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Color Superconductivity, BPS Z_k Strings and Monopole Confinement in $\mathcal{N}=2$ and $\mathcal{N}=4$ Super Yang-Mills Theories

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We review some recent developments on BPS string solutions and monopole confinement in the Higgs (or color) superconducting phase of $\mathcal{N}=2$ and $\mathcal{N}=4$ super Yang-Mills theories. In particular, the monopole magnetic fluxes are shown to be always integer linear combinations of string fluxes. Moreover, a bound for the threshold length of the string breaking is obtained. When the gauge group $SU(N)$ is broken to Z_N, the BPS string tension satisfies the Casimir scaling law. Furthermore, in the $SU(3)$ case the string solutions are such that they allow the formation of a confining system with three monopoles.

1 Introduction

It has long been believed that particle confinement at the strong coupling regime should be a phenomenon dual to monopole confinement in a (color) superconductor in weak coupling. Therefore, the study of monopole confinement in weak coupling may shed some light on particle confinement. Since dualities are better understood for supersymmetric theories, it is interesting to analyze monopole confinement in these theories.

We shall review the results in [1, 2, 3] where we analyzed monopole confinement in non-Abelian Yang-Mills-Higgs theories at the weak coupling regime with two symmetry breaking. In the first symmetry breaking the theory is in the Coulomb phase with solitonic monopole solutions which may fill representations of a non-Abelian group. Then, in the second symmetry breaking, the theory is in the Higgs or (color) superconducting phase with strings or flux tubes. We show explicitly that in these theories always the magnetic fluxes of the monopoles are integer multiple of the strings fluxes. The first symmetry breaking is due to the expectation value of a complex scalar ϕ_3 in the adjoint representation. Then, the second symmetry breaking is due to two complex scalars ϕ_1 and ϕ_2 in complex conjugated representations. In order to exist topological string solutions, two possible representations are considered: in [3] ϕ_1 and ϕ_2 are in the adjoint representation. On the other hand, in [1, 2] ϕ_1 is in the representation $R^{\rm sym}_{k\lambda_\phi}$ which is the symmetric part of the direct product of k fundamental representations R_{λ_ϕ}, with $k\geq 2$, and in ϕ_2 in the complex conjugated representation. In particular, if $k=2$, this is exactly the same representation as that of a diquark condensate, where by quark we mean a fermion in a fundamental representation R_{λ_ϕ} of the gauge group G. We have chosen the potential to be the bosonic part of $\mathcal{N}=4$ or $\mathcal{N}=2$ super Yang-Mill (SYM) theories with some deformation mass terms. These potentials appear naturally from the BPS string conditions. One of the main difference between the two analyzed representations for ϕ_1 and ϕ_2 mentioned above, is the following: when one considers the representation $R^{\rm sym}_{k\lambda_\phi}$, during the second symmetry breaking, just a $U(1)$ factor inside G is broken to a discrete subgroup, similarly to what happens in a superconductor, and it produces a monopole-antimonopole confinement. On the other hand, when ϕ_1 and ϕ_2 are in the adjoint representation, the full group G is broken to a discrete subgroup, which produce a color superconductor. For $G=SU(3)$ we showed that this kind of breaking produces a confining system with three different monopoles, besides the monopoles-antimonopole system. We also showed explicitly for $G=SU(N)$ that the BPS string tensions satisfy the Casimir scaling law. In [4, 5] was pointed out that this deformend $\mathcal{N}=4$ SYM theories should have a weakly coupled Higgs phase with magnetic flux tubes and this phase should be dual to a strongly coupled confining phase in the dual theory. One of our aims was to analyze many properties of these magnetic flux tubes.

When the scalars ϕ_1 and ϕ_2 are in the same representation as that of a diquark condensate, one could think of ϕ_1 and ϕ_2 as being themselves diquark condensates. In this case, we would have a situation quite similar to the one in an ordinary superconductor, described by the Abelian-Higgs theory with the scalar being a Cooper pair. If $G=SU(N)$, the scalar(s) in the adjoint representation could also be thought to be interpreted as quark-antiquark condensate(s). However it is important to note that all the results described here do not depend if the scalars are condensates or not. For $G=SU(3)$, these two kinds of condensates are the color

sextet and octet. They are expected to exist in the color superconducting phase of (dense) QCD at the weak coupling [6, 7]. The effective theory describing these condensates are not well known. One could think that the theory considered here when the gauge group is $G = SU(3)$, as been a toy model for an effective theory of these condensates. Then, one conclude that an effective theory for these condensates could have monopoles, flux tubes and monopole confinement, depending on the form of the potential. In the dual theory, one might conjecture that these scalars could be monopole-monopole and monopole-antimonopole condensates.

2 Deformed $\mathcal{N} = 2$ Super Yang-Mills theories

As is well known, the Abelian-Higgs in the broken phase is an effective theory for a superconductor with the complex scalar field ϕ being interpreted as a electron pair condensate. In this theory since the $U(1)$ gauge group is broken to a discrete subgroup there are topological flux tubes or string solutions with string tensions T satisfying

$$T \geq \frac{1}{2} q_\phi a^2 \left|\Phi_{\text{st}}\right| , \tag{1}$$

where

$$\Phi_{\text{st}} \equiv \int d^2x \, B_3 = \frac{2\pi}{q_\phi} n \, , \;\; n \in Z \tag{2}$$

is the string magnetic flux and $q_\phi = 2e$ is the electric charge of ϕ. The lower bound in (1) is attained by the BPS string. If one puts a (Dirac) monopole and antimonopole in a superconductor, their magnetic lines could not spread over space but must rather form a string which gives rise to a confining potential between the monopoles. This idea only makes sense since the (Dirac) monopole magnetic flux is $\Phi_{\text{mon}} = g = 2\pi/e$, which is an integer multiple of the string's magnetic flux quantization condition (2), allowing one to attach to the monopole two strings with $n = 1$. Then, using the electromagnetic duality of Maxwell theory one could map this monopole confining system in the weak coupling regime to an electric charge confining system in the strong coupling.

Let us generalize some of these ideas to a non-Abelian theory. Let us consider an arbitrary gauge group G, without $U(1)$ factors and such that $\Pi_0(G) = 0 = \Pi_1(G)$, like for example $G = SU(N)$. In [1, 2], we considered the Lagrangian

$$\begin{aligned} L &= -\frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu} + \frac{1}{2} \left(D_\mu \phi_3\right)_a^* \left(D^\mu \phi_3\right)_a + \\ &\quad + \frac{1}{2} \sum_{i=1}^{2} \left(D_\mu \phi_i^\dagger\right) \left(D^\mu \phi_i\right) - V(\phi) \end{aligned} \tag{3}$$

with potential given by

$$V(\phi) = \frac{1}{2} \left(\sum_{p=1}^{3} \left(d_a^p\right)^2 + \sum_{m=1}^{2} F_m^\dagger F_m \right)$$

where

$$\begin{aligned} d_a^3 &= \frac{e}{2} \left(\phi_{3b}^* i f_{bca} \phi_{3c} + \phi_m^\dagger \sigma_{mn}^3 T_a \phi_m - m \text{Re}\left(\phi_{3a}\right) \right) , \\ d_a^p &= \frac{e}{2} \left(\phi_m^\dagger \sigma_{mn}^p T_a \phi_n \right) , \quad p = 1, 2, \\ F_1 &= e \left(\phi_{3a}^\dagger T_a - \frac{\mu}{e} \right) \phi_1 , \\ F_2 &= e \left(\phi_{3a} T_a - \frac{\mu}{e} \right) \phi_2 , \end{aligned}$$

with σ^p being the Pauli matrices and T_a being the generators of G. This potential is the bosonic part of $\mathcal{N} = 2$ super Yang-Mills with one flavor and a breaking mass term. The scalar ϕ_{3a} in the adjoint representation belongs to the vector supermultiplet and the scalars ϕ_1 and ϕ_2 belong to a massive hypermultiplet. The real parameter μ gives a bare mass to ϕ_1 and ϕ_2 and m gives a bare mass to the real part of ϕ_3 and therefore breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 0$. In [1], we started with a generic potential and have shown that in order to obtain the BPS string conditions, the potential is almost constrained to have this form. We shall consider the theory in the weak coupling regime, and therefore we shall not consider the quantum corrections to the potential.

3 Phases of the theory

Let us review very quickly some of the Lie algebra conventions adopted. The Lie algebra generators satisfy the commutation relations

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_j, E_\alpha] &= \alpha^j E_\alpha , \\ [E_\alpha, E_{-\alpha}] &= \frac{2\alpha \cdot H}{\alpha^2} , \end{aligned}$$

where the upper index in α^j means the j component of the root α. Let us denote by α_i the simple roots and λ_i the fundamental weights which satisfy the relation

$$\frac{2\lambda_i \cdot \alpha_j}{\alpha_i^2} = \delta_{ij} .$$

The weights states $|\omega\rangle$ of a representation satisfy

$$v \cdot H \left|\omega\right\rangle = v \cdot \omega \left|\omega\right\rangle .$$

As mentioned in the introduction, in [1, 2], we considered ϕ_1 in $R_{k\lambda_\phi}^{\text{sym}}$, the symmetric part of the direct product of k fundamental representations $R_{\lambda_\phi}^{\text{sym}}$, where $k \geq 2$ and λ_ϕ is an arbitrary fundamental weight. This representation possess in particular the weight state $|k\lambda_\phi\rangle$, which will be responsible for one of the symmetry breakings as we shall see.

Returning to our physical problem. The vacua must be solutions of $V(\phi) = 0$ which is equivalent to

$$d^p = 0 = F_m \, . \tag{4}$$

In order to the topological string solutions to exist, we look for vacuum solutions of the form

$$\begin{aligned} \phi_1^{\rm vac} &= a\,|k\lambda_\phi\rangle\,, \\ \phi_2^{\rm vac} &= 0 \\ \phi_3^{\rm vac} &= b\lambda_\phi\cdot H, \\ W_\mu^{\rm vac} &= 0\,, \end{aligned} \tag{5}$$

where a is a complex constant and b is real. As explained in detail in [8, 1, 2], the above vacuum configuration produce a symmetry breaking

$$\begin{aligned} G \;&\to\; G_{\phi_3}\equiv[K\times U(1)]/Z_l \;\to \\ &\to\; G_{\phi_1}\equiv[K\times Z_{kl}]/Z_l \end{aligned} \tag{6}$$

where K is a subgroup of G and Z_l is a discrete subgroup of $U(1)$ and K. In the particular case $G = SU(N)$ and $\lambda_\phi = \lambda_1$, the fundamental weight of the N dimensional representation, we have the symmetry breaking

$$\begin{aligned} SU(N) \;&\to\; G_{\phi_3}\equiv[SU(N-1)\times U(1)]/Z_{N-1} \;\to \\ &\to\; G_{\phi_1}\equiv[SU(N)\times Z_{k(N-1)}]/Z_{N-1} \end{aligned} \tag{7}$$

The first symmetry breaking is due to $\phi_3^{\rm vac}$, with $b\neq 0$, and the second is due to $\phi_1^{\rm vac}$, with $a\neq 0$.

From the vacuum equations (4) one can conclude that

$$\begin{aligned} |a|^2 &= \frac{mb}{k}\,, \\ \left(kb\lambda_\phi^2-\frac{\mu}{e}\right)a &= 0\,. \end{aligned}$$

There are three possibilities:

(i) If $m\mu<0 \;\Rightarrow\; a=0=b$ and the gauge group G remains unbroken.

(ii) If $m=0,\ \mu\neq 0 \;\Rightarrow\; a=0$ and b can be any constant. In this case, $\phi_3^{\rm vac}$ produces the first symmetry breaking in (6) or (7) which corresponds to the Coulomb phase.

(iii) If $m\mu>0 \;\Rightarrow$

$$|a|^2=\frac{m\mu}{k^2e\lambda_\phi^2}\;,\qquad b=\frac{\mu}{ke\lambda_\phi^2} \tag{8}$$

and it happens the second symmetry breaking, which corresponds to the Higgs or superconducting phase.

Let us analyze each of these phases.

4 Coulomb phase

This phase occurs when G is broken to G_{ϕ_3}. The $U(1)$ factor in G_{ϕ_3} is generated by $\phi_3^{\rm vac}$. As we have seem, that symmetry breaking can happen only when $m=0$ and therefore the $\mathcal{N}=2$ symmetry is restored since m was a supersymmetry breaking parameter. In this phase, since $\Pi_2(G/G_{\phi_3})=Z$, there exist magnetic monopoles. The stable or fundamental BPS monopoles are those with lowest magnetic charge [9]. These fundamental monopoles, are believed to fill representations of the gauge subgroup $K^{\rm v}$ [10, 11]. The magnetic charges of monopoles for a general symmetry breaking has been obtained long time ago in [12, 9]. In particular for the first symmetry breaking in (6) or (7), the magnetic charge for the fundamental monopoles, can be written as [2]

$$g\equiv\frac{1}{|\phi_3^{\rm vac}|}\int dS_i{\rm Re}\,(\phi_3^a)\,B_i^a=\frac{2\pi}{e|\lambda_\phi|}=\frac{2\pi k}{q_\phi} \tag{9}$$

where the integral is taking over the closed surface surrounding the monopole, $B_i^a\equiv-\epsilon_{ijk}G_{jk}^a/2$ are the non-Abelian magnetic fields and

$$q_\phi=ek|\lambda_\phi|\,. \tag{10}$$

is electric charge of $\phi_1^{\rm vac}$ [2]. In Eq. (9) appears the real part of ϕ_{3a} since the vacuum configuration $\phi_3^{\rm vac}$ which produces the first symmetry breaking, only the real part is non-vanishing.

These monopoles fill supermultiplets of $\mathcal{N}=2$ supersymmetry and satisfy the mass formula

$$M_{\rm mon}=|\phi_3^{\rm vac}||g|\,. \tag{11}$$

In particular for the symmetry breaking

$$SU(N)\;\to\;[SU(N-1)\times U(1)]/Z_{N-1},$$

since $|\lambda_\phi|=|\lambda_1|=\sqrt{(N-1)/N}$, then[1]

$$g=\frac{2\pi}{e}\sqrt{\frac{N}{N-1}}.$$

In this case, the fundamental monopoles are expected to fill the $(N-1)$ representation of $SU(N-1)$ [10, 11].

5 Higgs or superconducting phase

5.1 The BPS Z_k-string solutions

This phase occurs when G is broken to G_{ϕ_1}. Moreover, since $m\neq 0$, $\mathcal{N}=2$ supersymmetry is broken to $\mathcal{N}=0$. In this phase, the $U(1)$ factor in G_{ϕ_3} is broken to the discrete subgroup Z_k and, like in the Abelian-Higgs theory, the magnetic flux lines associated to this $U(1)$ factor cannot spread over space. However, since G is broken in such a way that $\Pi_1(G/G_{\phi_1})=Z_k$, these flux lines may form topological Z_k-strings. In [1], a Z_k string ansatz was constructed, associated to each of the $(k-1)$ non trivial group elements of the discrete group Z_k. We have also obtained the BPS string conditions. Putting the ansatz into these BPS conditions we obtained that the functions which appear in the ansatz must satisfy exactly the same differential equations with same boundary conditions as for the BPS string in the Abelian-Higgs theory. The existence of non trivial solutions for these differential equations has been proven by Taubes[13].

[1] Note that for $G=SU(2)$, this result is obtained considering that $|\lambda_1|=1/\sqrt{2}$. However, for this group it is usually adopted the convention $|\lambda_1|=1/2$, which results the standard charge $g=4\pi/e$ for the stable monopole.

5.2 Z_k-string magnetic flux, monopole confinement and the string tension

In this phase, the monopole's magnetic lines associated to the broken $U(1)$ factor can no longer spread radially over space. However, these $U(1)$ could form flux tubes and the monopole get confined. In order for that to happen, the monopole flux Φ_{mon} in this $U(1)$ direction, which is equal to the magnetic charge (9), must be an integer multiple of the string fluxes Φ_{st} in this $U(1)$ direction. We define

$$\Phi_{\text{st}} \equiv \frac{1}{|\phi_3^{\text{vac}}|} \int d^2x \, \text{Re}(\phi_3^a) B_3^a \tag{12}$$

similarly to the monopole magnetic flux definition (9), but with surface integral taken over the plane perpendicular to the string. We obtained for our BPS Z_k string solutions that

$$\Phi_{\text{st}} = \frac{2\pi n}{q_\phi}, \quad n \in Z_k \tag{13}$$

where each value n is associated to a Z_k group elements used to construct the $(k-1)$ solutions.

Therefore we can conclude that Φ_{mono} can be equal for example to k times Φ_{st} for $n = 1$. This can be interpreted that for one monopole we could attach k Z_k-strings with $n = 1$. That is consistent with the fact the set k Z_k-strings with $n = 1$ belongs to the trivial sector of $\Pi_1(G/G_{\phi_1})$ and therefore can terminate at some point. However, since it has a non-vanishing magnetic flux it must terminate in a magnetic source, i.e., a monopole. It is important to stress the fact that being in the trivial topological sector does not mean that this set of strings has total vanishing flux. For the particular case $G = SU(2)$ and $k = 2$, the field ϕ_1 is in the three dimensional representation which is the adjoint of $SU(2)$. Then we can see that all these results are consistent with some well-known results for the Z_2 string of $SU(2)$ Yang-Mills-Higgs theory, as explained in [14, 15]. In this theory there are at least two complex scalars in the adjoint representation which produce the symmetry breakings $SU(2) \to U(1) \to Z_2$. In the Higgs phase, the stable Z_2 string has flux $2\pi/e$. In this phase, two strings get attached to a 't Hooft-Polyakov monopole with magnetic charge $g = 4\pi/e$, and can produce the monopole-antimonopole confinement.

We have shown that the string tension must satisfy the bound [1]

$$T \geq \frac{me}{2} |\phi_3^{\text{vac}}| \, |\Phi_{\text{st}}| = \frac{1}{2} q_\phi |a|^2 |\Phi_{\text{st}}|, \tag{14}$$

where $|a|$ given by Eq. (8) is the modulus of ϕ_1^{vac}, which produces the second symmetry breaking. That result is very similar to the $U(1)$ result given by Eq. (1). The string tension bound hold for the BPS string. Since the tension is constant, it produces a confining potential between monopoles increasing linearly with their distance. From string tension bound one can obtain easily that the threshold length d^{th} for the set of strings to break producing a new monopole-antimonopole pair, with masses (11), satisfies the bound[2]

$$d^{\text{th}} \leq \frac{4}{me} \, .$$

It is interesting to note that, unlike the Abelian-Higgs theory, in our theory the bare mass μ of ϕ_1 and ϕ_2 is not required to satisfy $\mu^2 < 0$ in order to happen the spontaneous symmetry breaking. Therefore, since one could interpret $\phi_1$and ϕ_2 as monopole condensates (when $k = 2$) in the dual theory, the monopole mass do not need to satisfy the problematic condition $M^2_{\text{mon}} < 0$ mentioned by 't Hooft [16]. The same thing happens in the theory where all the scalar are in the adjoint, analyzed in the next sections.

6 Deformed $\mathcal{N} = 4$ (or $\mathcal{N} = 2^*$) super Yang-Mills theories

Let us now analyze the monopole confinement in the theory with three complex scalars ϕ_s, $s = 1, 2, 3$, in the adjoint, as considered in [3]. Once more we shall consider a gauge group G without $U(1)$ factors and such that $\Pi_0(G) = 0 = \Pi_1(G)$. Let us consider the Lagrangian

$$L = -\frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu} + \frac{1}{2} \left(D_\mu \phi_s^*\right)_a \left(D^\mu \phi_s\right)_a - V(\phi).$$

We shall consider the potential

$$V(\phi) = \frac{1}{2} \left[(d_a)^2 + f_{sa}^\dagger f_{sa} \right] \tag{15}$$

where

$$d_a \equiv \frac{e}{2} \left(\phi_{sb}^* i f_{abc} \phi_{sc} - m \text{Re} \left(\phi_{3a} \right) \right), \tag{16}$$

and

$$\begin{aligned} f_1 &\equiv \frac{1}{2} \left(e \left[\phi_3, \phi_1 \right] - \mu \phi_1 \right) , \\ f_2 &\equiv \frac{1}{2} \left(e \left[\phi_3, \phi_2 \right] + \mu \phi_2 \right) , \\ f_3 &\equiv \frac{1}{2} \left(e \left[\phi_1, \phi_2 \right] - \mu_3 \phi_3 \right) . \end{aligned} \tag{17}$$

This is the potential of the bosonic part of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with some mass term deformations which break completely supersymmetry. If we set $m = 0$, $\mathcal{N} = 1$ supersymmetry is restored and we obtain the potential considered in [4]. If further $\mu_3 = 0$ we recover the potential of $\mathcal{N} = 2$ with a massive hypermultiplet in the adjoint representation. Finally, if also $\mu = 0$, we obtain $\mathcal{N} = 4$. As usual, we shall denote by $\mathcal{N} = 2^*$, $\mathcal{N} = 1^*$ and $\mathcal{N} = 0^*$ to the theories which are obtained by adding deformation mass terms to $\mathcal{N} = 4$ SYM theory.

7 Phases of the theory

The vacua of the theory are solutions of

$$G_{\mu\nu} = D_\mu \phi_s = V(\phi) = 0 \, . \tag{18}$$

The condition $V(\phi_s) = 0$ is equivalent to

$$d_a = 0 = f_{sa} \, . \tag{19}$$

We are looking for vacuum solutions which produce the symmetry breaking

$$G \to U(1)^r \to C_G,$$

where r is the rank of G and C_G its center. For the particular case of $G = SU(N)$, that corresponds to the symmetry breaking

$$SU(N) \to U(1)^{N-1} \to Z_N.$$

For the first phase transition magnetic monopoles will appear. Then, in the second phase transition magnetic flux tubes or strings (if C_G is non-trivial) will appear and the monopoles will become confined. In order to produce this symmetry breaking we shall look for vacuum solutions of the form

$$\begin{aligned} \phi_1^{\text{vac}} &= a_1 T_+ \,, \\ \phi_2^{\text{vac}} &= a_2 T_- \,, \\ \phi_3^{\text{vac}} &= a_3 T_3 \,, \\ W_\mu^{\text{vac}} &= 0 \,, \end{aligned} \tag{20}$$

where a_1 and a_2 are complex constants, a_3 is a real constant, and

$$\begin{aligned} T_3 &= \delta \cdot H \,, \quad \delta \equiv \sum_{i=1}^{r} \frac{2\lambda_i}{\alpha_i^2} = \frac{1}{2} \sum_{\alpha>0} \frac{2\alpha}{\alpha^2}, \\ T_\pm &= \sum_{i=1}^{r} \sqrt{c_i} E_{\pm\alpha_i}, \end{aligned}$$

with α_i and λ_i being simple roots and fundamental weights, respectively, and

$$c_i \equiv \sum_{j=1}^{r} \left(K^{-1}\right)_{ij},$$

with $K_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$ being the Cartan matrix. The generators T_3, $T_\pm$ form the so called principal $SU(2)$ subalgebra of G. The vacuum configuration ϕ_3^{vac} breaks G into $U(1)^{\text{r}}$ and then ϕ_1^{vac} or ϕ_2^{vac} breaks it further to C_G. Let

$$\alpha_i^{\text{v}} \equiv \frac{2\alpha_i}{\alpha_i^2} \quad , \quad \lambda_i^{\text{v}} \equiv \frac{2\lambda_i}{\alpha_i^2},$$

be the simple coroots and fundamental coweights, respectively. Then using the relations

$$\begin{aligned} \lambda_j^{\text{v}} &= \alpha_i^{\text{v}} \left(K^{-1}\right)_{ij}, \\ \lambda_i^{\text{v}} \cdot \alpha_j &= \delta_{ij}, \end{aligned}$$

we obtain from the vacuum equations $d_a = 0 = f_{sa}$, that

$$\begin{aligned} \left(a_3 - \frac{\mu}{e}\right) a_i &= 0 \,, \quad \text{for } i = 1, 2 \,, \\ a_1 a_2 &= \frac{\mu_3 a_3}{e} \,, \\ m a_3 &= |a_2|^2 - |a_1|^2 \,. \end{aligned}$$

Independently of the values of the mass parameters, this system always has the trivial solution $a_1 = a_2 = a_3 = 0$, which corresponds to the vacuum in which the G is unbroken. In [3] the symmetry breakings produced by the vacuum configuration given by Eq. (20) were analyzed depending on the values of mass parameters. We concluded that in the $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ theory (where $\mu \neq 0$), the gauge group G can be broken to $U(1)^r$ which corresponds to the Coulomb phase. Then, the gauge group can be further broken to C_G, if we add to the $\mathcal{N} = 2^*$ theory, a $\mathcal{N} = 1$ or $\mathcal{N} = 0$ deformation (or both). Let us analyze each of these phases in the next sections.

8 Coulomb phase

In this phase G is broken to $U(1)^r$ and there exist solitonic monopole solutions. As we have seen, this phase can only occur for the $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ cases. That could happen, for example, for energy scales in which one can consider $\mu_3 = 0 = m$. In this phase $a_1 = 0 = a_2$ and $a_3 \neq 0$. In principle a_3 is an arbitrary non-vanishing constant. However, we shall fix

$$a_3 = \frac{\mu}{e}$$

in order to have the same value as in the Higgs phase. The vacuum solution ϕ_3^{vac} is the generator of a particular $U(1)$ direction which we call $U(1)_\delta$. Since for any root α, $\delta \cdot \alpha \neq 0$, we can construct a monopole solution for each root α. The associated monopole magnetic charge is

$$g \equiv \frac{1}{|\phi_3^{\text{vac}}|} \int dS_i \, \text{Re}\left(\phi_3^a\right) B_i^a = \frac{2\pi}{e} \frac{\delta \cdot \alpha^{\text{v}}}{|\delta|}. \tag{21}$$

Clearly g is equal to the monopole magnetic flux in the $U(1)_\delta$ direction, Φ_{mon}. Similarly one can define magnetic fluxes $\Phi_{\text{mon}}^{(i)}$ associated to each $U(1)$ factor of the unbroken group $U(1)^r$ which gives

$$\Phi_{\text{mon}}^{(i)} = \frac{2\pi}{e} \lambda_i^{\text{v}} \cdot \alpha^{\text{v}}. \tag{22}$$

These are BPS monopoles with masses given by the central charge of the $\mathcal{N} = 2$ algebra [17, 18]. For monopoles with vanishing fermion number, their masses are $M_{\text{mon}} = |g||\phi_3^{\text{vac}}|$. Not all of these monopoles are stable. The stable or fundamental are the ones with lightest masses. For the present symmetry breaking, their masses are

$$M_{\text{mon}}^{\text{L}} = \frac{2\pi}{e|\delta|} |\phi_3^{\text{vac}}|. \tag{23}$$

Note that, since G is completely broken to $U(1)^{\text{r}}$, differently from from the monopoles considered in the previous sections, here the fundamental monopoles do not fill representations of a non-Abelian unbroken group.

9 Higgs or color superconducting phase

In the Higgs or color superconducting phase, G is broken to its center C_G. That can happen when $\mathcal{N} = 2^*$ is broken by an $\mathcal{N} = 1$ or $\mathcal{N} = 0$ deformation term (or both). In this phase, the monopole chromomagnetic flux lines cannot spread out radially over space. A phenomenon like that is expected to happen in the interior of very dense neutron stars [6]. However, since

$$\Pi_1 \left(G/C_G\right) = C_G, \tag{24}$$

if $C_G = Z_N$, these flux lines can form topologically nontrivial Z_N strings. Then, the monopoles of $\mathcal{N} = 2^*$ become confined in this phase, as shown below.

The string tension bound given by Eq. (14) holds for ϕ_1 and ϕ_2 in an arbitrary representation. In particular it holds for the adjoint representation, which is the case we are considering here. Therefore, since $|\phi_3^{\text{vac}}| = \mu|\delta|/e$ in this phase, it results that [3]

$$T \geq \frac{me}{2} \left|\phi_3^{\text{vac}}\right| \left|\Phi_{\text{st}}\right| = \frac{m\mu}{2} \left|\delta\right| \left|\Phi_{\text{st}}\right| \tag{25}$$

where, Φ_{st} is the string flux, given by Eq. (12). The bound in Eq. (25) holds for the BPS strings which satisfies the equations [3]

$$\begin{aligned} B_{3a} &= \mp d_{a,} && (26)\\ D_{\mp}\phi_s &= 0, && (27)\\ f_s &= 0, && (28)\\ E_{ia} &= B_{1a} = B_{2a} = D_0\phi_s = D_3\phi_s = 0, && (29) \end{aligned}$$

In order to have finite string tension, the string solution must satisfy the vacuum equations asymptotically, which implies that

$$\begin{aligned} \phi_s(\varphi, \rho \to \infty) &= g(\varphi)\phi_s^{\text{vac}} g(\varphi)^{-1}, \\ W_I(\varphi, \rho \to \infty) &= g(\varphi) W_I^{\text{vac}} g(\varphi)^{-1} - \frac{1}{ie}\left(\partial_I g(\varphi)\right) g(\varphi)^{-1}, \end{aligned}$$

where ρ is the radial coordinate and capital Latin letters I, J denote the coordinates 1 and 2; ϕ_s^{vac} and W_I^{vac} are given by Eq. (20) and $g(\varphi) \in G$. In order for the field configuration to be single valued, $g(\varphi + 2\pi)g(\varphi)^{-1} \in C_G$. Considering

$$g(\varphi) = \exp i\varphi M\,,$$

then $\exp 2\pi i M \in C_G$. That implies that M must be diagonalizable and we shall consider that

$$M = \omega \cdot H.$$

Then, in order to $\exp 2\pi i\omega \cdot H \in C_G$,

$$\omega = \sum_{i=1}^{r} l_i \lambda_i^{\text{v}},$$

where l_i are integer numbers; that is, ω must be a vector in the coweight lattice of G, which has the fundamental coweights λ_i^{v} as basis vectors. In principle, we could have other possibilities for M which however we shall not discuss here.

From this asymptotic configuration, in [3] we construct a string anstaz and obtained that

$$\Phi_{\text{st}} = \frac{2\pi}{e}\frac{\delta \cdot \omega}{|\delta|} \tag{30}$$

Similarly to the monopole, we can define string fluxes $\Phi_{\text{st}}^{(i)}$ associated with the generators of each $U(1)$ factor of $U(1)^r$ which results

$$\Phi_{\text{st}}^{(i)} = \frac{2\pi}{e}\lambda_i^{\text{v}} \cdot \omega\,. \tag{31}$$

Let us now check if the magnetic fluxes of the monopoles are compatible with the ones of the strings. Since an arbitrary coroot α^{v} can always be expanded in the coweight basis as $\alpha^{\text{v}} = \sum_{i=1}^{r} n_i \lambda_i^{\text{v}}$ where n_i are integer numbers, one can conclude that the magnetic fluxes (21) or (22) of the monopoles can be expressed as an integer linear combination of the string fluxes (30) or (31). Therefore, in the Higgs phase, the monopole magnetic flux lines can no longer spread radially over the space, since G is broken to the discrete group C_G. However, they can form one or more flux tubes or strings, and the monopoles can become confined. In the next section, some concrete examples are given for the case $G = SU(3)$. We shall call this set of strings attached to a monopole as confining strings. This set of confining strings must have total flux given by Eq. (30) or (31) with $\omega = \alpha^{\text{v}}$. That means that this set of confining magnetic strings belongs to the trivial topological sector of $\Pi_1(G/C_G)$ since $\exp 2\pi i\alpha^{\text{v}} \cdot H = 1$ in G. The fact that the set of confining strings must belong to the trivial sector is consistent with the fact that the set is not topologically stable and therefore can terminate at some point, like for the strings which appear in the other type of symmetry breaking. Once more, it is important to stress the fact that a string configuration belonging to the topological trivial sector does not imply that its flux must vanish as we can see from Eq. (30). Again all these results are generalizations of some results for the Z_2 string of $SU(2)$ Yang-Mills-Higgs theory. In the Higgs phase, string configurations can in principle exist with flux $2\pi n/e$ for any integer n, although only the ones with $n = \pm 1$ are topologically stable. The ones with odd n belong to the topologically nontrivial sector while the ones with even n belong to the trivial sector. Therefore string configurations belonging to the same topological sector do not have necessarily the same flux and therefore are not related by (nonsingular) gauge transformations [14, 19]. As we mentioned before, the string configuration with $n = 2$, belonging to the trivial sector and which can be formed by two strings with $n = 1$, is the one which can terminate in the 't Hooft-Polyakov monopole with magnetic charge $g = 4\pi/e$. In more algebraic terms one can say that this set of integer numbers n forms the coweight lattice Λ_{w} of $SU(2)$, the subset of even numbers $2n$ form the $SU(2)$ coroot lattice Λ_{r}, and the quotient $\Lambda_{\text{w}}/\Lambda_{\text{r}} \simeq Z_2$ corresponds to the center of $SU(2)$. Therefore this quotient has two elements which are represented by the

cosets $\Lambda_{\rm r}$ and $1+\Lambda_{\rm r}$. Each coset corresponds to a string topological sector, with $\Lambda_{\rm r}$ been the trivial one.

In [3], this result was generalized for an arbitrary G. Let us for simplicity consider the case $G = SU(N)$. Since $SU(N)$ is simply laced (i.e., $\alpha^2 = 2$ for all roots α), we do not need to distinguish between weights and coweights, roots and coroots. In this case, the string topological sectors are given by

$$\Pi_1\left(SU(N)/Z_N\right) = Z_N$$

and are associated with the N cosets

$$\Lambda_{\rm r}(SU(N)) \text{ and } \lambda_i + \Lambda_{\rm r}(SU(N)), \quad i = 1,\, 2,\, \ldots,\, N-1 \tag{32}$$

where λ_i are the fundamental weights of $SU(N)$ and $\Lambda_{\rm r}(SU(N))$ is the root lattice of $SU(N)$. The coset $\Lambda_{\rm r}(SU(N))$ corresponds to the trivial topological sector.

Since the confining string configuration linking a monopole to an antimonopole belongs to the trivial topological sector, it can break when it has enough energy to create a new monopole-antimonopole pair. As was done for the previous example of monopole confinement, one can obtain a bound for the threshold length $d^{\rm th}$for the string breaking, using the relation

$$2M^{\rm L}_{\rm mon} = E^{\rm th} = Td^{\rm th} \geq \frac{me}{2}\,|\phi_3^{\rm vac}|\,|\Phi_{\rm st}|\,d^{\rm th}\,, \tag{33}$$

where $E^{\rm th}$ is the string threshold energy and $M^{\rm L}_{\rm mon}$ is the mass of the lightest monopoles, given by Eq. (23). In the above relation we used the string bound given by Eq. (25) and did not consider a possible energy term proportional to the inverse of the monopole distance, known as the Lucher term. The modulus of the string flux, $|\Phi_{\rm st}|$, must be equal to the modulus of the magnetic charges $|g|$ of each confined monopoles. Let us consider that $|g| = 2\pi|\delta\cdot\beta^{\rm v}|/|\delta|$ with $\beta^{\rm v}$ being an arbitrary coroot. Therefore one can conclude from Eq. (33), using Eq. (23), that

$$d^{\rm th} \leq \frac{4}{me|\delta\cdot\beta^{\rm v}|}.$$

10 Monopole confinement for $SU(3)$ broken to Z_3

Let us consider $G = SU(3)$. We have seen that the magnetic lines of a given monopole can form a set of flux tubes or strings. However, there are countless different string configurations with this magnetic flux. It is not clear at the moment which could be the preferable one. The most "economical" sets would be the ones formed by a strings and an antistring as we shall see now.

For $SU(3)$, the different string topological sectors are associated with the cosets

$$\Lambda_{\rm r}(SU(3)),\ \lambda_1 + \Lambda_{\rm r}(SU(3)) \text{ and } \lambda_2 + \Lambda_{\rm r}(SU(3)).$$

One can, for example, construct string solutions associated with each of the three weights $\lambda_1,\ \lambda_1-\alpha_1,\ \lambda_1-\alpha_1-\alpha_2$ of the three dimensional fundamental representation. Since all of them belong to the coset $\lambda_1+\Lambda_{\rm r}(SU(3))$, these string solutions belong to the same topological sector. However, one can observe from Eq. (30) that they do not have the same flux $\Phi_{\rm st}$, similarly to the Z_2 strings of $SU(2)$ theory. Therefore these string solutions are *not* related by gauge transformations since $\Phi_{\rm st}$ is gauge invariant. One can construct the corresponding antistring solutions associated with the negative of these weights, which form the complex-conjugated representation $\overline{3}$ and which belong to the coset $\lambda_2+\Lambda_{\rm r}$. The magnetic fluxes of the monopoles associated with the six non-vanishing roots of $SU(3)$ can easily be written using these strings in the following way: for the monopole α_1 we can attach the strings λ_1 and $-\lambda_1+\alpha_1$. For the monopole α_2 we can attach strings $\lambda_1-\alpha_1$ and $-\lambda_1+\alpha_1+\alpha_2$. For the monopole $\alpha_1+\alpha_2$ we can attach the strings λ_1 and $-\lambda_1+\alpha_1+\alpha_2$. And similarly for the other three monopoles associated with the negative roots, just changing the signs. The remaining three combinations of strings and antistring have vanishing fluxes $\Phi_{\rm st}^{(i)}$.

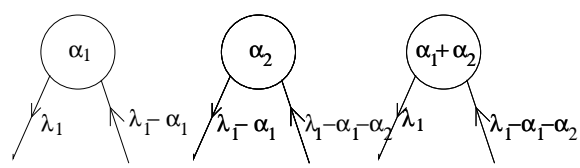

Figure 1. Strings attached to monopoles for $G = SU(3)$.

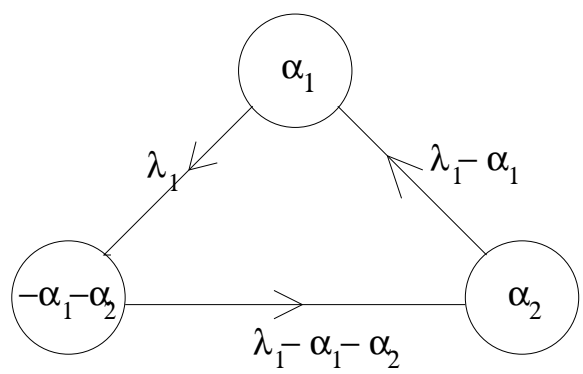

Figure 2. Confined system of three monopoles for $G = SU(3)$.

One could draw the above set of strings attached to monopoles as shown in Fig.1, where the circles represent the monopoles and the arrows are the string flux $\Phi_{\rm st}^{(i)}$. We represented the strings associated with weights in the fundamental representation by an arrow going out of the monopole and for the antistrings we reversed the sense of the arrow and simultaneously changed the sign of the weight. Then, in addition to the monopole-antimonopole pairs one could also conjecture about the formation of a confined system with the monopoles α_1, α_2 and $-\alpha_1-\alpha_2$ as shown in Fig. 2. Note that since these monopoles are not expected to fill the three dimensional fundamental representation of $SU(3)$, that system is not exactly like a baryon. With this configuration

of monopoles with strings attached, one could also think of putting one string in the north pole and the on the other in the south pole, forming a configuration similar to the bead described in [19]. One can easily extend this construction of strings attached to monopoles and monopole confined system to the $SU(N)$ case[3].

11 String tension and Casimir scaling law

The string tension is one of the main quantities to be determined in quark confinement in QCD. In these last 20 years quite a lot of work has been done trying to determine this quantity. There are mainly two conjectures for the string tension: the "Casimir scaling law" [20] and the "sine law" [21]. In these two conjectures the gauge group $G = SU(N)$ is considered and a string in the representation associated with the fundamental weight λ_k which can be obtained by the antisymmetric tensor product of k fundamental representations associated with λ_1. For the Casimir scaling conjecture, the string tension should satisfy

$$T_k = T_1 \frac{k(N-k)}{N-1}, \quad k = 1,\, 2,\, ...,\, N-1, \tag{34}$$

where T_1 would be the string tension in the λ_1 fundamental representation. On the other hand, for the sine law conjecture,

$$T_k = T_1 \frac{\sin\left(\frac{\pi k}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}, \quad k = 1,\, 2,\, ...,\, N-1.$$

All these conjectures are concerned with the chromoelectric strings. However, as we mentioned in the introduction, one expects that chromomagnetic strings could be related to chromoelectric strings by a duality transformation. Therefore one could ask if the tensions of our chromomagnetic strings satisfy one of the two conjectures.

For the case $G = SU(N)$, for a string associated with the weight ω, such that

$$\omega = \lambda_k - \beta_\omega,$$

where λ_k is a fundamental weight of $SU(N)$ and $\beta_\omega \in \Lambda_{\rm r}(SU(N))$, the string tension bound, given by Eq. (25), can be written as

$$T_{\lambda_k - \beta_\omega} \geq \frac{m\mu\pi}{e} \left| \frac{1}{2}\left[C(\lambda_k) - \lambda_k \cdot \lambda_k\right] - \delta \cdot \beta_\omega \right|, \tag{35}$$

where

$$C(\lambda_k) = \lambda_k \cdot (\lambda_k + 2\delta)$$

is the quadratic Casimir associated with the fundamental representation λ_k. That expression can be also written as

$$T_{\lambda_k - \beta_\omega} \geq \frac{m\mu\pi}{e} \left| \frac{1}{2}\left(\frac{(N-1)^2}{2N} \frac{k\,(N-k)}{N-1} \right) - \delta \cdot \beta_\omega \right| \tag{36}$$

Therefore the first term on the right-hand-side of this inequality or, equivalently, the BPS string tension associated with $\omega = \lambda_k$ can be written as

where

$$T^{\rm BPS}_{\lambda_1} = \frac{m\mu\pi}{2e} \frac{(N-1)^2}{2N}$$

is the BPS string tension associated with $\omega = \lambda_1$. Hence we explicitly showed that the BPS string tensions associated with an arbitrary $SU(N)$ fundamental weight λ_k satisfy the Casimir scaling conjecture, given by Eq. (34).

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References

[1] M.A.C. Kneipp and P. Brockill, Phys. Rev. D **64**, 125012 (2001).

[2] M.A.C. Kneipp, Phys. Rev. D**68**, 045009 (2003).

[3] M.A.C. Kneipp, hep-th/0308086, to appear in Phys. Rev. D**69**, (2004).

[4] C. Vafa and E. Witten, Nucl. Phys. B**431**, 3 (1994); R. Donagi and E. Witten, Nucl. Phys. B**460**, 299 (1996).

[5] M.J. Strassler, Prog. Theor. Phys. Suppl. **131**, 439 (1998); Nucl. Phys. Proc. Suppl. **73**, 120 (1999).

[6] D. Bailin and A. Love, Phys. Rept. **107**, 325 (1985); K. Rajagopal and F. Wilczek, hep-ph/0011333 and references therein.

[7] C. Wetterich, Phys. Lett. B**525**, 277 (2002), hep-ph/0011076; M. G. Alford, J. A. Bowers, J. M. Cheyne, and G. A. Cowan, Phys. Rev. D**67**, 054018 (2003), hep-ph/0210106.

[8] D. Olive and N. Turok, Phys. Lett. B **117**, 193 (1982).

[9] E. Weinberg, Nucl. Phys. B167, 500 (1980); Nucl. Phys. B**203**, 445 (1982).

[10] D. Olive, "Magnetic monopoles and Grand Unified Theories", in "Unification of the fundamental particle interactions" Proceedings Erice 1980.

[11] N. Dorey, C. Fraser, T. J. Hollowood, and M.A.C. Kneipp, "Non Abelian Duality in N=4 Supersymmetric Gauge Theories", hep-th/9512116.

[12] F.A. Bais, Phys. Rev. D**18**, 1206 (1978).

[13] C.H. Taubes, Commun. Math. Phys. **72**, 277 (1980).

[14] M. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).

[15] A. Vilenkin and E. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge University Press 1994.

[16] G. 't Hooft, Physica Scripta **25**, 133 (1982).

[17] E. Witten and D.I. Olive, Phys. Lett. B**78**, 97 (1978).

[18] N. Seiberg and E. Witten, Nucl. Phys. B**431**, 484 (1994).

[19] M. Hindmarsh and T.W.B. Kibble, Phys. Rev. Lett. **55**, 2398 (1985).

[20] J. Ambjorn, P. Olesen, and C. Peterson, Nucl. Phys. B**240**, 189 (1984); B**244**, 262 (1984); Phys. Lett. B**142**, 410 (1984).

[21] M.R. Douglas and S.H. Shenker, Nucl. Phys. B**447**, 271