Brazilian Journal of Physics
ISSN: 0103-9733
luizno.bjp@gmail.com
Sociedade Brasileira de Física
Brasil

Amorim, Ricardo; Boschi-Filho, Henrique; F. Braga, Nelson R.
Remarks on gauge fixing and BRST quantization of noncommutative gauge theories
Brazilian Journal of Physics, vol. 35, núm. 3A, september, 2005, pp. 645-651
Sociedade Brasileira de Física
Sâo Paulo, Brasil

Remarks on Gauge Fixing and BRST Quantization of Noncommutative Gauge Theories

Ricardo Amorim, Henrique Boschi-Filho, and Nelson R. F. Braga
Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, RJ 21941-972 – Brazil
Received on 12 May, 2005

We consider the BRST gauge fixing procedure of the noncommutative Yang-Mills theory and of the gauged $U(N)$ Proca model. An extended Seiberg-Witten map involving ghosts, antighosts and auxiliary fields for non-Abelian gauge theories is studied. We find that the extended map behaves differently for these models. For the Yang-Mills theory in the Lorentz gauge it was not possible to find a map that relates the gauge conditions in the noncommutative and ordinary theories. For the gauged Proca model we found a particular map relating the unitary gauge fixings in both formulations.

I. INTRODUCTION

Noncommutative gauge fields Y can be described in terms of ordinary gauge fields y by using the Seiberg-Witten (SW) map[1] defined by

$$\delta Y = \bar{\delta} Y[y] \tag{1.1}$$

where δ and $\bar{\delta}$ are the gauge transformations for noncommutative and ordinary theories respectively and $Y[y]$ means that we express the noncommutative fields Y in terms of ordinary ones y. In this work we will consider gauge theories whose algebra closes as

$$[\delta_1, \delta_2]\, Y = \delta_3 Y \tag{1.2}$$

without the use of equations of motion. In terms of the mapped quantities this condition reads

$$[\bar{\delta}_1, \bar{\delta}_2]\, Y[y] = \bar{\delta}_3 Y[y]\ . \tag{1.3}$$

This implies that the mapped noncommutative gauge parameters must satisfy a composition law in such a way that they depend in general on the ordinary parameters and also on the fields y. Usually this noncommutative parameter composition law is the starting point for the construction of SW maps. Interesting properties of Yang-Mills noncommutative theories where discussed in[2–4], where these points are considered in detail. Other noncommutative theories with different gauge structures are also studied in [5, 6].

Originally the Seiberg-Witten map has been introduced relating noncommutative and ordinary gauge fields and the corresponding actions. When one considers the gauge fixing procedure one enlarges the space of fields by introducing ghosts, antighosts and auxiliary fields. In this case one can define an enlarged BRST-SW map[7]

$$sY = \bar{s} Y[y] \tag{1.4}$$

where s and $\bar{s}$ are the BRST differentials for the noncommutative and ordinary theories respectively and Y and y here include ghosts, antighosts and auxiliary fields. It is interesting to note that in this BRST approach the closure relation (1.3) is naturally contained in the above condition for the ghost field. This means that it is not necessary to construct a SW map for the parameter.

Here we will investigate the extension of this map also to the gauge fixed actions. Observe that in ref.[7] the Hamiltonian formalism was used while here we consider a Lagrangian approach. In particular we study the consistency of such maps with the gauge fixing process. Considering the BRST quantization of a noncommutative theory we will find that some usual gauge choices for the noncommutative theories are mapped in a non trivial way in the ordinary model.

Recent results coming from string theory are motivating an increasing interest in noncommutative theories. The presence of an antisymmetric tensor background along the D-brane [8] world volumes (space time region where the string endpoints are located) is an important source for noncommutativity in string theory[9, 10]. Actually the idea that spacetime may be noncommutative at very small length scales is not new[11]. Originally this has been thought just as a mechanism for providing space with a natural cut off that would control ultraviolet divergences[12], although these motivations have been eclipsed by the success of the renormalization procedures.

Gauge theories can be defined in noncommutative spaces by considering actions that are invariant under gauge transformations defined in terms of the Moyal structure[1]. In this case the form of the gauge transformations imply that the algebra of the generators must close under commutation and anticommutation relations. That is why $U(N)$ is usually chosen as the symmetry group for noncommutative extensions of Yang-Mills theories in place of $SU(N)$, although other symmetry structures can also be considered [3][13][14]. Once one has constructed a noncommutative gauge theory, it is possible to find the Seiberg-Witten map relating the noncommutative fields to ordinary ones[2]. The mapped Lagrangian is usually written as a nonlocal infinite series of ordinary fields and their space-time derivatives but the noncommutative Noether identities are however kept by the Seiberg-Witten map. This assures that the mapped theory is still gauge invariant.

In this work we will first consider (section **II**) the case of the noncommutative $U(N)$ Yang-Mills theory and investigate the BRST gauge fixing procedure in the Lorentz gauge. Then the construction of the SW map between the noncommutative and ordinary Yang-Mills fields, including ghosts, antighosts and auxiliary fields is discussed. We will see that imposing the Lorentz gauge in the noncommutative theory does not imply an equivalent condition in the ordinary theory. Conversely, imposing the Lorentz gauge condition in the ordinary gauge fields would not correspond to the same condition in the noncommutative theory.

Another model that will present an interesting behavior is the gauged version of noncommutative non-Abelian Proca field, discussed in sections **III** and **IV**. For this model we study the BRST gauge fixing in the unitary gauge. The model is obtained by introducing an auxiliary field which promotes the massive vector field to a gauge field. This auxiliary field can be seen as a pure gauge "compensating vector field" , defined in terms of $U(N)$ group elements and having a null curvature [15, 16]. In this model we find that the general SW map relates in a non trivial way noncommutative and ordinary gauge fixing conditions. However it is possible to find a particular SW map that relates the unitary gauges in noncommutative and ordinary theories. Some of these points have been partially considered in [5].

Regardless of these considerations, the Fradkin-Vilkovisky theorem[17] assures that the physics described by any non anomalous gauge theory is independent of the gauge fixing, without the necessity of having the gauge fixing functions mapped. This means that the quantization can be implemented consistently in noncommutative and ordinary theories. Note that the SW map is defined for the gauge invariant action. This places a relation between the noncommutative and ordinary theories before any gauge fixing. Once a gauge fixing is chosen one does not necessarily expect that the theories would still be related by the same map. In the Yang-Mills case with Lorentz gauge we could not relate the complete theories by a BRST-SW map after gauge fixing. However, for the gauged Proca theory we could find a map relating the unitary gauges in both noncommutative and ordinary theories.

II. GAUGE FIXING THE NONCOMMUTATIVE $U(N)$ YANG-MILLS THEORY

In order to establish notations and conventions that will be useful later, let us start by considering the ordinary U(N) Yang-Mills action (denoting ordinary actions by the upper index $^{(0)}$)

$$S_0^{(0)} = tr\int d^4x \left(-\frac{1}{2} f_{\mu\nu} f^{\mu\nu}\right) \tag{2.1}$$

where

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu] \tag{2.2}$$

is the curvature. We assume that the connection a_μ takes values in the algebra of U(N), with generators T^A satisfying the trace normalization

$$tr(T^A T^B) = \frac{1}{2}\delta^{AB} \tag{2.3}$$

and the (anti)commutation relations

$$\begin{aligned} [T^A, T^B] &= i f^{ABC} T^C \\ \{T^A, T^B\} &= d^{ABC} T^C \end{aligned} \tag{2.4}$$

where f^{ABC} and d^{ABC} are assumed to be completely antisymmetric and completely symmetric respectively.

The action (2.1) is invariant under the infinitesimal gauge transformations

$$\bar{\delta} a_\mu = D_\mu \alpha \equiv \partial_\mu \alpha - i[a_\mu, \alpha] \tag{2.5}$$

which closes in the algebra

$$[\bar{\delta}_1, \bar{\delta}_2]\, a_\mu = \bar{\delta}_3\, a_\mu \tag{2.6}$$

with parameter composition rule given by

$$\alpha_3 = i[\alpha_2, \alpha_1] \tag{2.7}$$

The gauge structure displayed above leads to the definition of the BRST differential $\bar{s}$ such that

$$\begin{aligned} \bar{s}c &= ic^2 \\ \bar{s}a_\mu &= D_\mu c \\ &\equiv \partial_\mu c - i[a_\mu, c] \\ \bar{s}\bar{c} &= \gamma \\ \bar{s}\gamma &= 0 \end{aligned} \tag{2.8}$$

As $\bar{s}$ is an odd derivative acting from the right, it is easy to verify from the above definitions that it indeed is nilpotent. Naturally c and $\bar{c}$ are grassmannian quantities with ghost numbers respectively $+1$ and -1. $\bar{c}$ and γ form a trivial pair necessary to implement the gauge fixing.

The functional BRST quantization starts by defining the total action

$$S^{(0)} = S_0^{(0)} + S_1^{(0)} \tag{2.9}$$

where $S_0^{(0)}$ is given by (2.1) and

$$S_1^{(0)} = -2\, tr\, \bar{s} \int d^4x\, \bar{c} \left(-\frac{\gamma}{\beta} + \partial_\mu a^\mu\right) \tag{2.10}$$

appropriated to fix the (Gaussian) Lorentz condition, is BRST exact. This assures that $S_1^{(0)}$ is BRST invariant, due to the nilpotency of $\bar{s}$. Since $\bar{s} f_{\mu\nu} = i[c, f_{\mu\nu}]$ according to (2.2) and (2.8), it follows that $S_0^{(0)}$ is also BRST invariant. In (2.10), β is a free parameter, as usual.

In general $S_1^{(0)}$ can be written as

$$S_1^{(0)} = S_{gh}^{(0)} + S_{gf}^{(0)} \tag{2.11}$$

with the ghost action given by

$$S_{gh}^{(0)} = -2tr\int d^4x\,\bar{c}Mc \tag{2.12}$$

and the gauge fixing one by

$$S_{gf}^{(0)} = -2tr\int d^4x\,\gamma\left(-\frac{\gamma}{\beta}+\mathcal{F}\right) \tag{2.13}$$

where $\mathcal{F}=\mathcal{F}(a)$ is a gauge fixing function and $M=\delta\mathcal{F}/\delta\alpha$. The Lorentz gauge condition corresponds to $\mathcal{F}_1=\partial_\mu a^\mu$, $M_1=\partial_\mu D^\mu$. The functional quantization is constructed by functionally integrating the exponential of iS over all the fields, with appropriate measure and the external source terms in order to generate the Green's functions.

The noncommutative version of this theory comes from replacing $\underline{y}=\{a_\mu, c, \bar{c}, \gamma\}$ by the noncommutative fields $Y=\{A_\mu, C, \bar{C}, \Gamma\}$ as well as the ordinary products of fields by $\star$-Moyal products, defined through

$$\star \equiv exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right) \tag{2.14}$$

where $\theta^{\mu\nu}$ is a constant and antisymmetric matrix. The Moyal product is associative and cyclic under the integral sign when appropriate boundary conditions are adopted. We also assume that the group structure is deformed by this product. In this way, the group elements are constructed from the exponentiation of elements of the algebra of U(N) by formally using $\star$ in the series, that is, $g=1+\lambda^A T^A+\frac{1}{2}\lambda^A T^A\star\lambda^B T^B+\ldots$. General consequences of this deformation in field theories can be seen in [1, 4]. In particular, the appearance of $\theta^{0i}\neq 0$ breaks the unitarity of the corresponding quantum theory. Furthermore in a Hamiltonian formalism this would imply higher order time derivatives which demand a non standard canonical treatment. This last aspect does not show up here since we are using a Lagrangian formalism.

The noncommutative action corresponding to (2.1) can be written as

$$S_0 \;=\; tr\int d^4x\left(-\frac{1}{2}F_{\mu\nu}\star F^{\mu\nu}\right) \tag{2.15}$$

where now

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu - i\left[A_\mu \stackrel{\star}{,} A_\nu\right] \tag{2.16}$$

As expected, the noncommutative gauge transformations $\delta A_\mu=\partial_\mu\varepsilon - i[A_\mu \stackrel{\star}{,} \varepsilon]$ close in an algebra like (1.2) with composition rule for the parameters given by

$$\varepsilon_3 = i[\varepsilon_2 \stackrel{\star}{,} \varepsilon_1] \tag{2.17}$$

The total action is given by

$$S = S_0 + S_1 \tag{2.18}$$

where

$$S_1 = -2\,trs\int d^4x\,\bar{C}\left(-\frac{\Gamma}{\beta}+\partial_\mu A^\mu\right) \tag{2.19}$$

The BRST differential s is defined through

$$\begin{aligned} sC &= -iC\star C\\ sA_\mu &= D_\mu C\\ &\equiv \partial_\mu C - i[A_\mu \stackrel{\star}{,} C]\\ s\bar{C} &= \Gamma\\ s\Gamma &= 0 \end{aligned} \tag{2.20}$$

Obviously both S_0 and S_1 are BRST invariant.

The BRST Seiberg-Witten map is obtained from the condition (1.4). In this work we will consider only the expansion of noncommutative fields in terms of ordinary fields to first order in the noncommutative parameter θ:

$$Y[y] = y + Y^{(1)}[y] + O(\theta^2)\,,$$

where Y represents the noncommutative fields $A_\mu, C, \bar{C}, \Gamma$. Then we find from (1.4) and (2.20) that

$$\begin{aligned} \bar{s}C^{(1)} + i\{c, C^{(1)}\} &= \frac{1}{4}\theta^{\alpha\beta}[\partial_\alpha c, \partial_\beta c]\\ \bar{s}A_\mu^{(1)} + i[A_\mu^{(1)}, c] &= \partial_\mu C^{(1)} + i[C^{(1)}, a_\mu]\\ &- \frac{1}{2}\theta^{\alpha\beta}\{\partial_\alpha c, \partial_\beta a_\mu\}\\ \bar{s}\bar{C}^{(1)} &= \Gamma^{(1)}\\ \bar{s}\Gamma^{(1)} &= 0 \end{aligned} \tag{2.21}$$

The corresponding solutions for the ghost and the gauge field are

$$C^{(1)} = \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu c, a_\nu\right\} + \lambda_1\theta^{\mu\nu}\left[\partial_\mu c, a_\nu\right] \tag{2.22}$$

$$\begin{aligned} A_\mu^{(1)} &= -\frac{1}{4}\theta^{\alpha\beta}\left\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\right\} + \sigma\theta^{\alpha\beta}D_\mu f_{\alpha\beta}\\ &+ \frac{\lambda_1}{2}\theta^{\alpha\beta}D_\mu[a_\alpha, a_\beta]\,, \end{aligned} \tag{2.23}$$

where σ and λ_1 are arbitrary constants.

It is important to remark that when we extend the space of fields in order to implement BRST quantization we could in principle find solutions for $C^{(1)}$ and $A_\mu^{(1)}$ depending on $\bar{c}$ and γ. However there are two additional conditions to be satisfied, besides eqs. (2.21): the ghost number and the dimension of all first order corrections must be equal to those of the corresponding zero order field. It can be checked that there are no other possible contributions to the solutions (2.22) and (2.23) involving $\bar{c}$ and γ and satisfying these criteria.

For the trivial pair we find

$$\begin{aligned} \bar{C}^{(1)} &= \theta^{\alpha\beta} H_{\alpha\beta} \\ \Gamma^{(1)} &= \theta^{\alpha\beta} \bar{s} H_{\alpha\beta} \end{aligned} \tag{2.24}$$

where $H_{\alpha\beta}$ is a function of the fields $a_\mu, c, \bar{c}$ and γ with ghost number $= -1$ (taking the convention that the ghost numbers of c and $\bar{c}$ are $+1$ and -1 respectively). Note that the sum of the mass dimensions of c and $\bar{c}$ is 2. Then the mass dimension of $H_{\alpha\beta}$ will be the dimension of $\bar{c}$ plus 2. Considering these points we find that the general form for this quantity is

$$H_{\alpha\beta} = \omega_1 \bar{c} a_{[\alpha} a_{\beta]} + \omega_2 \bar{c} \partial_{[\alpha} a_{\beta]} + \omega_3 \partial_{[\alpha} \bar{c} a_{\beta]} \tag{2.25}$$

where ω_1, ω_2 and ω_3 are arbitrary parameters. From eqs. (2.8), (2.24) and (2.25) we see that

$$\begin{aligned} \Gamma^{(1)} &= \theta^{\alpha\beta} \Big(\omega_1 \gamma a_\alpha a_\beta + \omega_2 \gamma \partial_\alpha a_\beta + \omega_3 \partial_\alpha \gamma a_\beta \\ &+ \omega_1 \bar{c} (D_\alpha c a_\beta + a_\alpha D_\beta c) + \omega_2 \bar{c} \partial_\alpha D_\beta c \\ &+ \omega_3 \partial_\alpha \bar{c} D_\beta c \Big) . \end{aligned} \tag{2.26}$$

The usual Seiberg-Witten map is defined for the S_0 (gauge invariant) part of the action. When we gauge fix by including S_{gh} and S_{gf} we find a total action that is no more gauge invariant but rather BRST invariant. This poses a non trivial problem of whether it would still be possible to relate noncommutative and ordinary gauge fixed theories by a SW map.

Let us consider the Lorentz gauge condition appearing in (2.19):

$$\mathcal{F}_1 = \partial^\mu A_\mu = 0 \tag{2.27}$$

If we use the Seiberg-Witten map found above we see that this condition would correspond to

$$\begin{aligned} \partial^\mu a_\mu &= 0 && (2.28) \\ \partial^\mu A_\mu^{(1)}[a_\mu] &= 0 . && (2.29) \end{aligned}$$

That means, besides the ordinary Lorentz condition (2.28), we find the additional non linear conditions on a_μ from (2.23) and (2.29). If we were adopting in our expansions terms up to order N in θ, we would find a set of conditions $\partial^\mu A_\mu^{(n)}[a_\mu] = 0$, $n = 0, 1, .., N$. So it seems that the condition $\partial^\mu A_\mu = 0$ is not compatible with the solution (2.23), and its higher order extensions, for the SW mapping.

Alternatively we could choose in the noncommutative theory the non trivial non linear gauge condition (to first order in θ)

$$\begin{aligned} \mathcal{F}_2 &= \partial^\mu A_\mu + \frac{1}{4} \theta^{\alpha\beta} \partial^\mu \{ A_\alpha, \partial_\beta A_\mu + F_{\beta\mu} \} \\ &- \sigma \theta^{\alpha\beta} \partial^\mu D_\mu F_{\alpha\beta} - \frac{\lambda_1}{2} \theta^{\alpha\beta} \partial^\mu D_\mu [A_\alpha, A_\beta] \\ &= 0 . \end{aligned} \tag{2.30}$$

If we assume the map (2.23) to hold, this gauge fixing condition would correspond just to the ordinary Lorentz condition (2.28) to first order in the noncommutative parameter θ. We see that if we impose the Lorentz gauge in one of the theories, the SW map would lead to a somehow complicated and potentially inconsistent gauge in the other.

Instead of just considering the map of the gauge fixing functions, a more general approach is to consider the behaviour of the action S_1 eq. (2.19) under the SW map. This action can be written, to first order in the noncommutative parameter, as

$$S_1 = S_1^{(0)} + S_1^{(1)} , \tag{2.31}$$

where $S_1^{(0)}$ is given by (2.10) and

$$\begin{aligned} S_1^{(1)} &= -2\, tr\, \bar{s} \int d^4x \Big(\bar{C}^{(1)} (-\frac{\gamma}{\beta} + \partial_\mu a^\mu) \\ &+ \bar{c} (-\frac{\Gamma^{(1)}}{\beta} + \partial_\mu A^{(1)\mu}) \Big) . \end{aligned} \tag{2.32}$$

Note that the condition $\mathcal{F}_1 = \partial^\mu A_\mu = 0$ in the noncommutative theory would be effectively mapped into $\mathcal{F}_3 = \partial^\mu a_\mu = 0$ if $S_1^{(1)}$ could vanish. In order to see if this is possible we introduce (2.23) and (2.26) in (2.32) and examine the terms with the same field content. The part of $S_1^{(1)}$ independent of the ghost sector is

$$\begin{aligned} S^{(1)}_{1\,no\,ghost} &= -2\, \theta^{\alpha\beta} tr \int d^4x \Big[\Big(\partial_\mu a^\mu - \frac{2\gamma}{\beta} \Big) \\ &\times \Big(\omega_1 \gamma a_\alpha a_\beta + \omega_2 \gamma \partial_\alpha a_\beta + \omega_3 \partial_\alpha \gamma a_\beta \Big) \\ &+ \gamma \partial^\mu \Big(-\frac{1}{4} \Big\{ a_\alpha, \partial_\beta a_\mu + f_{\beta\mu} \Big\} + \sigma D_\mu f_{\alpha\beta} \\ &+ \frac{\lambda_1}{2} D_\mu [a_\alpha, a_\beta] \Big) \Big] . \end{aligned} \tag{2.33}$$

The terms linear in the connection a_μ in the integrand are

$$-\frac{2\gamma}{\beta} \Big(\omega_2 \gamma \partial_{[\alpha} a_{\beta]} + \omega_3 \partial_{[\alpha} \gamma a_{\beta]} \Big) + 2\sigma\gamma \Box \partial_{[\alpha} a_{\beta]} \tag{2.34}$$

which will vanish modulo total differentials only if $\sigma = 0$ and $\omega_3 = 2\omega_2$. Using these results the quadratic part in a_μ of the integrand can be written as

$$-\frac{2}{\beta}\gamma^2\omega_1\, a_{[\alpha}a_{\beta]} - \gamma\Big(\omega_2\,(\,\partial_{[\alpha}\, a_{\beta]}\,\partial_\mu a^\mu + 2\, a_{[\beta}\partial_{\alpha]}\,\partial_\mu a^\mu\,) + \frac{1}{4}\partial^\mu(2\{a_{[\alpha},\partial_{\beta]}\, a_\mu\} + \{\partial_\mu a_{[\alpha}, a_{\beta]}\} - 4\lambda_1\,\partial_\mu(a_{[\alpha}\, a_{\beta]}))\Big) \tag{2.35}$$

modulo a total differential. The quadratic term in γ can be set to zero choosing $\omega_1 = 0$. However, there is no choice for λ_1 and ω_2 which makes the linear terms in γ in the expression above vanish or be written as a total derivative. Since the terms from the ghost sector can not cancel the above ones, we conclude that $S_1^{(1)}$ can not vanish. So it is not possible to relate the Lorentz gauge conditions in the ordinary and noncommutative theories by the SW map. Anyway, quantization can be consistently implemented in both theories as pointed out in the end of section **I**. We will see in section **IV** that for the gauge invariant Proca model it is possible to find a particular BRST-SW map relating the gauge fixing unitary conditions for both sectors.

III. THE NON-ABELIAN PROCA MODEL

Before considering the gauge invariant noncommutative $U(N)$ Proca model, it will be useful to briefly present some basic properties of the ordinary Abelian Proca theory with action

$$S_0^{(0)} = \int d^4x \left(-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} m^2 a_\mu a^\mu \right) . \tag{3.1}$$

Here a_μ represents the massive Abelian vector field. Variation of (3.1) with respect to a_μ gives the equation of motion

$$\partial_\mu f^{\mu\nu} + m^2 a^\nu = 0 \tag{3.2}$$

which, by symmetry, implies that

$$\partial_\mu a^\mu = 0 \tag{3.3}$$

Substituting (3.3) into (3.2) we find that the vector field satisfies a massive Klein-Gordon equation, as expected:

$$(\Box + m^2) a_\mu = 0 \tag{3.4}$$

This model can be described in a gauge invariant way by introducing a compensating field λ. This kind of mechanism is useful, for instance, when calculating anomalies[15, 16]. In this case, the action (3.1) is replaced by

$$S_0^{(0)} = \int d^4x \left(-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} m^2 (a_\mu - \partial_\mu\lambda)(a^\mu - \partial^\mu\lambda) \right) . \tag{3.5}$$

This action is invariant under the local transformations $\delta a_\mu = \partial_\mu\alpha$ and $\delta\lambda = \alpha$. The equations of motion for a_μ and λ are

$$\begin{aligned} \partial_\mu f^{\mu\nu} + m^2(a^\nu - \partial^\nu\lambda) &= 0 \\ \partial_\mu(a^\mu - \partial^\mu\lambda) &= 0 \end{aligned} \tag{3.6}$$

Note that applying ∂_ν to the first equation, one reobtains the second one. In this case (3.3) does not come from the equations of motion. However as now the model is gauge invariant, we must impose a gauge fixing function. We may choose for instance one of the gauge fixing functions $\mathcal{F}_1 = \partial_\mu a^\mu$, $\mathcal{F}_2 = \Box\lambda$ or $\mathcal{F}_3 = \lambda$ in order to recover the original Proca theory.

We now consider the non-Abelian generalization of this model. Now a_μ takes values in the algebra of U(N), exactly as in the Yang-Mills case of the previous section.

In place of (3.2) one finds

$$D_\mu f^{\mu\nu} + m^2 a^\nu = 0 \tag{3.7}$$

Applying D_ν to this equation and using the property

$$[D_\mu, D_\nu] y = i[\, f_{\mu\nu}\, , y] \tag{3.8}$$

where y is any arbitrary function with values in the algebra, we find as in the Abelian case that

$$D_\mu a^\mu = \partial_\mu a^\mu = 0\,. \tag{3.9}$$

However, the equations of motion present non linear terms. Using the "Lorentz identity" (3.9) in eq. (3.7) we obtain the equations of motion for a^μ

$$(\Box + m^2) a^\rho - i[a_\mu, \partial^\mu a^\rho + f^{\mu\rho}] = 0 \tag{3.10}$$

which is no longer a Klein Gordon equation.

It is worth to mention that contrarily to the Abelian case, the non-Abelian Proca model is not renormalizable, although the divergencies at one-loop level cancel in an unexpected way [18],[19]. Nonrenormalizability is, in any way, an almost general property of noncommutative field theories[4].

The next step would be to consider the gauged version of this non-Abelian model. This will be done in the noncommutative context in the next section. We will also discuss there the gauge fixing procedure and the BRST formalism.

IV. THE GAUGE INVARIANT NONCOMMUTATIVE $U(N)$ PROCA MODEL

The gauge invariant version of the U(N) noncommutative Proca model can be written as[5]

$$\begin{aligned} S_0 \;=\; & tr\int d^4x \Big(-\frac{1}{2} F_{\mu\nu} \star F^{\mu\nu} \\ & + \; m^2 (A_\mu - B_\mu) \star (A^\mu - B^\mu) \Big) \end{aligned} \tag{4.1}$$

where the curvature is again given by eq. (2.16) and

$$B_\mu \equiv -i\,\partial_\mu G \star G^{-1} \tag{4.2}$$

In the above expressions G is an element of the noncommutative U(N) group and B_μ is a "pure gauge" vector field in the sense that its curvature, analogous to (2.16), vanishes identically. Note that B_μ is the noncommutative $U(N)$ version of $\partial_\mu\lambda$ discussed in the previous section. We assume that the algebra generators satisfy, as in the Yang-Mills case, the trace normalization and (anti)commutation relations (2.3) and (2.4).

By varying action (4.1) with respect to A_μ and G, we get the equations of motion

$$\begin{aligned} D_\mu F^{\mu\nu} + m^2(A^\nu - B^\nu) &= 0 && (4.3)\\ \bar{D}_\mu(A^\mu - B^\mu) &= 0 && (4.4)\end{aligned}$$

where we have defined the two covariant derivatives

$$\begin{aligned} D_\mu X &= \partial_\mu X - i[A_\mu \stackrel{\star}{,} X]\\ \bar{D}_\mu X &= \partial_\mu X - i[B_\mu \stackrel{\star}{,} X] && (4.5)\end{aligned}$$

for any quantity X with values in the algebra. By taking the covariant divergence of equation (4.3) and using the noncommutative Bianchi identity

$$[D_\mu, D_\nu]X = i[F_{\mu\nu} \stackrel{\star}{,} X] \tag{4.6}$$

one finds

$$D_\mu(A^\mu - B^\mu) = 0 \tag{4.7}$$

which is equivalent to equation (4.4), as can be verified. Actually, we can rewrite (4.4) or (4.7) in the convenient form

$$\partial_\mu A^\mu = D_\mu B^\mu\,. \tag{4.8}$$

So we see that if we choose, among the possible gauging fixing functions, $\mathcal{F}_1 = \partial_\mu A^\mu$ or $\mathcal{F}_4 = G - 1$, the compensating field is effectively eliminated and the Lorentz condition is implemented.

Action (4.1) is invariant under the infinitesimal gauge transformations

$$\begin{aligned} \delta A_\mu &= D_\mu \varepsilon\\ \delta G &= i\varepsilon \star G && (4.9)\end{aligned}$$

which also implies that

$$\begin{aligned} \delta B_\mu &= \bar{D}_\mu \varepsilon\\ \delta F_{\mu\nu} &= i[\varepsilon \stackrel{\star}{,} F_{\mu\nu}] && (4.10)\end{aligned}$$

As expected, the noncommutative gauge transformations listed above close in an algebra as (1.2), for the fields A_μ, G , B_μ or $F_{\mu\nu}$. The composition rule for the parameters is as (2.17).

The associated BRST algebra obtained again by introducing the trivial pair $\bar{C}$ and Γ, corresponds to the Yang-Mills algebra of eq. (2.20) plus the transformation of the compensating field

$$sG = iC \star G\,, \tag{4.11}$$

that corresponds to

$$sB_\mu = \bar{D}_\mu C = \partial_\mu C - i[B_\mu \stackrel{\star}{,} C]\,. \tag{4.12}$$

A gauge fixed action is constructed from (4.1) by adding S_{gh} and S_{gf} from eqs. (2.12) and (2.13) as in the Yang-Mills case.

For the Lorentz gauge we choose as in the Yang-Mills case $\mathcal{F}_1 = \partial_\mu A^\mu$, $M_1 = \partial_\mu D^\mu$ and we get formally the same ghost and gauge fixing actions. The complete action is BRST invariant, as expected.

For the unitary gauge corresponding to the choice $\mathcal{F}_4 = G - 1$, we find the ghost and gauge fixing actions

$$\begin{aligned} S_1 &= S_{gh} + S_{gf}\\ &= -2tr\,s\int d^4x\,\bar{C}(G-1) && (4.13)\end{aligned}$$

which is obviously BRST invariant. Since $sS_0 = 0$, as can be verified, it follows the invariance of the complete action.

Let us now build up the BRST Seiberg-Witten map for this model. We start again by imposing [7] that $sY = \bar{s}Y[y]$ submitted to the condition $Y[y]_{|\theta=0} = y$ that lead us again to equations (2.21) and also to

$$\begin{aligned} \bar{s}G^{(1)} - icG^{(1)} &= -\frac{1}{2}\theta^{\mu\nu}\partial_\mu c\,\partial_\nu g + iC^{(1)}g\\ \bar{\delta}B^{(1)}_\mu + i[B^{(1)}_\mu, c] &= \partial_\mu C^{(1)} + i[C^{(1)}, b_\mu]\\ &- \frac{1}{2}\theta^{\alpha\beta}\{\partial_\alpha c, \partial_\beta b_\mu\} && (4.14)\end{aligned}$$

The solutions of these equations are again obtained by searching for all the possible contributions with the appropriate dimensions and Grassmaniann characters. However now we have the extra compensating field b_μ. So the solutions for $A^{(1)}_\mu$ and $C^{(1)}$ are no longer the Yang-Mills solutions (2.22) and (2.23) but rather

$$\begin{aligned} C^{(1)} &= \frac{1}{4}(1-\rho)\theta^{\mu\nu}\left\{\partial_\mu c, a_\nu\right\} + \lambda_1\theta^{\mu\nu}\left[\partial_\mu c, a_\nu\right]\\ &+ \frac{1}{4}\rho\theta^{\mu\nu}\left\{\partial_\mu c, b_\nu\right\} + \lambda_2\theta^{\mu\nu}\left[\partial_\mu c, b_\nu\right] && (4.15)\end{aligned}$$

$$
\begin{aligned}
A_\mu^{(1)} &= -\frac{1-\rho}{4}\theta^{\alpha\beta}\left\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\right\} + \sigma\theta^{\alpha\beta}D_\mu f_{\alpha\beta} \\
&+ \frac{\lambda_1}{2}\theta^{\alpha\beta}D_\mu[a_\alpha, a_\beta] + \frac{\rho}{4}\theta^{\alpha\beta}\left\{b_\alpha, D_\mu b_\beta - 2\partial_\beta a_\mu\right\} \\
&+ \lambda_2\theta^{\alpha\beta}D_\mu\Big(b_\alpha b_\beta\Big) \qquad (4.16)
\end{aligned}
$$

In these equations λ_1, λ_2 , ρ and σ are arbitrary constants. These solutions are a generalization of those in ref. [5].

For the compensating field we find

$$
\begin{aligned}
G^{(1)} &= -\frac{1}{2}(1-\rho)\theta^{\alpha\beta}a_\alpha\left(\partial_\beta g - \frac{i}{2}a_\beta g\right) \\
&+ i\lambda_1\theta^{\alpha\beta}a_\alpha a_\beta g + \gamma\theta^{\alpha\beta}f_{\alpha\beta}g \\
&+ i(\lambda_2 - \frac{\rho}{4})\theta^{\alpha\beta}b_\alpha b_\beta g + O(\theta^2) \qquad (4.17)
\end{aligned}
$$

with arbitrary γ.

Now we may consider the compatibility of the BRST-SW map and the unitary gauge fixing, in the same spirit of what was discussed in the last section. First we observe that the condition $\mathcal{F}_4 = G-1$ is mapped in $g-1$ plus complicated first order corrections in θ. However, if we choose the parameters of the map as

$$
\begin{aligned}
\lambda_1 &= \frac{1}{4}(\rho-1) \\
\gamma &= 0 \qquad (4.18)
\end{aligned}
$$

we find that the conditions $g=1$ and $G=1$ are equivalent to first order in θ, as can be verified from (4.17). So that for this particular solution of the map, the unitary gauge in the noncommutative theory is mapped in the ordinary unitary gauge in the mapped theory.

Additionally, if we choose $\rho=1$ implying $\lambda_1=0$ and also $\sigma=0$ we find that in the unitary gauge $A_\mu^{(1)}=0$. So that A_μ is mapped just in a_μ. In this case we find that the non gauge invariant noncommutative $U(N)$ Proca theory is recovered in the unitary gauge and at the considered order in θ.

V. CONCLUSION

We have investigated the extension of the Seiberg-Witten map to ghosts, antighosts and auxiliary fields in the BRST gauge fixing procedure. Two non-Abelian gauge models that present different behaviors under the BRST-SW map of the gauge fixing conditions have been considered. For the Yang-Mills theory we found that it is not possible to map the Lorentz gauge condition in the noncommutative and ordinary theories. On the other hand, for the gauged $U(N)$ Proca model we were able to find a SW map that relates the unitary gauge fixing in the noncommutative and ordinary theories. It would be interesting to investigate how the BRST-SW map act in other gauge fixing conditions and models.

Acknowledgment: This work is supported in part by CAPES and CNPq (Brazilian research agencies).

[1] N. Seiberg and E. Witten, JHEP **09**, 32 (1999).

[2] T. Asakawa and I. Kishimoto, JHEP **11**, 024 (1999).

[3] B. Jurco, J. Moller, S. Schraml, P. Schupp, and J. Wess, Eur. Phys. J. C **21**, 383 (2001); D. Brace, B. L. Cerchiai, A. F. Pasqua, U. Varadarajan, and B. Zumino, JHEP **0106**, 047 (2001).

[4] For a review see R.J. Szabo, Phys. Rep. **378**, 207 (2003), and references therein.

[5] R. Amorim, N. R. F. Braga, and C. N. Ferreira, Phys. Lett. B **591**, 181 (2004).

[6] R. Amorim, C. N. Ferreira, and C. Godinho, Phys. Lett. B **593**, 203 (2004).

[7] R. Amorim and F. A. Farias, Phys. Rev. D **69**, 045013 (2004). See also O. F. Dayi, Phys. Lett. B **481**, 408 (2000).

[8] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995).

[9] C. Hofman , E. Verlinde, J. High Energy Phys. **12**, 10 (1998).

[10] C.-S.Chu , P.-M. Ho, Nucl. Phys. B **550**, 151 (1999).

[11] W. Pauli, Scientific Correspondence, Vol II p.15, Ed. Karl von Meyenn, Springer-Verlag, 1985.

[12] H.S. Snyder *Quantized Space-Time*, Phys. Rev. **71**, (1) 38 (1947).

[13] L. Bonora and L. Salizzoni, Phys. Lett. B **504**, 80 (2001); A. Armoni, Nucl Phys. B **593**, 229 (2001); M.M. Sheikh-Jabbari, Nucl.Phys.Proc.Suppl. **108**, 113 (2002).

[14] R. Amorim and F. A. Farias, Phys. Rev. D **65**, 065009 (2002).

[15] B de Wit and M. T. Grisaru, *Compensating Fields and Anomalies* in Quantum Field Theory and Quantum Statistics, Vol.2, eds. I.A.Batalin, C.J.Isham and G.A. Vilkovisky, Adam Hilger, 1987.

[16] R. Amorim, N. R. F. Braga, and M. Henneaux, Phys. Lett. B **436**, 125 (1998).

[17] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1992.

[18] C. Itzykson and J-B Zuber, *Quantum Field Theory*, McGraw-Hill Book Company, 1980.

[19] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, 1996.