from equation

$$a_j = 3A_{51} - |A_{52}|\langle S_j S_k\rangle_{1D}. \tag{168}$$

In the study, the four-spin interaction on the square lattice [6] was used three different models (denoted by A, B, and C). Model A is similar to that used in the hexagonal lattice and models B and C have been studied before [83]. Let S_0 denotes the center spin and S_1, S_2, S_3, and S_4 denote the nearest neighbors spins of site 0; and S_5, S_6, S_7, and S_8 denote the next-nearest neighbors of site 0 (in Fig. 3 we show the square lattice with two- and four-spin interactions). The four-spin interaction in model A couples any three sites nearest neighbors of the center site 0 in a square lattice; in other words, it considers $S_0S_1S_2S_3$, $S_0S_2S_3S_4$, $S_0S_3S_4S_1$, and $S_0S_4S_1S_2$. The four-spin interaction in model B couples any four next-nearest neighbors sites of the vertices of all unit squares; or, it considers $S_0S_1S_2S_5$, $S_0S_2S_3S_6$, $S_0S_3S_4S_7$, and $S_0S_1S_4S_8$. In model C, only the spins on the vertices of alternate unit squares are coupled with four-spin interaction, or $S_0S_1S_2S_5$ and $S_0S_3S_4S_7$. The resulting equations from which one obtains the upper bounds are (Fig. 3):

(i) Model A

$$a_j = 4A_{53} - 4|A_{54}|\langle S_j S_k\rangle_{1D}. \tag{169}$$

(ii) Model B

$$a_j = 4A_{55} - 4|A_{56}|\langle S_j S_k\rangle_{1D} + 5A_{57}. \tag{170}$$

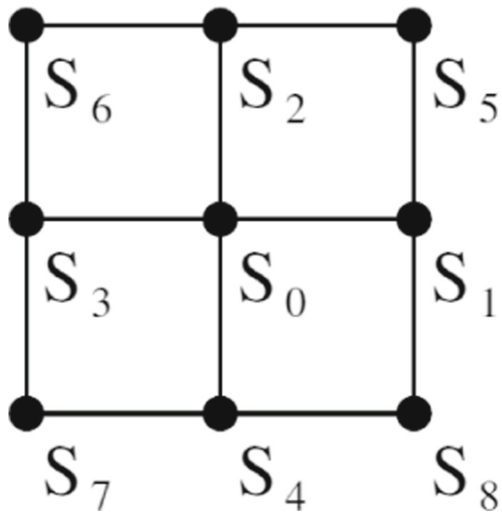

Fig. 3 Square lattice with two-and four-spin interactions

(iii) Model C:

$$a_j = 4A_{58} - 4|A_{59}| \langle S_j S_k \rangle_{1D} + 5A_{510}. \qquad (171)$$

where $\langle S_j S_k \rangle_{1D} = \tanh^2(\beta J)$ is the one dimensional spin-spin correlation separated by two lattice parameters for the spin-1/2 Ising model. See coefficients A_{5i} in reference [6]. Evaluating numerically the value of the critical temperature T_c , it was obtained the following upper bounds, shown in Table 2.

5.6 Spin-1 Blume-Capel Model. One-Site Cluster

For the spin-1 Blume-Capel model, the upper bounds for T_c in the hexagonal, square, and cubic lattices were obtained [10], showing the following results

5.6.1 Hexagonal lattice (d = 2, z = 3)

$$a_j = A_{61} - |A_{62}| - |A_{63}| \langle S_j S_k \rangle_{1D} + A_{64} \qquad (172)$$

5.6.2 Square lattice (d = 2, z = 4)

$$a_j = A_{65} - |A_{66}| \langle S_j S_k \rangle_{1D} + A_{67} + A_{68} \langle S_j S_k \rangle_{1D} + A_{69}. \qquad (173)$$

5.6.3 Cubic lattice (d = 3, z = 6)

$$a_j = A_{610} - |A_{611}| - |A_{612}| \langle S_j S_k \rangle_{1D} + A_{613} + A_{614} \langle S_j S_k \rangle_{1D} + A_{615}$$
$$+ A_{616} + A_{617} + A_{618} + A_{619} + A_{620}. \qquad (174)$$

where $\langle S_j S_k \rangle_{1D}$ is the one dimensional spin-spin correlation for the spin-1 Blume Capel model, separate by two parameter sites, given by

$$\langle S_j S_k \rangle_{1D} = (1 + \sqrt{1 - 2f(2\beta J)})/(f(2\beta J)), \qquad (175)$$

with $f(x)$ given by (48). See coefficients A_{6i} in reference [10]. Evaluating numerically the value of the critical temperature T_c, the following upper bounds for the limits $D = 0$ (spin-1 Ising model) and $D = \infty$ (spin-1/2 Ising model) were obtained and shown in Table 3.

5.7 Spin-1 Blume-Capel Model in a Kagome Lattice

For the Blume-Capel model in a Kagome lattice, the upper bounds on T_c were determined, the details of the calculation will be presented elsewhere, and the results are

$$\begin{aligned} a_j = {} & A_{71} - |A_{72}| - |A_{73}| - |A_{74}| - |A_{75}| - |A_{76}| - |A_{77}| - |A_{78}| \\ & -|A_{79}| - |A_{710}| - |A_{711}| - |A_{712}| < S_{11}S_{12} > -|A_{713}| < S_{11}S_{21} > \\ & +(-|A_{714}| - |A_{715}| - |A_{716}|) < S_{11}S_{12} > +(|A_{717}| - |A_{718}| - |A_{719}|) \\ & \times < S_{11}S_{21} > +(-|A_{720}| - |A_{721}|) < S_{11}S_{12} > +(-|A_{722}| - |A_{723}|) \\ & \times < S_{11}S_{21} > +(-|A_{724}| - |A_{725}|) < S_{11}S_{12} >^2 -|A_{726}| < S_{11}S_{12} > \\ & \times < S_{21}S_{31} > -|A_{727}| < S_{11}S_{12} >^2 -|A_{728}| < S_{11}S_{12} >< S_{21}S_{31} > \\ & -|A_{729}| < S_{11}S_{12} >^3 +5(A_{730} + A_{731}) < S_{11}S_{12}S_{21}S_{22} > +A_{732} \\ & +A_{733} + A_{734} + A_{735} + A_{733} + A_{737} + A_{738} + A_{739} + A_{740} + A_{741} \\ & +A_{742} + A_{743} + A_{744} + A_{745} + A_{746} + A_{747} + A_{748} + A_{749} + A_{750} \\ & +A_{751} + A_{752} + A_{753} + A_{754} + A_{755} + A_{756}, \qquad (176) \end{aligned}$$

where the two- and four- spins correlation functions appearing in the coefficient a_j were exactly determined in a cluster with nine spins (see [82]). Evaluating numerically the value of the critical temperature T_c such that $\sum_{|j|=1} a_j \leq 1,\ a_j > 0$, the following upper bounds were obtained : $k_B T_c/J = 1.8541$ for $D/J = 0$ and $k_B T_c/J = 2.6533$ for $D/J \to \infty$.

Table 2 Upper Bounds on kT_C/J

four-spin interaction	$z = 3$	$z_A = 4$	$z_B = 4$	$z_C = 4$
K = 0	1.998	3.014	3.014	3.014
K/J = 0.2	1.94	2.97	2.98	2.99

5.8 Z_2 Lattice Gauge Model

Sá Barreto and Carroll [16] determined the correlation equalities for Z_2 lattice gauge theories and applied them to obtain area decay of the Wilson loop observable in a range of the coupling parameter. They found the following results for β which guarantees the exponential decay of the Wilson loop, therefore, implying the confinement.

5.8.1 For the $d = 3$

Let β be such that $4A_{81} < 1$. Then

$$\langle W(C)\rangle \leq \mathrm{e}^{(-|ln[4A_{81}]|A)}. \quad (177)$$

5.8.2 For the $d = 4$

Let β be such that $6(A_{82} + A_{83}) < 1$. Then

$$\langle W(C)\rangle \leq \mathrm{e}^{(-|ln[6(A_{82}+A_{83})]|A)}. \quad (178)$$

$\langle W(C)\rangle$, the well-known Wilson loop observable of a Z_d lattice gauge theory, is the finite lattice Gibbs ensemble average with Wilson's action Boltzmann factor $e^{\beta\sum_P \chi_P}$ and C is a planar rectangle of area A. See coefficients A_{8i} in reference [16].

5.9 Transverse Ising Model

For the transverse Ising Model, the upper bounds for T_c in the square and cubic lattices were determined [18] giving the following results

5.9.1 For the square lattice $d = 2$, $z = 4$

$$a_j = A_{91} - |A_{92}| \left\langle S_j^z S_k^z \right\rangle_{1D}. \quad (179)$$

Table 3 Upper Bounds on kT_C/J

Value anisotropy	z = 3	z = 4	z = 6
D = 0	1.591	2.322	3.678
D = ∞	1.999	3.070	5.084

Table 4 Upper Bounds on k_BT_c/J and Ω_c

	z=3		z=6	
	k_BT_c/J	Ω_c	k_BT_c/J	Ω_c
$\Omega_c = 0$	3.014	–	5.423	–
$k_BT_c/J = 0$	–	1.3755J	–	2.4466J

5.9.2 For the cubic lattice $d = 2$, $z = 6$

$$a_j = A_{93} - |A_{94}| \left\langle S_j^z S_k^z \right\rangle_{1D} + 5A_{95}. \quad (180)$$

where $\left\langle S_j S_k \right\rangle_{1D} = \tanh^2(\beta J)$ is the one dimensional spin-spin correlation for the spin-1/2 Ising model. See coefficients A_{9i} in reference [18]. Evaluating numerically the value of the critical temperature T_c , the following upper bounds were obtained, where Ω is the transverse field (Table 4),

6 Concluding Remarks

In this review, we presented a methodology to obtain rigorous bounds on the critical temperature of spin systems described by various models. The methodology is based on three theories or concepts: spin correlation identities, rigorous (theorems) correlation inequalities, and the decay of the spin correlation functions. First, in Section 2, we derived the generalization of the spin correlation function identity obtained from finite size clusters containing n sites. We end up with a representation formula for the spin-spin correlation function $\langle S_i S_r\rangle$ in terms of higher order correlations $\langle S_i S_j \cdots S_k S_r\rangle$. The identity thus obtained is a generalization of Callen's identity [2] for the spin-1/2 Ising model, and it was applied, in the same Section 2, to other classical spin models in various lattices. Second, in Section 3, we presented the theorems for the spin correlation inequalities, which would be used latter to decouple the higher order spin correlations appearing in the r.h.s of the identities of the spin-spin correlation function $\langle S_0 S_r\rangle$ obtained in the previous Section 2. Following, in Section 4, the theorems on the decay of the spin correlation functions were presented [1]. Later, in Section 5, the identity obtained in Section 2 was used for an iteration procedure, which allowed the estimate of the exponential decay rates, as a function of βJ, as $\|i - r\| \to \infty$. In other words, we replaced the identity obtained in Section 2, by the inequality $\langle S_i S_r\rangle \leq \left[\sum_j a_j(T)\right]\langle S_j S_r\rangle$. The above procedure could now be iterated and the exponential decay of correlations will follow as long as the condition $\sum_j a_j(T) < 1$ is satisfied. Defining $\overline{T}_c$ so that $\sum_j a_j(\overline{T}_c) = 1$, it comes out that the critical temperature T_c must satisfy the bound $T_c < \overline{T}_c$. The results obtained are rigorous upper bounds on the critical temperature. The rigorous upper bounds on the critical temperature improve over the results of effective-type theories (mean field- and effective field- approximations). Also, the upper bounds obtained from the n-site ($n > 1$) cluster spin correlation identities and the rigorous correlation inequalities are better than those obtained by the one-site cluster. The methodology presented here can be applied to other models in various lattices and for cluster with different number of n sites.

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