Noncommutative Configuration Space. Classical and Quantum Mechanical Aspects

F. J. Vanhecke, C. Sigaud, and A. R. da Silva
Instituto de Física, Instituto de Matemática, UFRJ, Rio de Janeiro, Brazil

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In this work we examine noncommutativity of position coordinates in classical symplectic mechanics and its quantisation. In coordinates $\{q^i, p_k\}$ the canonical symplectic two-form is $\omega_0 = dq^i \wedge dp_i$. It is well known in symplectic mechanics [5, 6, 9] that the interaction of a charged particle with a magnetic field can be described in a Hamiltonian formalism without a choice of a potential. This is done by means of a modified symplectic two-form $\omega = \omega_0 - e\mathbf{F}$, where e is the charge and the (time-independent) magnetic field $\mathbf{F}$ is closed: $\mathbf{dF} = 0$. With this symplectic structure, the canonical momentum variables acquire non-vanishing Poisson brackets: $\{p_k, p_l\} = eF_{kl}(q)$. Similarly a closed two-form in p-space $\mathbf{G}$ may be introduced. Such a *dual magnetic field* $\mathbf{G}$ interacts with the particle's *dual charge* r. A new modified symplectic two-form $\omega = \omega_0 - e\mathbf{F} + r\mathbf{G}$ is then defined. Now, both p- and q-variables will cease to Poisson commute and upon quantisation they become noncommuting operators. In the particular case of a linear phase space $\mathbf{R}^{2N}$, it makes sense to consider constant $\mathbf{F}$ and $\mathbf{G}$ fields. It is then possible to define, by a linear transformation, global Darboux coordinates: $\{\xi^i, \pi_k\} = \delta^i{}_k$. These can then be quantised in the usual way $[\widehat{\xi}^i, \widehat{\pi}_k] = i\hbar\delta^i{}_k$. The case of a quadratic potential is examined with some detail when N equals 2 and 3.

Keywords: Noncommutativity; Symplectic mechanics; Quantization

I. INTRODUCTION

The idea to consider non vanishing commutation relations between position operators $[\mathbf{x}, \mathbf{y}] = i\ell^2$, analogous to the canonical commutation relations between position and conjugate momentum $[\mathbf{x}, \mathbf{p}_x] = i\hbar$, is ascribed to Heisenberg, who saw there a possibility to introduce a fundamental lenght ℓ which might control the short distance singularities of quantum field theory. However, noncommutativity of coordinates appeared first nonrelativistically in the work of Peierls [2] on the diamagnetism of conduction electrons. In the limit of a strong magnetic field in the z-direction, the gap between Landau levels becomes large and, to leading order, one obtains $[\mathbf{x}, \mathbf{y}] = i\hbar c/eB$. In relativistic quantum mechanics, noncommutativity was first examined in 1947 by Snyder [3] and, in the last five years, inspired by string and brane-theory, many papers on field theory in noncommutative spaces appeared in the physics literature. The apparent unitarity problem related to time-space noncommutativity in field theory was studied and solved in [10]. Also (nonrelativistic) quantum mechanics on noncommutative twodimensional spaces has been examined more thorougly in the recent years: [11–16]. The above mentionned unitarity problem in quantum physics is also examined in Balachandran et al. [17].

In this work we discuss noncommutativity of configuration space Q in classical mechanics on the cotangent bundle $T^*(Q)$ and its canonical quantisation in the most simple case. In section II we review the classical theory of a non relativistic particle interacting with a time-independent magnetic field $\mathbf{F} = 1/2F_{ij}(q)\,dq^i \wedge dq^j$; $\mathbf{dF} = 0$. This is done in every textbook introducing a potential in a Lagrangian formalism. The Legendre transformation defines then the Hamiltonian and the canonical symplectic two-form $dq^i \wedge dp_i$ implements the corresponding Hamiltonian vector field. We also recall the less well known procedure of avoiding the introduction of a potential using a modified symplectic structure: $\omega = dq^i \wedge dp_i - e\mathbf{F}$. The coupling with the charge e is hidden in the symplectic structure and does not show up in the Hamiltonian: $H_0(q, p) = \delta^{kl}\,p_k\,p_l/2m + \mathcal{V}(q)$. In section III, a closed two-form in p-space, the *dual field*: $\mathbf{G} = 1/2\,G^{kl}(p)\,dp_k \wedge dp_l$, is added to the symplectic structure $\omega = dq^i \wedge dp_i - e\mathbf{F} + r\mathbf{G}$, where r is a *dual charge*.

Such an approach with a modified symplectic structure has been previously considered by Duval and Horvathy [11, 14] emphasizing the $N = 2$-dimensional case in connection with the quantum Hall effect. We should also mention Plyushchay's interpretation [18] of such a dual charge r when $N = 2$ as the anyon spin. Considering here an arbitrary number of dimensions N, no such interpretation of r is assumed. The crucial point is that, now, both p- and q-variables cease to Poisson commute and upon quantisation they should become noncommuting operators. In the particular case of a linear phase space $\mathbf{R}^{2N}$, it makes sense to consider constant $\mathbf{F}$ and $\mathbf{G}$ fields. It is then possible to define global Darboux coordinates with Poisson brackets $\{\xi^i, \pi_k\} = \delta^i{}_k$. These can then be quantised uniquely [1] in the usual way: $[\widehat{\xi}^i, \widehat{\pi}_k] = i\hbar\delta^i{}_k$. However, in general, the dynamics become non-linear and there is no guarantee that the Hamiltonian vector field is complete. It is then not trivial to quantise the Hamiltonian, which becomes nonlocal. However, for a linear or quadratic Hamiltonian, this is possible and it is seen that the noncommutativity generates a magnetic moment type interaction. The cases $N = 2$ and $N = 3$ are discussed in detail in section IV. In section V we examine the problem of symmetries in the modified symplectic manifold. Finally, in section VI general comments are made and further developments are suggested. In appendix A we recall basic notions in symplectic geometry and in appendix B we give a brief account of the Gotay-Nester-Hinds algorithm [7] for constrained Hamiltonian systems.

II. NON RELATIVISTIC PARTICLE INTERACTING WITH A TIME-INDEPENDENT MAGNETIC FIELD

A particle of mass m and charge e, with potential energy $\mathcal{V}$, moving in a Euclidean configuration space Q, with cartesian coordinates q^i, interacts with a (time-independent) magnetic field given by a closed two-form $\mathbf{F}(q) = \frac{1}{2} F_{ij}(q)\, \mathbf{d}q^i \wedge \mathbf{d}q^j$. The dynamics is given by the Laplace equation:

$$m\, \frac{\mathbf{d}^2 q^i}{\mathbf{d}t^2} = \delta^{ij} \left(e F_{jk}(q)\, \frac{\mathbf{d}q^k}{\mathbf{d}t} - \frac{\partial \mathcal{V}(q)}{\partial q^j} \right) . \qquad \text{(II.1)}$$

Assuming Q to be Euclidean avoids topological subtleties, so that there exists a global potential one-form $\mathbf{A}(q) = A_i(q)\, \mathbf{d}q^i$ such that $\mathbf{F} = \mathbf{dA}$. A global Lagrangian formalism can then be established with a Lagrangian function on the tangent bundle $\{\tau : T(Q) \to Q\}$:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m\, \delta_{ij}\, \dot{q}^i \dot{q}^j + e\, \dot{q}^i A_i(q) - \mathcal{V}(q) .$$

The Euler-Lagrange equation is obtained as:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\mathbf{d}}{\mathbf{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = -\frac{\partial \mathcal{V}}{\partial q^i} + e\, \dot{q}^k \frac{\partial A_k(q)}{\partial q^i} - \frac{\mathbf{d}}{\mathbf{d}t} \left(m\, \delta_{ij}\, \dot{q}^j + e A_i(q) \right) \\ &= -\frac{\partial \mathcal{V}}{\partial q^i} + e\, \dot{q}^k \left(\frac{\partial A_k(q)}{\partial q^i} - \frac{\partial A_i(q)}{\partial q^k} \right) - m\, \frac{\mathbf{d}}{\mathbf{d}t}\, \delta_{ij}\, \dot{q}^j \\ &= -\frac{\partial \mathcal{V}}{\partial q^i} + e\, \mathbf{F}_{ik}(q)\, \dot{q}^k - m\, \delta_{ij}\, \ddot{q}^j , \end{aligned} \qquad \text{(II.2)}$$

and coincides with the Laplace equation (II.1).

The Legendre transform

$$(q^i, \dot{q}^j) \to \left(q^i, p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = m\, \delta_{kl}\, \dot{q}^l + e A_k(q) \right) ,$$

defines the Hamiltonian on the cotangent bundle $\{T^*(Q) \xrightarrow{\kappa} Q\}$:

$$\mathcal{H}_{\mathbf{A}}(q, p) = -\mathcal{L}(q, \dot{q}) + p_i\, \dot{q}^i =$$

$$\frac{1}{2m} \delta^{kl} (p_k - e A_k(q))\, (p_l - e A_l(q)) + \mathcal{V}(q) .$$

With the canonical symplectic two-form

$$\omega_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i , \qquad \text{(II.3)}$$

the Hamiltonian vector field of $\mathcal{H}_{\mathbf{A}}$ is:

$$\mathbf{X}_{\mathcal{H}} = \frac{\delta^{ij}}{m} (p_j - e A_j)\, \frac{\partial}{\partial \mathbf{q}^i} + \left(\frac{e}{m} \delta^{kl}\, \frac{\partial A_k}{\partial q^i} (p_l - e A_l) - \frac{\partial \mathcal{V}}{\partial q^i} \right) \frac{\partial}{\partial \mathbf{p}_i} .$$

Its integral curves are solutions of:

$$\frac{\mathbf{d}\, q^i}{\mathbf{d}t} = \frac{\delta^{ij}}{m} (p_j - e A_j) \ , \ \frac{\mathbf{d}\, p_i}{\mathbf{d}t} = \frac{e}{m}\, \delta^{kl}\, \frac{\partial A_k}{\partial q^i} (p_l - e A_l) - \frac{\partial \mathcal{V}}{\partial q^i} , \qquad \text{(II.4)}$$

which is again equivalent to (II.1).

If the second de Rham cohomology were not trivial, $H^2_{dR}(Q) \neq 0$, there is no global potential $\mathbf{A}$ and a local Lagrangian formalism is needed. This can be done enlarging the configuration space Q to the total space $\mathcal{P}$ of a principal $U(1)$ bundle over Q with a connection, given locally by $\mathbf{A}$[19]. This can be avoided using a global Hamiltonian formalism[20] in the cotangent bundle $T^*(Q)$ using a modified symplectic two-form:

$$\omega = \omega_0 - e\mathbf{F} = \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2} e F_{ij}(q)\, \mathbf{d}q^i \wedge \mathbf{d}q^j , \qquad \text{(II.5)}$$

and a "charge-free" Hamiltonian:

$$\mathcal{H}_0(p, q) = \frac{1}{2m}\, \delta^{kl}\, p_k\, p_l \, + \, \mathcal{V}(q) .$$

The Hamiltonian vector fields corresponding to an observable $f(q, p)$ are now defined relative to ω as $\iota_{\mathbf{X}^{\mathbf{F}}_f}\, \omega = \mathbf{d} f$ and given by:

$$\mathbf{X}^{\mathbf{F}}_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial \mathbf{q}^i} - \left(\frac{\partial f}{\partial q^l} + \frac{\partial f}{\partial p_k} e F_{kl}(q) \right) \frac{\partial}{\partial \mathbf{p}_l} .$$

With the Hamiltonian $\mathcal{H}_0$, the dynamics are again given by the Laplace equation (II.1) in the form:

$$\frac{\mathbf{d}\, q^i}{\mathbf{d}t} = \frac{\delta^{ij}}{m} p_j \ ; \ \frac{\mathbf{d}\, p_l}{\mathbf{d}t} = -\, \delta^{ki} \left(\frac{\partial \mathcal{V}}{\partial q^i} + \frac{e}{m} p_i\, F_{kl}(q) \right) . \qquad \text{(II.6)}$$

The Poisson brackets, relative to the symplectic structure II.5, are:

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} + \frac{\partial f}{\partial p_k}\, eF_{kl}(q)\,\frac{\partial g}{\partial p_l}\,. \qquad \text{(II.7)}$$

In particular, the coordinates themselves have Poisson brackets:

$$\{q^i,q^j\} = 0\,,\ \{q^i,p_l\} = \delta^i{}_l\,,$$
$$\{p_k,q^j\} = -\delta_k{}^j\,,\ \{p_k,p_l\} = eF_{kl}(q)\,. \qquad \text{(II.8)}$$

Obviously, the meaning of the $\{q,p\}$ variables in (II.3) and (II.5) are different. However both formalisms $(\omega_0,\mathcal{H}_{\mathbf{A}})$ and $(\omega,\mathcal{H}_0)$ lead to the same equations of motion and thus, they must be equivalent. Indeed, in each open set U homeomorphic to $\mathbf{R}^6$, the vanishing $\mathbf{dF}=0$ implies the existence of $\mathbf{A}$ such that $\mathbf{F}=\mathbf{dA}$ in U and, locally:

$$\omega = \mathbf{d}q^i\wedge\mathbf{d}p_i - \frac{1}{2}eF_{ij}\,\mathbf{d}q^i\wedge\mathbf{d}q^j = -\mathbf{d}[(p_i+eA_i)\,\mathbf{d}q^i].$$

Thus there exist local Darboux coordinates:

$$\xi^i = q^i\,,\ \pi_k = p_k + eA_k(q)\,, \qquad \text{(II.9)}$$

such that $\omega = \mathbf{d}\xi^i\wedge\mathbf{d}\pi_i$, which is the form (II.3).

The dynamics defined by the Hamiltonian $\mathcal{H}_0(q,p) = p^2/2m + \mathcal{V}(q)$, with symplectic two-form ω, is equivalent to the dynamics defined by the Hamiltonian $\mathcal{H}_{\mathbf{A}}(\xi,\pi) = (\pi - eA(\xi))^2/2m + \mathcal{V}(\xi)$ and canonical symplectic structure $\omega = \mathbf{d}\xi^i\wedge\mathbf{d}\pi_i$. Equivalence is trivial since both symplectic two-forms are equal, but expressed in different coordinates $\{q,p\}$ and $\{\xi,\pi\}$, related by (II.9). It seems worthwhile to note that a gauge transformation $\mathbf{A}\to\mathbf{A}' = \mathbf{A}+\mathbf{grad}\phi$ corresponds to a change of Darboux coordinates

$$\{\xi^i,\pi_k\}\ \Rightarrow\ \{\xi^{i\prime}=\xi^i,\pi'_k=\pi_k+e\partial_k\phi\}\,,$$

i.e. a symplectic transformation.

III. NONCOMMUTATIVE COORDINATES

Let us consider an affine configuration space $\mathcal{Q} = \mathbf{A}^N$ so that points of phase space, identified with $\mathcal{M} \doteq \mathbf{R}^{2N} = \mathbf{R}^N_q\times\mathbf{R}^N_p$, may be given by linear coordinates (q,p). Together with the (usual) magnetic field $\mathbf{F}$, we may introduce a (dual) magnetic field $\mathbf{G} = 1/2\,G^{kl}(p)\,\mathbf{d}p_k\wedge\mathbf{d}p_l$, a closed two-form, $\mathbf{dG}=0$, in $\mathbf{R}^n_p$ space. Let e be the usual electric charge and r, a dual charge, which couples the particle with $\mathbf{F}$ and $\mathbf{G}$. Consider the closed two-form:

$$\begin{aligned}\omega &= \omega_0 - e\mathbf{F} + r\mathbf{G}\\ &= \mathbf{d}q^i\wedge\mathbf{d}p_i - \frac{1}{2}eF_{ij}(q)\,\mathbf{d}q^i\wedge\mathbf{d}q^j + \frac{1}{2}rG^{kl}(p)\,\mathbf{d}p_k\wedge\mathbf{d}p_l\,.\end{aligned} \qquad \text{(III.1)}$$

In matrix notation this two-form (III.1) is represented as:

$$\begin{aligned}(\Omega) &= \begin{pmatrix} -e\mathbf{F} & \mathbf{1}\\ -\mathbf{1} & +r\mathbf{G}\end{pmatrix}\\ &= \begin{pmatrix} 0 & \mathbf{1}\\ \mathbf{1} & +r\mathbf{G}\end{pmatrix}\begin{pmatrix} -\Psi & 0\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix} \mathbf{1} & 0\\ -e\mathbf{F} & \mathbf{1}\end{pmatrix}\\ &= \begin{pmatrix} e\mathbf{F} & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\begin{pmatrix} -\mathbf{1} & 0\\ 0 & \Phi\end{pmatrix}\begin{pmatrix} \mathbf{1} & -r\mathbf{G}\\ 0 & \mathbf{1}\end{pmatrix}.\end{aligned} \qquad \text{(III.2)}$$

where[21] $\Phi = (\mathbf{1} - e\mathbf{F}\,r\mathbf{G})$; $\Psi = (\mathbf{1} - r\mathbf{G}\,e\mathbf{F})$.
The fundamental Hamiltonian equation $\iota_{\mathbf{X}}\omega = \mathbf{d}f$, in (A.1), reads:

$$(X^i - r\,G^{ij}X_j)\,\mathbf{d}p_i - (X_k - eF_{kl}X^l)\,\mathbf{d}q^k = \frac{\partial f}{\partial q^k}\mathbf{d}q^k + \frac{\partial f}{\partial p_i}\mathbf{d}p_i\,. \qquad \text{(III.3)}$$

This can be rewritten as

$$\left(\frac{\partial f}{\partial p_i} - r\,G^{ij}\frac{\partial f}{\partial q^j}\right) = \Psi^i{}_j X^j\ ;\ \left(\frac{\partial f}{\partial q^k} - eF_{kl}\frac{\partial f}{\partial p^l}\right) = -\Phi_k{}^l X_l\,. \qquad \text{(III.4)}$$

Obviously, from (III.2) or (III.4), the closed two-form ω will be non degenerate, and hence symplectic, if $\mathbf{det}(\Omega) = \mathbf{det}(\Psi) = \mathbf{det}(\Phi) \neq 0$, so that (Ω) has an inverse:

$$\begin{aligned}(\Omega)^{-1} &= \begin{pmatrix} \mathbf{1} & 0\\ +e\mathbf{F} & \mathbf{1}\end{pmatrix}\begin{pmatrix} -\Psi^{-1} & 0\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix} -r\mathbf{G} & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\\ &= \begin{pmatrix} +\Psi^{-1}r\mathbf{G} & -\Psi^{-1}\\ +e\mathbf{F}\Psi^{-1}r\mathbf{G}+\mathbf{1} & -e\mathbf{F}\Psi^{-1}\end{pmatrix};\end{aligned} \qquad \text{(III.5)}$$

$$\begin{aligned}&= \begin{pmatrix} \mathbf{1} & +r\mathbf{G}\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix} -\mathbf{1} & 0\\ 0 & \Phi^{-1}\end{pmatrix}\begin{pmatrix} 0 & \mathbf{1}\\ \mathbf{1} & -e\mathbf{F}\end{pmatrix}\\ &= \begin{pmatrix} +r\mathbf{G}\Phi^{-1} & -r\mathbf{G}\Phi^{-1}e\mathbf{F}-\mathbf{1}\\ \Phi^{-1} & -\Phi^{-1}e\mathbf{F}\end{pmatrix}.\end{aligned} \qquad \text{(III.6)}$$

Explicitely:

$$\omega^\flat : \mathbf{d}f \to \begin{cases} (X_f)^i &= (\Psi^{-1})^i{}_j\left(\partial f/\partial p_j - r\,G^{jk}\partial f/\partial q^k\right)\\ (X_f)_k &= -(\Phi^{-1})_k{}^l\left(\partial f/\partial q^l - eF_{lj}\partial f/\partial p_j\right)\end{cases} \qquad \text{(III.7)}$$

The corresponding Poisson brackets are given by:

$$\{f,g\} = \omega(\mathbf{X}_f,\mathbf{X}_g) = (\partial_q f\ \ \partial_p f)\ (\Lambda)\ \begin{pmatrix}\partial_q g\\ \partial_p g\end{pmatrix} \qquad \text{(III.8)}$$

with the matrix

$$(\Lambda) = -(\Omega)^{-1} = \begin{pmatrix} -(\Psi^{-1}\, r\mathbf{G} = r\mathbf{G}\,\Phi^{-1}) & +\Psi^{-1} \\ -\Phi^{-1} & +(\Phi^{-1}\, e\mathbf{F} = e\mathbf{F}\,\Psi^{-1}) \end{pmatrix} . \qquad \text{(III.9)}$$

Explicitely:

$$\{f,g\} = -\frac{\partial f}{\partial q}(\Psi^{-1} rG)\frac{\partial g}{\partial q} - \frac{\partial f}{\partial p}(\Phi^{-1})\frac{\partial g}{\partial q} + \frac{\partial f}{\partial q}(\Psi^{-1})\frac{\partial g}{\partial p} + \frac{\partial f}{\partial p}(\Phi^{-1} eF)\frac{\partial g}{\partial p} \, . \qquad \text{(III.10)}$$

In particular, for the coordinates (q^i, p_k), we have:

$$\begin{aligned} \{q^i, q^j\} &= -(\Psi^{-1})^i{}_k\, rG^{kj} = -rG^{ik}\,(\Phi^{-1})_k{}^j , \\ \{q^i, p_l\} &= (\Psi^{-1})^i{}_l , \\ \{p_k, q^j\} &= -(\Phi^{-1})_k{}^j , \\ \{p_k, p_l\} &= (\Phi^{-1})_k{}^j\, eF_{jl} = eF_{kj}\,(\Psi^{-1})^j{}_l \, . \end{aligned} \qquad \text{(III.11)}$$

With $\mathcal{H}(q,p) = (\delta^{kl}\, p_k\, p_l/2m) + \mathcal{V}(q)$, the equations of motion read:

$$\begin{aligned} \frac{dq^i}{dt} &= \{q^i, \mathcal{H}\} = (\Psi^{-1})^i{}_j \left(-rG^{jk}\,\frac{\partial \mathcal{H}}{\partial q^k} + \frac{\partial \mathcal{H}}{\partial p_j}\right) , \\ &= (\Psi^{-1})^i{}_j \left(-rG^{jk}\,\frac{\partial \mathcal{V}}{\partial q^k} + \frac{p^j}{m}\right) , \\ \frac{dp_k}{dt} &= \{p_k, \mathcal{H}\} = (\Phi^{-1})_k{}^l \left(-\frac{\partial \mathcal{H}}{\partial q^l} + eF_{lj}\,\frac{\partial \mathcal{H}}{\partial p_j}\right) \\ &= (\Phi^{-1})_k{}^l \left(-\frac{\partial \mathcal{V}}{\partial q^l} + eF_{lj}\,\frac{p^j}{m}\right) . \end{aligned} \qquad \text{(III.12)}$$

The celebrated Darboux theorem guarantees the existence of local coordinates (ξ^i, π_k), such that $\omega = \mathbf{d}\xi^i \wedge \mathbf{d}\pi_i$. When one of the charges (e,r) vanishes, such Darboux coordinates are easily obtained using the potential one-forms $\mathbf{A} = A_i(q)\mathbf{d}q^i$ and $\widetilde{\mathbf{A}} = \widetilde{A}^k(p)\mathbf{d}p_k$, such that $\mathbf{F} = \mathbf{dA}$ and $\mathbf{G} = \mathbf{d}\widetilde{\mathbf{A}}$.
Indeed, if $r = 0$, as in section II, Darboux coordinates are provided by $\xi^i = q^i ; \pi_k = p_k + eA_k(q)$. A modified symplectic potential and two-form are defined by:

$$\theta = (p_k + eA_k)\,\mathbf{d}q^k \; ; \; \omega = -\,\mathbf{d}\theta \, . \qquad \text{(III.13)}$$

The Hamiltonian and corresponding equations of motion are:

$$\mathcal{H}(\xi,\pi) = \frac{1}{2}\delta^{kl}(\pi_k - eA_k(\xi))(\pi_l - eA_l(\xi)) + \mathcal{V}(\xi) , \qquad \text{(III.14)}$$

$$\frac{\mathbf{d}\xi^i}{\mathbf{d}t} = \delta^{ij}\,(\pi_j - eA_j(\xi)) \, , \; \frac{\mathbf{d}\pi_i}{\mathbf{d}t} = e\,\delta^{kl}(\pi_k - eA_k)\,\frac{\partial A_l}{\partial \xi^i} - \frac{\partial \mathcal{V}}{\partial \xi^i} \, , \qquad \text{(III.15)}$$

which yields the second order equation in ξ, as in (II.1):

$$\frac{\mathbf{d}^2\xi^i}{\mathbf{d}t^2} = \delta^{ij}\left(-\frac{\partial \mathcal{V}(\xi)}{\partial \xi^j} + eF_{jl}(\xi)\,\frac{\mathbf{d}\xi^l}{\mathbf{d}t}\right) . \qquad \text{(III.16)}$$

When $e = 0$, Darboux variables are

$$\xi^i = q^i + r\widetilde{A}^i(p) \, ; \pi_k = p_k \, , \qquad \text{(III.17)}$$

and we define

$$\theta = p_k\,\mathbf{d}(q^k + r\widetilde{A}^k) \; ; \; \omega = -\,\mathbf{d}\theta \, . \qquad \text{(III.18)}$$

The Hamiltonian and equations of motion are now given by:

$$\mathcal{H}(\xi,\pi) = \frac{1}{2}\delta^{kl}\,\pi_k\,\pi_l + \mathcal{V}(\xi - r\widetilde{A}(\pi)) , \qquad \text{(III.19)}$$

$$\frac{\mathbf{d}\xi^i}{\mathbf{d}t} = \delta^{ij}\,\pi_j - r\partial_k\mathcal{V}(q)\,\frac{\partial \widetilde{A}^k}{\partial \pi_i} \, , \; \frac{\mathbf{d}\pi_i}{\mathbf{d}t} = -\frac{\partial \mathcal{V}}{\partial q^i}(q) \, . \qquad \text{(III.20)}$$

The second order equation, obeyed by π **(!)**, is given by

$$\frac{\mathbf{d}^2\pi_i}{\mathbf{d}t^2} = \partial^2_{ij}\mathcal{V}(q)\left(-\delta^{jk}\pi_k + rG^{jk}(\pi)\,\frac{\mathbf{d}\pi_l}{\mathbf{d}t}\right) . \qquad \text{(III.21)}$$

Here the q-variable is assumed to be solved in terms of $\dot{\pi}$ from equation $\dot{\pi}_k = -\partial\mathcal{V}(q)/\partial q^k$ and this is possible if $\mathbf{det}(\partial^2_{ij}\mathcal{V}(q)) \neq 0$!
In the case of nonzero charges (e,r) and non-constant $\mathbf{F}$ and $\mathbf{G}$ fields, there is no generic formula to define global Darboux coordinates (ξ^i, π_k). However, if the fields $\mathbf{F}$ and $\mathbf{G}$ are constant, the Poisson matrix (III.2) is brought in canonical Darboux form by a linear symplectic orthogonalization procedure, à la Hilbert-Schmidt. In the next section this is done explicitely for $N = 2$ and $N = 3$. Obviously such a linear transformation: $(q^i, p_k) \Rightarrow (\xi^i, \pi_k)$ is defined up to a linear symplectic map of $\mathbf{Sp}(2n)$. These variables $(\xi^i, \pi_k) \in \mathbf{R}^{2n}$ can be canonically quantised as operators obeying the commutation relations

$$\left[\widehat{\xi^i}, \widehat{\xi^j}\right] = 0 \, ; \; \left[\widehat{\xi^i}, \widehat{\pi_l}\right] = i\hbar\delta^i{}_l \; ; \; [\widehat{\pi_k}, \widehat{\pi_l}] = 0 \, . \qquad \text{(III.22)}$$

As von Neumann taught us in [1], they are realised on the Hilbert space of square integrable functions of the variable ξ as

$$(\widehat{\xi^i}\Psi)(\xi) = \xi^i\,\Psi(\xi) \; ; \; (\widehat{\pi_k}\Psi)(\xi) = \frac{\hbar}{i}\,\frac{\partial \Psi(\xi)}{\partial \xi^k} \, . \qquad \text{(III.23)}$$

The original variables (q^i, p_k) being linear functions of the (ξ^i, π_k) are then also quantised.
When $\mathbf{det}(\Psi) = \mathbf{det}(\Phi) = 0$, the closed two-form ω is singular. When its rank is constant, ω defines a presymplectic structure on phase space which we call the primary constraint manifold denoted by $\mathcal{M}_1$. The consistency of the resulting constrained Hamiltonian system will be examined in the $N = 2$ and $N = 3$ cases.

IV. EXAMPLES: $N = 2$ AND 3

In the two examples below, we consider a classical Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2m}\delta^{kl}\, p_k\, p_l + \mathcal{V}(q)\,. \tag{IV.1}$$

A complete resolution will be given for a harmonic oscillator potential:

$$\mathcal{V}(q) \doteq \frac{\kappa}{2}\delta_{ij}\, q^i q^j\ . \tag{IV.2}$$

Also of interest is the case of a constant "electric field": $\mathcal{V}(q) = -\mathbf{E}_k q^k$, which is exactly solvable and left to the reader.

A. Dynamics in the noncommutative plane

The magnetic fields in two dimensions, are written as:

$$e F_{ij} = B\varepsilon_{ij}\ ;\ r\, G^{kl} = C\,\varepsilon^{kl}\,, \tag{IV.3}$$

where B and C are pseudoscalars. The closed two-form (III.1) becomes

$$\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i - B\,\mathbf{d}q^1 \wedge \mathbf{d}q^2 + C\,\mathbf{d}p_1 \wedge \mathbf{d}p_2\,. \tag{IV.4}$$

The equation $\imath_X \omega = \mathbf{d}f$ reads

$$X^i - C\varepsilon^{ij} X_j = \frac{\partial f}{\partial p_i}\ ;\ X_k - B\varepsilon_{kl} X^l = -\frac{\partial f}{\partial q^k}\ . \tag{IV.5}$$

Denoting $\chi \doteq (1 + CB)$, the matrices Φ and Ψ are written as $\Phi_i{}^j = \chi\,\delta_i{}^j$ and $\Psi^k{}_l = \chi\,\delta^k{}_l$. The matrix (III.2) is then invertible if χ does not vanish.

1. The non degenerate case

Here, we will assume χ to be strictly positive. The above equation **(IV.5)** can then be inverted with Hamiltonian vector fields given by:

$$X^i = \chi^{-1}\left(\frac{\partial f}{\partial p_i} - C\varepsilon^{ij}\frac{\partial f}{\partial q^j}\right)\,,\ X_k = -\chi^{-1}\left(\frac{\partial f}{\partial q^k} - B\varepsilon_{kl}\frac{\partial f}{\partial p^l}\right). \tag{IV.6}$$

The Poisson brackets **(III.11)** become:

$$\{q^i, q^j\} = -C\chi^{-1}\varepsilon^{ij}\ \ ;\ \ \{q^i, p_l\} = \chi^{-1}\,\delta^i{}_l\,,$$
$$\{p_k, q^j\} = -\chi^{-1}\,\delta_k{}^j\ \ ;\ \ \{p_k, p_l\} = B\chi^{-1}\varepsilon_{kl}\ . \tag{IV.7}$$

Substitution of the Ansatz

$$\xi^i = \alpha\, q^i + \beta\,\frac{C}{2}\, p_k\,\varepsilon^{ki}\ ;\ \pi_k = \gamma\,\frac{B}{2}\, q^j\,\varepsilon_{jk} + \delta\, p_k\,, \tag{IV.8}$$

in the canonical Poison brackets, leads to the equations

$$\alpha^2 - \alpha\beta - \frac{CB}{4}\beta^2 = 0\ ,\ \delta^2 - \delta\gamma - \frac{CB}{4}\gamma^2 = 0\,,$$
$$\alpha\delta + \frac{CB}{2}(\alpha\gamma + \delta\beta) - \frac{CB}{4}\beta\gamma = \chi\,. \tag{IV.9}$$

We choose the solution:

$$\alpha = \delta = \sqrt{u}\ ;\ \beta = \gamma = \frac{1}{\sqrt{u}}\ ;\ u = \frac{1}{2}(1 + \sqrt{\chi})\,, \tag{IV.10}$$

such that (IV.8) reduces to (II.9) when $C = 0$ or to (III.17) in case $B = 0$. The 2-form (III.1) has the canonical Darboux form $\omega = d\xi^i \wedge d\pi_i$ in the variables

$$\xi^i = \sqrt{u}\left(q^i - \frac{C}{2u}\varepsilon^{ik}\, p_k\right)\ ;\ \pi_k = \sqrt{u}\left(p_k - \frac{B}{2u}\varepsilon_{ki}\, q^i\right)\,. \tag{IV.11}$$

These have an inverse if, and only if $\chi \neq 0$:

$$\sqrt{\chi}\, q^i = \sqrt{u}\left(\xi^i + \frac{C}{2u}\varepsilon^{ik}\pi_k\right)\ ;\ \sqrt{\chi}\, p_k = \sqrt{u}\left(\pi_k + \frac{B}{2u}\varepsilon_{ki}\,\xi^i\right)\,. \tag{IV.12}$$

With the complex variables

$$q = q^1 + \mathbf{i}\, q^2\ ,\ p = p_1 + \mathbf{i}\, p_2\ ;\ \xi = \xi^1 + \mathbf{i}\,\xi^2\ ,\ \pi = \pi_1 + \mathbf{i}\,\pi_2\ , \tag{IV.13}$$

the above changes of variables are written as:

$$\xi = \sqrt{u}\left(q + \mathbf{i}\,\frac{C}{2u}p\right)\ ;\ \pi = \sqrt{u}\left(p + \mathbf{i}\,\frac{B}{2u}q\right)\,. \tag{IV.14}$$

The inverse transformations are:

$$q = \sqrt{u/\chi}\left(\xi - \mathbf{i}\frac{C}{2u}\pi\right)\ ;\ p = \sqrt{u/\chi}\left(\pi - \mathbf{i}\frac{B}{2u}\xi\right)\,. \tag{IV.15}$$

The Hamiltonian (IV.2) becomes

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m'}\delta^{kl}\,\pi_k\pi_l + \frac{\kappa'}{2}\delta_{ij}\,\xi^i\xi^j - \omega'_L\Lambda \\ &= \frac{1}{2m'}\frac{\pi^\dagger\pi + \pi\pi^\dagger}{2} + \frac{\kappa'}{2}\frac{\xi^\dagger\xi + \xi\xi^\dagger}{2} - \omega'_L\Lambda\end{aligned} \tag{IV.16}$$

where Λ is angular momentum

$$\begin{aligned}\Lambda &= \frac{1}{2}\left(\varepsilon_{ij}\,\xi^i\,\delta^{jk}\pi_k - \varepsilon^{kl}\,\pi_k\,\delta_{lj}\xi^j\right) \\ &= \frac{1}{2}\left((\xi^1\pi_2 - \xi^2\pi_1) - (\pi_1\xi^2 + \pi_2\xi^1)\right) \\ &= \frac{1}{4\mathbf{i}}\left((\xi^\dagger\pi - \xi\pi^\dagger) - (\pi\xi^\dagger + \pi^\dagger\xi)\right)\,.\end{aligned} \tag{IV.17}$$

The "renormalised" mass and elasticity constant are given by:

$$\frac{1}{m'} = \frac{1}{m}\frac{u}{\chi}\left(1 + \frac{c^2}{4u^2}\right)\ ;\ \kappa' = \kappa\frac{u}{\chi}\left(1 + \frac{b^2}{4u^2}\right)\,. \tag{IV.18}$$

where

$$b = \frac{B}{\sqrt{m\kappa}}\ ;\ c = C\sqrt{m\kappa}\,. \tag{IV.19}$$

The corresponding frequency $\omega'_0 = \sqrt{\kappa'/m'}$ is given in terms of the "bare" frequency $\omega_0 = \sqrt{\kappa/m}$ by:

$$\omega'_0 = \frac{\omega_0}{2\chi}\left((b - c)^2 + 4\chi\right)^{1/2}\,. \tag{IV.20}$$

and ω'_L, the induced Larmor frequency, by:

$$\omega'_L = \frac{\omega_0}{2\chi}(b-c)\ . \tag{IV.21}$$

The solution of Hamiltonian's equations with (IV.16) is standard. With[22]

$$m'\omega'_0 = \sqrt{m'\kappa'} = \sqrt{m\kappa}\left(\left(1+\frac{b^2}{4u^2}\right)\left(1+\frac{c^2}{4u^2}\right)^{-1}\right)^{1/2} \tag{IV.22}$$

reduced variables are introduced by:

$$Q \doteq (m'\omega'_0)^{1/2}\,\xi\ ;\ P \doteq (m'\omega'_0)^{-1/2}\,\pi\ . \tag{IV.23}$$

The original (q,p) are expressed as:

$$\begin{aligned} q &= \sqrt{u/\chi}\,(m'\omega'_0)^{-1/2}\left(Q - \mathbf{i}\,\frac{c'}{2u}P\right)\ ,\\ p &= \sqrt{u/\chi}\,(m'\omega'_0)^{+1/2}\left(P - \mathbf{i}\,\frac{b'}{2u}Q\right)\ , \end{aligned} \tag{IV.24}$$

where

$$c' = C\,(m'\omega'_0) = C\sqrt{m'\kappa'}\ ,\ b' = B/(m'\omega'_0) = B/\sqrt{m'\kappa'}\ . \tag{IV.25}$$

The symplectic structure and the Poisson brackets are:

$$\begin{aligned} \omega &= \frac{1}{2}\left(\mathbf{d}Q^\dagger\wedge\mathbf{d}P + \mathbf{d}Q\wedge\mathbf{d}P^\dagger\right)\\ \{f,g\} &= 2\left(\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P^\dagger} + \frac{\partial f}{\partial Q^\dagger}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q^\dagger} - \frac{\partial f}{\partial P^\dagger}\frac{\partial g}{\partial Q}\right)\ . \end{aligned} \tag{IV.26}$$

The fundamental nonzero Poisson bracket is

$$\{Q,P^\dagger\} = 2\ . \tag{IV.27}$$

In these variables, the Hamiltonian **(IV.16)** reads:

$$\mathcal{H} = \frac{\omega'_0}{4}\left((P^\dagger P + PP^\dagger) + (Q^\dagger Q + QQ^\dagger)\right) - \omega'_L\,\Lambda\ , \tag{IV.28}$$

where

$$\Lambda = \frac{1}{4\mathbf{i}}\left((Q^\dagger P - QP^\dagger) - (PQ^\dagger + P^\dagger Q)\right)\ . \tag{IV.29}$$

The corresponding equations of motion are:

$$\begin{aligned} \frac{dQ}{dt} = \{Q,\mathcal{H}\} &= 2\,\frac{\partial\mathcal{H}}{\partial P^\dagger} = \omega'_0 P - \mathbf{i}\,\omega'_L Q\\ \frac{dP}{dt} = \{Q,\mathcal{H}\} &= -2\,\frac{\partial\mathcal{H}}{\partial Q^\dagger} = -\omega'_0 Q - \mathbf{i}\,\omega'_L P \end{aligned} \tag{IV.30}$$

With the shift variables

$$A_{(+)} = \frac{1}{2}(Q + \mathbf{i}P)\ ;\ A_{(-)} = \frac{1}{2}\left(Q^\dagger + \mathbf{i}P^\dagger\right)\ , \tag{IV.31}$$

the symplectic structure and the Poisson brackets are given by:

$$\omega = -\mathbf{i}\left(\mathbf{d}A^\dagger_{(+)}\wedge\mathbf{d}A_{(+)} + \mathbf{d}A^\dagger_{(-)}\wedge\mathbf{d}A_{(-)}\right)\ , \tag{IV.32}$$

$$\begin{aligned} \{f,g\} = -\mathbf{i}\Bigg(&\frac{\partial f}{\partial A_{(+)}}\frac{\partial g}{\partial A^\dagger_{(+)}} + \frac{\partial f}{\partial A_{(-)}}\frac{\partial g}{\partial A^\dagger_{(-)}}\\ &- \frac{\partial f}{\partial A^\dagger_{(+)}}\frac{\partial g}{\partial A_{(+)}} - \frac{\partial f}{\partial A^\dagger_{(-)}}\frac{\partial g}{\partial A_{(-)}}\Bigg) \end{aligned} \tag{IV.33}$$

with fundamental nonzero brackets:

$$\{A_{(\pm)},A^\dagger_{(\pm)}\} = -\mathbf{i}\ . \tag{IV.34}$$

The Hamiltonian, with the (positive !) frequencies

$$\omega_{(\pm)} = (\omega'_0 \pm \omega'_L)\,, \tag{IV.35}$$

reads now:

$$\mathcal{H} = \frac{\omega_{(+)}}{2}\left(A^\dagger_{(+)}A_{(+)} + A_{(+)}A^\dagger_{(+)}\right) + \frac{\omega_{(-)}}{2}\left(A^\dagger_{(-)}A_{(-)} + A_{(-)}A^\dagger_{(-)}\right)\ . \tag{IV.36}$$

The corresponding equations of motion and their solutions are given by:

$$\frac{dA_{(\pm)}}{dt} = \{A_{(\pm)},\mathcal{H}\} = -\mathbf{i}\,\frac{\partial\mathcal{H}}{\partial A^\dagger_{(\pm)}} = -\mathbf{i}\,\omega_{(\pm)}A_{(\pm)}\ ; \tag{IV.37}$$

$$A_{(\pm)}(t) = \mathbf{exp}\left\{-\mathbf{i}\,\omega_{(\pm)}t\right\}A_{(\pm)}(0)\,. \tag{IV.38}$$

The relations between variables are given by:

$$\begin{aligned} A_{(+)} &= \frac{1}{2}(Q+\mathbf{i}P) \\ &= \frac{\sqrt{u}}{2}\left((m'\omega_0')^{+1/2}(1-\frac{b'}{2u})\,q+\mathbf{i}\,(m'\omega_0')^{-1/2}(1+\frac{c'}{2u})\,p\right) \\ A^\dagger_{(-)} &= \frac{1}{2}(Q-\mathbf{i}P) \\ &= \frac{\sqrt{u}}{2}\left((m'\omega_0')^{+1/2}(1+\frac{b'}{2u})\,q-\mathbf{i}\,(m'\omega_0')^{-1/2}(1-\frac{c'}{2u})\,p\right). \end{aligned} \quad \text{(IV.39)}$$

The inverse transformations are:

$$\begin{aligned} q &= (m'\omega_0')^{-1/2}\sqrt{u/\chi}\left(Q-\mathbf{i}\frac{c'}{2u}P\right), \\ &= (m'\omega_0')^{-1/2}\sqrt{u/\chi}\left((1-\frac{c'}{2u})A_{(+)}+(1+\frac{c'}{2u})A^\dagger_{(-)}\right), \\ p &= (m'\omega_0')^{+1/2}\sqrt{u/\chi}\left(P-\mathbf{i}\frac{b'}{2u}Q\right) \\ &= \mathbf{i}\,(m'\omega_0')^{+1/2}\sqrt{u/\chi}\left((1-\frac{b'}{2u})A^\dagger_{(-)}-(1+\frac{b'}{2u})A_{(+)}\right). \end{aligned} \quad \text{(IV.40)}$$

Quantisation is trivial though the substitution of the fundamental Poison brackets (IV.27),(IV.34) by operator commutators

$$\left[\mathbf{Q},\mathbf{P}^\dagger\right]=2\mathbf{i}\hbar\,;\ \left[\mathbf{A}_{(\pm)},\mathbf{A}^\dagger_{(\pm)}\right]=\hbar. \quad \text{(IV.41)}$$

Having kept the initial ordering, the quantum Hamiltonian has eigenvalues:

$$E(n_{(+)},n_{(-)})=\hbar\omega_{(+)}(n_{(+)}+1/2)+\hbar\omega_{(-)}(n_{(-)}+1/2), \quad \text{(IV.42)}$$

where $n_{(\pm)}$ are nonnegative integers. The corresponding eigenvectors are denoted by $|n_{(+)},n_{(-)}>$.

2. The degenerate or constraint case

The condition $\chi \doteq (1+BC)=0$ determines ω as a presymplectic structure on $\mathcal{M}$ and shall be called the primary constraint. Again, the notation is simplified using complex variables[23]. The presymplectic two-form reads

$$\begin{aligned} \omega &= \frac{1}{2}\left(dq^\dagger\wedge dp+dq\wedge dp^\dagger\right) \\ &\quad -\frac{B}{4\mathbf{i}}\left(dq^\dagger\wedge dq-dq\wedge dq^\dagger\right)+\frac{C}{4\mathbf{i}}\left(dp^\dagger\wedge dp-dp\wedge dp^\dagger\right). \end{aligned} \quad \text{(IV.43)}$$

The Hamiltonian (IV.2) becomes

$$\mathcal{H}=\frac{1}{2m}\frac{p^\dagger p+p\,p^\dagger}{2}+\frac{\kappa}{2}\frac{q^\dagger q+q\,q^\dagger}{2}, \quad \text{(IV.44)}$$

Writing a vector field as

$$\mathbf{X}=X^i\,\partial/\partial q^i+X_k\,\partial/\partial p_k=U\,\partial/\partial q+U^\dagger\,\partial/\partial q^\dagger+V\,\partial/\partial p+V^\dagger\,\partial/\partial p^\dagger,$$

$$\begin{aligned} \imath_X\omega &= \frac{1}{2}\Big((U+\mathbf{i}CV)\,dq^\dagger+(U^\dagger-\mathbf{i}CV^\dagger)\,dq \\ &\quad -(V+\mathbf{i}BU)\,dp^\dagger-(V^\dagger-\mathbf{i}BU^\dagger)\,dp\Big). \end{aligned} \quad \text{(IV.45)}$$

The homogeneous equation, $\imath_{\mathbf{Z}}\omega=0$ has nontrivial solutions. Indeed, with $U_0=Z^1+\mathbf{i}Z^2$ and $V_0=Z_1+\mathbf{i}Z_2$, equation (IV.45) yields the system:

$$U_0+\mathbf{i}CV_0=0\,;\ \text{or}\quad V_0+\mathbf{i}BU_0=0, \quad \text{(IV.46)}$$

of which the determinant is $\chi = 1 + BC = 0$.
The inhomogeneous equation $\imath_{\mathbf{X}}\omega = \mathbf{d}\mathcal{H}$, i.e. the Hamiltonian dynamics, reads

$$U + \mathbf{i}CV = 2\frac{\partial \mathcal{H}}{\partial p^\dagger} = \frac{p}{m} \; ; \; V + \mathbf{i}BU = -2\frac{\partial \mathcal{H}}{\partial q^\dagger} = \kappa q \, . \quad \text{(IV.47)}$$

It will have a solution if

$$\langle \mathbf{d}\mathcal{H} | \mathbf{Z} \rangle = 0 \, . \quad \text{(IV.48)}$$

This condition, termed secondary constraint, is explicitely given by:

$$\frac{\partial \mathcal{H}}{\partial p} - \mathbf{i}C\frac{\partial \mathcal{H}}{\partial q} = 0 \; ; \text{ or } \quad \frac{\partial \mathcal{H}}{\partial q} - \mathbf{i}B\frac{\partial \mathcal{H}}{\partial p} = 0 \, . \quad \text{(IV.49)}$$

For the Hamiltonian (IV.44) this condition (IV.49) is linear:

$$\frac{1}{m}\,p + \mathbf{i}C\kappa q = 0 \; ; \text{ or } \quad \kappa q + \mathbf{i}B\frac{1}{m}\,p = 0 \, . \quad \text{(IV.50)}$$

and defines the secondary constraint manifold $\mathcal{M}_2$.
On $\mathcal{M}_2$, a particular solution of $\imath_{\mathbf{X}}\omega = \mathbf{d}\mathcal{H}$ is given by:

$$U_P = \frac{p}{m} \; ; \; V_P = 0 \, . \quad \text{(IV.51)}$$

The general solution is given by:

$$U = \frac{p}{m} + U_0 \; ; \; V = V_0 \, . \quad \text{(IV.52)}$$

where (U_0, V_0) is restricted to obey (IV.46). This vector field, restricted to $\mathcal{M}_2$, should conserve the constraints i.e. must be tangent to $\mathcal{M}_2$:

$$0 = \langle \frac{1}{m}\,\mathbf{d}p + \mathbf{i}C\,\kappa\,\mathbf{d}q | X \rangle \, , \quad \text{(IV.53)}$$

The vector fields U and V are completely defined on $\mathcal{M}_2$, with ensuing equations of motion:

$$\begin{aligned} \frac{dq}{dt} &= U = -\mathbf{i}\frac{\sqrt{m\kappa}C}{1+m\kappa C^2}\,\omega_0\,q = \frac{1}{1+m\kappa C^2}\,\frac{p}{m} \, , \\ \frac{dp}{dt} &= V = -\mathbf{i}\frac{\sqrt{m\kappa}C}{1+m\kappa C^2}\,\omega_0\,p = -\frac{m\kappa C^2}{1+m\kappa C^2}\,\kappa q \, . \end{aligned} \quad \text{(IV.54)}$$

In terms of the frequency:

$$\omega_r = -\frac{\sqrt{m\kappa}C}{1+m\kappa C^2}\,\omega_0 = \frac{B/\sqrt{m\kappa}}{1+B^2/m\kappa}\,\omega_0 \, , \quad \text{(IV.55)}$$

the solution is given by

$$q(t) = \mathbf{exp}\{\mathbf{i}\,\omega_r\,t\}\,q_0 \; ; \; p(t) = \mathbf{exp}\{\mathbf{i}\,\omega_r\,t\}\,p_0 \, . \quad \text{(IV.56)}$$

Obviously, if q_0 and p_0 obey the secondary constraints (IV.50), $q(t)$ and $p(t)$ obey them at all times.
The same result can be obtained by symplectic reduction, restricting the pre-symplectic two-form (IV.43) to $\mathcal{M}_2$:

$$\omega_{|\mathcal{M}_2} = -\mathbf{i}\,\frac{(1+m\kappa C^2)^2}{2C}\,dq^\dagger \wedge dq \, . \quad \text{(IV.57)}$$

$$\{f,g\}_{\mathcal{M}_2} = \frac{2\mathbf{i}C}{(1+m\kappa C^2)^2}\left(\frac{\partial f}{\partial q^\dagger}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial q^\dagger}\right) \, . \quad \text{(IV.58)}$$

The fundamental Poisson bracket is

$$\{q, q^\dagger\}_{\mathcal{M}_2} = \frac{-2\mathbf{i}C}{(1+m\kappa C^2)^2} \quad \text{(IV.59)}$$

The dynamics are given by:

$$\frac{dq}{dt} = -\frac{2\mathbf{i}C}{(1+m\kappa C)^2}\,\frac{\partial \mathcal{H}_r}{\partial q^\dagger} \, . \quad \text{(IV.60)}$$

And, with the reduced Hamiltonian $\mathcal{H}_r$ given by

$$\mathcal{H}_r = (1+m\kappa C^2)\,\frac{\kappa}{2}\,q^\dagger\,q \, , \quad \text{(IV.61)}$$

this yields equation (IV.56). When $B > 0$, hence $C < 0$, we define

$$a = \frac{(1+m\kappa C^2)}{|2C|}\,q^\dagger \, , \quad \text{(IV.62)}$$

such that

$$\{a, a^\dagger\} = -\mathbf{i} \; ; \; \mathcal{H}_r = \frac{\omega_r}{2}\,(a^\dagger\,a + a\,a^\dagger) \, . \quad \text{(IV.63)}$$

Quantisation is again trivial introducing operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, obeying

$$[\mathbf{a}, \mathbf{a}^\dagger] = \hbar \quad \text{(IV.64)}$$

such that the quantum Hamiltonian

$$\mathbf{H}_r = \frac{\omega_r}{2}\,(\mathbf{a}^\dagger\,\mathbf{a} + \mathbf{a}\,\mathbf{a}^\dagger) \, . \quad \text{(IV.65)}$$

has eigenvalues:

$$E(n) = \hbar\omega_r\,(n + 1/2) \, . \quad \text{(IV.66)}$$

3. The $\chi \to 0$ limit of **(IV A 1)**.

We need the expansion of

$$(m'\omega_0') = (m\omega_0) \times \left(\left(1 + \frac{b^2}{4u^2}\right) \left(1 + \frac{c^2}{4u^2}\right)^{-1} \right)^{1/2} , \quad \text{(IV.67)}$$

in powers of $\varepsilon = \sqrt{\chi}$, where $1 + bc = \varepsilon^2$ and $2u = 1 + \varepsilon$.

$$\begin{aligned} (m'\omega_0') &= \frac{m\omega_0}{|c|} \left(1 + \frac{c^2-1}{c^2+1}\varepsilon + \cdots\right) \\ &= \frac{1}{|C|} \left(1 + \frac{c^2-1}{c^2+1}\varepsilon + \cdots\right) \\ (m'\omega_0')^{-1} &= \frac{(m\omega_0)^{-1}}{|b|} \left(1 + \frac{b^2-1}{b^2+1}\varepsilon + \cdots\right) \\ &= \frac{1}{|B|} \left(1 + \frac{b^2-1}{b^2+1}\varepsilon + \cdots\right) . \end{aligned} \quad \text{(IV.68)}$$

Also, from **(IV.25)**, we obtain

$$\begin{aligned} \frac{c'}{2u} &= \frac{(m'\omega_0')C}{2u} = \frac{C}{|C|} \left(1 - \frac{2}{c^2+1}\varepsilon + \cdots ,\right) \\ \frac{b'}{2u} &= \frac{B}{(m'\omega_0')\,2u} = \frac{B}{|B|} \left(1 - \frac{2}{b^2+1}\varepsilon + \cdots ,\right) \end{aligned} \quad \text{(IV.69)}$$

For definitenees, we assume in the following $B > 0$ and so $C < 0$ in the limit $\varepsilon \to 0$. We obtain

$$\begin{aligned} 1 - \frac{b'}{2u} &= \frac{2}{1+b^2}\varepsilon + \cdots \quad ; \\ 1 + \frac{b'}{2u} &= 2 - \frac{2}{1+b^2}\varepsilon + \cdots \\ 1 + \frac{c'}{2u} &= \frac{2}{1+c^2}\varepsilon + \cdots \quad ; \\ 1 - \frac{c'}{2u} &= 2 - \frac{2}{1+b^2}\varepsilon + \cdots . \end{aligned} \quad \text{(IV.70)}$$

Also:

$$\omega_0' = \frac{\omega_0}{2\varepsilon^2}(b-c)\left(1 + \frac{2\varepsilon^2}{(b-c)^2}\right) , \; \omega_L' = \frac{\omega_0}{2\varepsilon^2}(b-c) , \quad \text{(IV.71)}$$

$$\begin{aligned} \omega_{(+)} &= \omega_0' + \omega_L' = -\omega_0 \frac{1+(m\omega_0)^2C^2}{(m\omega_0)C} \frac{1}{\varepsilon^2} , \\ \omega_{(-)} &= \omega_0' - \omega_L' = -\omega_0 \frac{(m\omega_0)C}{1+(m\omega_0)^2C^2} . \end{aligned} \quad \text{(IV.72)}$$

One of the frequencies $\omega_{(+)}$ diverges, while the other $\omega_{(-)}$ tends to ω_r defined in (IV.55). The relations in (IV.39) yield the initial conditions:

$$\begin{aligned} A_{(+)}(0) &\approx \sqrt{\frac{|B|}{2}} (1+b^2)^{-1} \left(q_0 + \mathbf{i}\frac{B}{(m\omega_0)^2} p_0\right) (\varepsilon + O(\varepsilon^2)) \\ A^{\dagger}_{(-)}(0) &\approx \sqrt{\frac{|B|}{2}} \left(q_0 - \mathbf{i}\frac{1}{|B|} p_0\right) (1 + O(\varepsilon^2)) . \end{aligned} \quad \text{(IV.73)}$$

The solutions (IV.40), in the $\varepsilon \to 0$ limit are then written as

$$\begin{aligned} q(t) &\approx \sqrt{\frac{2}{|B|}} \left(\frac{1}{\varepsilon} A_{(+)}(0)\,\mathbf{exp}\{-\mathbf{i}\,\omega_{(+)}t\} + \frac{1}{1+c^2} A^{\dagger}_{(-)}(0)\,\mathbf{exp}\{\mathbf{i}\,\omega_r t\}\right) \\ &\approx (1+b^2)^{-1} \left(q_0 + \mathbf{i}\frac{|B|}{(m\omega_0)^2} p_0\right) \mathbf{exp}\{-\mathbf{i}\,\omega_{(+)}t\} \\ &\quad + (1+c^2)^{-1} \left(q_0 - \mathbf{i}\,|B|^{-1} p_)\right) \mathbf{exp}\{+\mathbf{i}\,\omega_r t\} ; \end{aligned} \quad \text{(IV.74)}$$

The first term is a fast oscillating function with diverging frequency and so averages to zero. Furthermore, if the initial conditions are on $\mathcal{M}_2$, i.e. if $\left(q_0 + \mathbf{i}\,|B|\,p_0/(m\omega_0)^2\right) = 0$, this first term behaves as $O(\varepsilon)\,\mathbf{exp}\{\mathbf{i}\nu t/\varepsilon^2\}$ converging to zero. The second term is then reduced to the expression (IV.56) of $q(t)$. Similar considerations hold for $p(t)$ in such a way that the solution stays on M_2.

B. Noncommutative $\mathbf{R^3}$

In $\mathbf{R^3}$, the magnetic fields $\mathbf{F}$ and $\mathbf{G}$ are written in terms of pseudovectors $\overline{\mathbf{B}} = \{B^k\}$ and $\underline{\mathbf{C}} = \{C_k\}$ as:

$$e F_{ij} = \varepsilon_{ijk} B^k \; ; \; r G^{ij} = \varepsilon^{ijk} C_k \, . \tag{IV.75}$$

The closed two-form (III.1) is written as:

$$\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2}\varepsilon_{ijk} B^k \mathbf{d}q^i \wedge \mathbf{d}q^j + \frac{1}{2}\varepsilon^{klm} C_m \mathbf{d}p_k \wedge \mathbf{d}p_l \, . \tag{IV.76}$$

The fundamental equation $\imath_X \omega = \mathbf{d} f$ reads

$$X^i - C_k \varepsilon^{ijk} X_j = \frac{\partial f}{\partial p_i} \; ; \; X_k - B^i \varepsilon_{kli} X^l = -\frac{\partial f}{\partial q^k} \, . \tag{IV.77}$$

Defining $\vartheta = \underline{\mathbf{C}} \cdot \overline{\mathbf{B}} = C_k B^k$ and $\chi = 1 + \vartheta$, this is also written as

$$\begin{aligned} \chi X^i &= (\delta^i{}_j + B^i C_j)\frac{\partial f}{\partial p_j} - C_k \varepsilon^{ijk} \frac{\partial f}{\partial q^j} \\ \chi X_k &= -\left((\delta_k{}^l + C_k B^l)\frac{\partial f}{\partial q^l} - B^i \varepsilon_{kli} \frac{\partial f}{\partial p_l} \right) . \end{aligned} \tag{IV.78}$$

The 3×3 matrices Φ and Ψ read:

$$\Phi_i{}^j = \chi \delta_i{}^j - C_i B^j \; ; \; \Psi^k{}_l = \chi \delta^k{}_l - B^k C_l \, ,$$

with $\det \Phi = \det \Psi = \chi^2$. Assuming again $\chi \neq 0$[24], these matrices have inverses:

$$(\Phi^{-1})_i{}^j = \frac{1}{\chi}\left(\delta_i{}^j + C_i B^j\right) \, , \; (\Psi^{-1})^k{}_\ell = \frac{1}{\chi}\left(\delta^k{}_\ell + B^k C_\ell\right) \, .$$

The Hamiltonian vector fields are obtained from (IV.78):

$$\begin{aligned} X^i &= \chi^{-1}\left((\delta^i{}_j + B^i C_j)\frac{\partial f}{\partial p_j} - C_k \varepsilon^{ijk} \frac{\partial f}{\partial q^j} \right) , \\ X_k &= -\chi^{-1}\left((\delta_k{}^l + C_k B^l)\frac{\partial f}{\partial q^l} - B^i \varepsilon_{kli} \frac{\partial f}{\partial p_l} \right) . \end{aligned} \tag{IV.79}$$

The Poissson brackets are given by:

$$\begin{aligned} \{q^i, q^j\} &= -\chi^{-1} \varepsilon^{ijk} C_k \quad , \quad \{q^i, p_l\} = \chi^{-1}\left(\delta^i{}_l + B^i C_l\right) , \\ \{p_k, q^j\} &= -\chi^{-1}\left(\delta_k{}^j + C_k B^j\right) \quad , \quad \{p_k, p_l\} = \chi^{-1} \varepsilon_{klm} B^m \, . \end{aligned} \tag{IV.80}$$

The Ansatz (IV.8) has to be generalised to

$$\begin{aligned} \xi^i &= \alpha q^i + \alpha' B^i (C_k q^k) - \beta \frac{1}{2}\varepsilon^{ijk} p_j C_k ; \\ \pi_k &= \alpha p_k + \alpha' (p_i B^i) C_k + \beta \frac{1}{2}\varepsilon_{klm} B^l q^m \, . \end{aligned} \tag{IV.81}$$

For α, β similar equations as in (IV.9) are obtained:

$$\alpha^2 - \alpha\beta - \frac{\vartheta}{4}\beta^2 = 0 \, , \; \alpha^2 + \vartheta(\alpha\beta) - \frac{\vartheta}{4}\beta^2 = \chi \, , \tag{IV.82}$$

with a the same solution (χ assumed to be strictly positive):

$$\alpha = \sqrt{u} \; ; \; \beta = \frac{1}{\sqrt{u}} \; ; \; u = \frac{1}{2}(1 + \sqrt{\chi}) \, . \tag{IV.83}$$

Furthermore, there is an additional equation for α':

$$\chi\left(\vartheta \alpha'^2 + 2\alpha\alpha'\right) + \left(\alpha^2 - \alpha\beta + \frac{1}{4}\beta^2\right) = 0 \, . \tag{IV.84}$$

Substituting (IV.83), one obtains

$$\vartheta \alpha'^2 + 2\sqrt{u}\alpha' + \frac{1}{4u} = 0 \, ,$$

with solution, remaining finite when $\vartheta \to 0$,:

$$\alpha' = \sqrt{u}\gamma = \frac{(1 - \sqrt{u})}{\vartheta} \, . \tag{IV.85}$$

The formulae (IV.81) are finally written as:

$$\begin{aligned} \xi^i &= \sqrt{u}\left(q^i + \gamma B^i (C_k q^k) - \frac{1}{2u}\varepsilon^{ijk} p_j C_k \right) ; \\ \pi_k &= \sqrt{u}\left(p_k + \gamma (p_i B^i) C_k + \frac{1}{2u}\varepsilon_{klm} B^l q^m \right) . \end{aligned} \tag{IV.86}$$

In old fashioned vector notation, this appears as:

$$\begin{aligned} \overline{\xi} &= \sqrt{u}\left(\overline{\mathbf{q}} + \gamma \overline{\mathbf{B}} (\underline{\mathbf{C}} \cdot \overline{\mathbf{q}}) - \frac{1}{2u} \underline{\mathbf{p}} \times \underline{\mathbf{C}} \right) ; \\ \underline{\pi} &= \sqrt{u}\left(\underline{\mathbf{p}} + \gamma (\underline{\mathbf{p}} \cdot \overline{\mathbf{B}}) \underline{\mathbf{C}} + \frac{1}{2u} \overline{\mathbf{B}} \times \overline{\mathbf{q}} \right) . \end{aligned} \tag{IV.87}$$

The inverse formulae of (IV.86) are obtained as:

$$\begin{aligned} q^i &= \frac{\sqrt{u}}{\sqrt{\chi}}\left(\xi^i + \gamma' B^i (C_k \xi^k) + \frac{1}{2u}\varepsilon^{ijk} \pi_j C_k \right) ; \\ p_k &= \frac{\sqrt{u}}{\sqrt{\chi}}\left(\pi_k + \gamma' C_k (\pi_l B^l) - \frac{1}{2u}\varepsilon_{klm} B^l \xi^m \right) \end{aligned} \tag{IV.88}$$

Or, in vector notation:

$$\begin{aligned} \overline{\mathbf{q}} &= \frac{\sqrt{u}}{\sqrt{\chi}}\left(\overline{\xi} + \gamma' \overline{\mathbf{B}} (\underline{\mathbf{C}} \cdot \overline{\xi}) + \frac{1}{2u} \underline{\pi} \times \underline{\mathbf{C}} \right) ; \\ \underline{\mathbf{p}} &= \frac{\sqrt{u}}{\sqrt{\chi}}\left(\underline{\pi} + \gamma' \underline{\mathbf{C}} (\underline{\pi} \cdot \overline{\mathbf{B}}) - \frac{1}{2u} \overline{\mathbf{B}} \times \overline{\xi} \right) , \end{aligned} \tag{IV.89}$$

where

$$\gamma' = \frac{\sqrt{\chi} - \sqrt{u}}{\vartheta \sqrt{u}} . \qquad \text{(IV.90)}$$

Again, for sake of simplicity, we consider a configuration space which is Euclidean $Q = \mathbf{E}^3$ with metric $< \overline{\mathbf{v}}; \overline{\mathbf{w}} >= \delta_{ij} v^i w^j = (\underline{\mathbf{v}} \cdot \overline{\mathbf{w}})$ such that $v_i = \delta_{ij} v^i$. Substitution of (IV.88) in a Hamiltonian of the form (IV.2), leads to a Hamiltonian quadratic in (ξ, π) and to a system of linear evolution equations. In the case when $\overline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ point in the same direction:

$$\overline{\mathbf{B}} = B \overline{\mathbf{e}}_Z \; ; \; \underline{\mathbf{C}} = C \underline{\mathbf{e}}_Z \; , \qquad \text{(IV.91)}$$

a particularly simple Hamiltonian is obtained. Parallel coordinates are defined by ξ^3, π_3 and transverse coordinate vectors by $\overline{\xi}_\perp = \overline{\xi} - \xi^3 \overline{\mathbf{e}}_Z$ and $\underline{\pi}_\perp = \underline{\pi} - \pi_3 \underline{\mathbf{e}}_Z$. Indeed, eq. (IV.88) becomes

$$\begin{aligned} q^1 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\xi^1 + \frac{1}{2u} \pi_2 C \right) \; , \; p_1 = \frac{\sqrt{u}}{\sqrt{\chi}} \left(\pi_1 + \frac{1}{2u} \xi^2 B \right) , \\ q^2 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\xi^2 - \frac{1}{2u} \pi_1 C \right) \; , \; p_2 = \frac{\sqrt{u}}{\sqrt{\chi}} \left(\pi_2 - \frac{1}{2u} B \xi^1 \right) , \\ q^3 &= \xi^3 \; , \; p_3 = \pi_3 . \end{aligned} \qquad \text{(IV.92)}$$

The Hamiltonian is:

$$\mathcal{H}(\xi, \pi) = \left(\frac{1}{2 m_\perp} (\underline{\pi}_\perp)^2 + \frac{k_\perp}{2} (\overline{\xi}_\perp)^2 \right) + \left(\frac{1}{2m} (\pi_3)^2 + \frac{k}{2} (\xi^3)^2 \right) + \mathcal{H}_{int}(\xi, \pi) . \qquad \text{(IV.93)}$$

The transverse degrees of freedom are seen to have a renormalised[25] mass and elasticity constant which are given by the same expressions as in (IV.18):

$$\frac{1}{m_\perp} == \frac{1}{m} \frac{u}{\chi} \left(1 + \frac{c^2}{4u^2} \right) ; \; \kappa_\perp = \kappa \frac{u}{\chi} \left(1 + \frac{b^2}{4u^2} \right) , \qquad \text{(IV.94)}$$

where

$$b = \frac{B}{\sqrt{m\kappa}} \; ; \; c = C \sqrt{m\kappa} .$$

The fields $\overline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ induce a sort of magnetic moment interaction along the Z-axis with the same Larmor frequency as before:

$$\widetilde{\mathcal{H}}_{ind}(\xi, \pi) = -\omega'_L \Lambda_3 \; , \qquad \text{(IV.95)}$$

where $\Lambda_3 = \xi^1 \pi_2 - \xi^2 \pi_1$. Acrtually, the condition (IV.91) reduces the $(N = 3)$ case to a sum $(N = 2) \oplus (N = 1)$. The three relevant frequencies of our oscillator are:

$$\omega_3 = \sqrt{k/m} \; ; \; \omega_\perp = \sqrt{k_\perp / m_\perp} \; ; \; \omega'_L = \frac{1}{\chi} \omega_0 (b - c) . \qquad \text{(IV.96)}$$

The spectrum of the quantum Hamiltonian is easily obtained as

$$E(n_{(+)}, n_{(-)}, n_3) = \hbar\omega_{(+)} (n_{(+)} + 1/2) +$$

$$\hbar\omega_{(-)} (n_{(-)} + 1/2) + \hbar\omega_3 (n_3 + 1/2) , \qquad \text{(IV.97)}$$

where $n_{(\pm)}, n_3$ are nonnegative integers. Corresponding eigenvectors are denoted by $|n_{(+)}, n_{(-)}, n_3 >$.

V. SYMMETRIES

For Euclidean configuration space $Q \equiv \mathbf{E}^N$, with metric δ_{ij}, an infinitesimal rotation is written as:

$$\varphi : q^i \to q'^i = q^i + \frac{1}{2} \delta\varepsilon^{\alpha\beta} \left(M_{\alpha\beta} \right)^i{}_j q^j \; , \qquad \text{(V.98)}$$

where $\left(M_{\alpha\beta} \right)^i{}_j = \delta^i{}_\alpha \delta_{\beta j} - \delta^i{}_\beta \delta_{\alpha j}$ are the generators of the rotation group obeying the Lie algebra relations:

$$[M_{\alpha\beta}, M_{\mu\nu}] = -\delta_{\alpha\mu} M_{\beta\nu} + \delta_{\alpha\nu} M_{\beta\mu} - \delta_{\beta\nu} M_{\alpha\mu} + \delta_{\beta\mu} M_{\alpha\nu} . \qquad \text{(V.99)}$$

This induces the push forward in $T^*(Q)$:

$$\begin{aligned} \widetilde{\varphi} &: T^*(Q) \to T^*(Q) : (q^i, p_k) \to (q'^i, p'_k) , \\ q'^i &= q^i + \frac{1}{2} \delta\varepsilon^{\alpha\beta} \left(M_{\alpha\beta} \right)^i{}_j q^j \; ; \\ p'_k &= p_k - \frac{1}{2} \delta\varepsilon^{\alpha\beta} p_l \left(M_{\alpha\beta} \right)^l{}_k . \end{aligned} \qquad \text{(V.100)}$$

In a basis[26] $\{\mathbf{e}_{\alpha\beta}\}$ of $\mathcal{L}(SO(N))$, let $\mathbf{u} = (1/2) \mathbf{e}_{\alpha\beta} u^{\alpha\beta}$ denote a generic element. With $\mathcal{R}(\mathbf{u}) = \exp\left\{ \frac{1}{2} u^{\alpha\beta} M_{\alpha\beta} \right\}$, finite rotations are written as

$$q^i \to q'^i = \mathcal{R}(\mathbf{u})^i{}_j q^j \; ; \; p_k \to p'_k = p_l \mathcal{R}^{-1}(\mathbf{u})^l{}_k . \qquad \text{(V.101)}$$

The vector field $\mathbf{X_u}$ (see appendix **A**) is given by its components:

$$(X_\mathbf{u})^i = \frac{1}{2} u^{\alpha\beta} (M_{\alpha\beta})^i{}_j q^j \; ; \; (X_\mathbf{u})_k = -\frac{1}{2} u^{\alpha\beta} p_l (M_{\alpha\beta})^l{}_k . \qquad \text{(V.102)}$$

It conserves the canonical symplectic potential and two-form:

$$\mathcal{L}_{X_\mathbf{u}} \theta_0 = 0 \; ; \; \mathcal{L}_{X_\mathbf{u}} \omega_0 = 0 .$$

The action is in fact Hamiltonian for the *canonical symplectic structure*. With the notation of appendix **A**, we have

$$\begin{aligned}
&\mathbf{X}_{\mathbf{u}} = \omega_0^\sharp(\mathbf{d}\,\Xi(\mathbf{u}))\ ,\\
&\Xi(\mathbf{u}) = \frac{1}{2}\,u^{\alpha\beta}\,\mathcal{J}^0_{\alpha\beta}(q,p)\ ,\\
&\mathcal{J}^0 : T^*(Q) \to \mathcal{L}^*(SO(N)) : (q,p) \to \frac{1}{2}\mathcal{J}^0_{\alpha\beta}(q,p)\,\mathbf{e}^{\alpha\beta}\,,\\
&\mathcal{J}^0_{\alpha\beta}(q,p) = p_k\,(M_{\alpha\beta})^k{}_j\,q^j\ . \qquad \text{(V.103)}
\end{aligned}$$

In terms of the momenta $\mathcal{J}^0_{\alpha\beta}$, the rotation (V.98) reads

$$\delta q^i = \frac{1}{2}\,\delta\varepsilon^{\alpha\beta}\,\{q^i,\mathcal{J}^0_{\alpha\beta}\}_0\ ;\ \delta p_k = \frac{1}{2}\,\delta\varepsilon^{\alpha\beta}\,\{p_k,\mathcal{J}^0_{\alpha\beta}\}_0\,. \quad \text{(V.104)}$$

The Lie algebra relations **(V.99)** become Poisson brackets:

$$\left\{\mathcal{J}^0_{\alpha\beta},\mathcal{J}^0_{\mu\nu}\right\}_0 = -\,\delta_{\alpha\mu}\,\mathcal{J}^0_{\beta\nu} + \delta_{\alpha\nu}\,\mathcal{J}^0_{\beta\mu} - \delta_{\beta\nu}\,\mathcal{J}^0_{\alpha\mu} + \delta_{\beta\mu}\,\mathcal{J}^0_{\alpha\nu}\,. \quad \text{(V.105)}$$

Naturally, for the modified symplectic structure (III.1), the action (V.100) will be symplectic if, and only if, the magnetic fields obey:

$$\begin{aligned}
F_{kl}(q) &= F_{ij}(\mathcal{R}(\mathbf{u})\,q)\,(\mathcal{R}(\mathbf{u}))^i{}_k\,(\mathcal{R}(\mathbf{u}))^j{}_l\,, && \text{(V.106)}\\
G^{kl}(p) &= (\mathcal{R}^{-1}(\mathbf{u}))^k{}_i\,(\mathcal{R}^{-1}(\mathbf{u}))^l{}_j\,G^{ij}(p\,\mathcal{R}^{-1}(\mathbf{u}))\,. && \text{(V.107)}
\end{aligned}$$

For constant magnetic fields, this holds if $\mathcal{R}(\mathbf{u})$ belongs to the intersection of the isotropy groups of **F** and **G**, which, in three dimensions, is not empty if both magnetic fields are along the same axis. A rotation along this "z-axis" is then symplectic. However, in general it will not be Hamiltonian and there will be no momentum $\mathcal{J}_Z$ such that $\delta q = \{q,\mathcal{J}_Z\}$. Again the discussion simplifies when one of the charges r or e vanishes. If the potentials $\mathbf{A}$ or $\widetilde{\mathbf{A}}$ are invariant under $\mathcal{R}(\mathbf{u})$, then the action is Hamiltonian[27] with momentum defined by the symplectic potentials (III.13) or (III.18) as

$$\langle\mathcal{J}(q,p)|\mathbf{u}\rangle = \langle\theta_{(e,0)}|X_{\mathbf{u}}\rangle \ \text{or} \quad \langle\theta_{(0,r)}|X_{\mathbf{u}}\rangle\ . \qquad \text{(V.108)}$$

Obviously there is always an $SO(N)$ group action on the (ξ,π) coordinates which is Hamiltonian with respect to **(III.1)** and momentum given by:

$$\mathcal{J}_{\alpha\beta}(\xi,\pi) = \pi_k\,(M_{\alpha\beta})^k{}_j\,\xi^j\ . \qquad \text{(V.109)}$$

However, the hamiltonian (IV.2), looking apparently $SO(N)$ symmetric, is explicitely seen not to be so when expressed in the (ξ,π) variables.

VI. FINAL COMMENTS

The symplectic structure in cotangent space, $T^*(Q)\xrightarrow{\kappa}Q$, was modified through the introduction of a closed two-form **F** on T^*Q, which has the geometic meaning of the pull-back of the magnetic field F, a closed two-form on Q: $\mathbf{F} = \kappa^*(F)$. A first caveat warns us that the other closed two-form **G** does not have such an intrinsic interpretation. Indeed, it is obvious that a mere change of coordinates in Q will spoil the form **(III.1)** of ω. This means that our approach must be restricted to configuration spaces with additional properties, which have to be conserved by coordinate changes. The most simple example is a flat linear[28] space $Q = \mathbf{E}^N$, when (III.1) is assumed to hold in linear coordinates. Obviously, a linear change in coordinates will then conserve this particular form. Although the restriction to constant fields **F** and **G** is a severe limitation[29], it allowed us to find explicit Darboux coordinates **(IV.8)** when $N = 2$ and (IV.81) when $N = 3$.

Finally, when $\det\{\mathbf{1} - r\mathbf{G}\,e\mathbf{F}\} = 0$, the closed two-form ω is degenerate with constant rank and defines a pre-symplectic structure on $T^*(Q)$. Its null-foliation decomposes $T^*(Q)$ in disjoint leaves and on the space of leaves, ω projects to a unique symplectic two-form. In two dimensions, the representations of the corresponding quantum algebra in Hilbert space and its reduction in the degeneracy case were studied in [11–14, 18].

APPENDIX A: ESSENTIAL SYMPLECTIC MECHANICS

Let $\{\mathcal{M},\omega\}$ be a symplectic manifold with symplectic structure defined by a two-form ω which is closed, $\mathbf{d}\omega = 0$, and nondegenerate such that the induced mapping $\omega^\flat : T(\mathcal{M}) \to T^*(\mathcal{M}) : \mathbf{X} \to \imath_{\mathbf{X}}\omega$ has an inverse $\omega^\sharp : T^*(\mathcal{M}) \to T(\mathcal{M}) : \alpha \to \omega^\sharp(\alpha)$. The paradigm of a (non-compact) symplectic manifold is a cotangent bundle $T^*(Q)$ of a differential configuration space Q. In a coordinate system $\{q^i\}$ of Q, a cotangent vector may be written as $\alpha_q = p_i\,\mathbf{d}q^i$. This defines coordinates $z \Rightarrow \{q^i,p_k\}$ of points $z \in \mathcal{M} \equiv T^*(Q)$ and an associated holonomic basis $\{\mathbf{d}p_k,\mathbf{d}q^i\}$ of $T^*_z(\mathcal{M})$. The canonical one-form is defined as $\theta_0 \doteq p_i\,\mathbf{d}q^i$. Obviously, the exact two-form $\omega_0 \doteq -\,\mathbf{d}\theta_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i$ is symplectic.

To each observable, which is a differentiable function f on $\{\mathcal{M},\omega\}$, the symplectic structure associates a *Hamiltonian vector field*:

$$\mathbf{X}_f \doteq \omega^\sharp(\mathbf{d}f) \quad \text{or} \quad \imath_{\mathbf{X}_f}\omega = \mathbf{d}f\ . \qquad \text{(A.1)}$$

Such a vector field generates a one-parameter (local) transformation group: $\mathcal{T}_f(t) : \mathcal{M} \to \mathcal{M} : z_0 \to z(t)$, solution of $\mathbf{d}z(t)/\mathbf{d}t = \mathbf{X}_f(z(t))$, $z(0) = z_0$.

In particular, *the* Hamiltonian $\mathcal{H}$ generates the dynamics of the associated mechanical system. With the usual interpretation of time, $\mathbf{X}_{\mathcal{H}}$ is assumed to be complete such that its flux is defined for all $t \in [-\infty,+\infty]$. Transformations, induced by an Hamiltonian vector field $\mathbf{X}_f$, conserve the symplectic structure[30]:

$$\mathcal{T}_f(t)^*\omega = \omega \ \text{or, locally:}\ \mathcal{L}_{\mathbf{X}_f}\omega = 0\,. \qquad \text{(A.2)}$$

More generally, the transformations conserving the symplectic structure form the group $Sympl(\mathcal{M})$ of *symplectomorphisms* or *canonical transformations*. Vector fields obeying $\mathcal{L}_{\mathbf{X}}\omega = 0$, generate canonical transformations and are called *locally Hamiltonian*, since [31] $\mathbf{d}\,\imath_{\mathbf{X}}\omega = 0$ implies that, locally in some $U \subset \mathcal{M}$, there exists a function f such that $\mathbf{d}f_{|U} = (\imath_{\mathbf{X}}\omega)_{|U}$.

The *Darboux theorem* guarantees the existence of local charts $U \subset \mathcal{M}$ with coordinates $\{q^i, p_k\}$ such that, in each U, ω is written as:

$$\omega_{|U} = \mathbf{d}q^i \wedge \mathbf{d}p_i \,. \tag{A.3}$$

In the natural basis $\{\partial/\partial \mathbf{q^i}, \partial/\partial \mathbf{p_k}\}$ of $T_z(\mathcal{M})$, the Hamiltonian vector fields corresponding to f reads

$$\mathbf{X}_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial \mathbf{q}^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial \mathbf{p}_i} \,.$$

The *Poisson bracket* of two observables is defined by: $\{f, g\} \doteq \omega(\mathbf{X}_f, \mathbf{X}_g)$, with the following properties:

$$\begin{aligned} &\{f_1, f_2\} = -\{f_2, f_1\} \\ &\{f_1, g_1 \cdot g_2\} = \{f, g_1\} \cdot g_2 + g_1 \cdot \{f, g_2\} \\ &\{f, \{g_1, g_2\}\} = \{\{f, g_1\}, g_2\} + \{g_1, \{f, g_2\}\} \end{aligned}$$

These properties, relating the pointwise product $g_1 \cdot g_2$ with the bracket $\{f, g\}$, are said to endow the set of differentiable functions on $\mathcal{M}$ with the structure of a *Poisson algebra* $\mathcal{P}(\mathcal{M})$. In a coordinate system (z^A), where $\omega = \frac{1}{2}\omega_{AB}\,\mathbf{d}z^A \wedge \mathbf{d}z^B$, it is given by:

$$\{f, g\} = \frac{\partial f}{\partial z^A}\Lambda^{AB}\frac{\partial g}{\partial z^B} \,, \tag{A.4}$$

where Λ is minus ω^{-1}. In Darboux coordinates it reads:

$$\{f, g\}_0 = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \,. \tag{A.5}$$

The Poisson brackets of the Darboux coordinates themselves are:

$$\{q^i, q^j\}_0 = 0 \,,\ \{q^i, p_l\}_0 = \delta^i{}_l \,,\ \{p_k, q^j\}_0 = -\delta_k{}^j \,,\ \{p_k, p_l\}_0 = 0 \,. \tag{A.6}$$

The dynamical evolution of an observable is given by:

$$\frac{\mathbf{d}f}{\mathbf{d}t} = \overrightarrow{\mathbf{X}}_{\mathcal{H}}(f) = \imath_{\mathbf{X}_{\mathcal{H}}}\mathbf{d}f = \imath_{\mathbf{X}_{\mathcal{H}}}\imath_{\mathbf{X}_f}\omega = \omega(\mathbf{X}_f, \mathbf{X}_{\mathcal{H}}) = \{f, \mathcal{H}\} \,. \tag{A.7}$$

A Lie group G acts as a symmety group on a symplectic manifold $\mathcal{M}$, if there is a group homomorphism $\mathcal{T} : G \to Sympl(\mathcal{M}) : g \to \mathcal{T}(g)$. An infinitesimal action defined by a Lie algebra element $\mathbf{u} \in \mathcal{G}$ is given by the locally Hamiltonian vector field

$$\mathbf{X_u}(z) = \frac{d}{dt}\left(\mathcal{T}(\exp(t\mathbf{u}))z\right)_{|t=0} \,. \tag{A.8}$$

When each $\mathbf{X_u}$ is Hamiltonian, the group action is said to be *almost Hamiltonian* and $\{\mathcal{M}, \omega\}$ is called a *symplectic G-space*. In such a case, a linear map $\Xi : \mathcal{G} \to \mathcal{P}(\mathcal{M}) : \mathbf{u} \to \Xi(\mathbf{u})$ can always be constructed such that $\mathbf{X_u} = \omega^\sharp(\mathbf{d}\Xi(\mathbf{u}))$. When there is a Ξ which is also a Lie algebra homomorphism: $\Xi([\mathbf{u}, \mathbf{v}]) = \{\Xi(\mathbf{u}), \Xi(\mathbf{v})\}$, the group is said to have a *Hamiltonian action* and $\{\mathcal{M}, \omega, \Xi\}$ is called a *Hamiltonian G-space*. Since Ξ is linear in $\mathcal{G}$, it defines a *momentum mapping* $\mathcal{J}$ from $\mathcal{M}$ to the dual $\mathcal{G}^*$ of the Lie algebra defined by: $\langle \mathcal{J}(z)|\mathbf{u}\rangle = \Xi(\mathbf{u}, z)$. When $\mathcal{M}$ is a Hamiltonian G-space, the momentum mapping is equivariant under the action of G on $\mathcal{M}$ and its co-adjoint action on $\mathcal{G}^*$.

In general there may be topological obstructions to such a Lie algebra homomorphism. However, when G acts on Q: $\varphi : G \to Diff(Q) : g \to \varphi(g) : q \to q' = \varphi(g)q$, the action is extended to a symplectic action in $\{\mathcal{M} = T^*(Q), \omega_0\}$: $\widetilde{\varphi} : G \to Sympl(\mathcal{M}) : g \to \widetilde{\varphi}(g) : (q, p) \to (q', p')$, where p' is defined by $p = (\varphi(g))^*_{|q}\,p'$. It follows that $\widetilde{\varphi}(g)^*\theta_0 = \theta_0$; $\widetilde{\varphi}(g)^*\omega_0 = \omega_0$. The infinitesimal action is given by $\mathbf{X_u}(z) = (\mathbf{d}\widetilde{\varphi}(\exp(t\mathbf{u}))z/dt)_{|t=0}$ and $\mathcal{L}_{\mathbf{X_u}}\theta_0 = 0$; $\mathcal{L}_{\mathbf{X_u}}\omega_0 = 0$. From $\omega_0^\flat(\mathbf{X_u}) = \mathbf{d}\langle\theta_0|\mathbf{X_u}\rangle$, it follows that the action is almost Hamiltonian with $\Xi(\mathbf{u}) = \langle\theta_0|\mathbf{X_u}\rangle$. Moreover, since $\langle\theta_0|\mathbf{X}_{[\mathbf{u},\mathbf{v}]}\rangle = \omega_0(\mathbf{X_u}, \mathbf{X_v}) = \{\Xi(\mathbf{u}), \Xi(\mathbf{v})\}$, the action is Hamiltonian and $\{T^*(Q), \omega_0, \Xi\}$ is a Hamiltonian G-space.

APPENDIX B: PRESYMPLECTIC MECHANICS

A manifold $\mathcal{M}_1$, endowed with a closed but degenerate[32] 2-form ω, with constant rank, is said to be presymplectic. The mapping $\omega^\flat$ has a nonvanishing kernel, given by those nonzero vector fields $\mathbf{X}_0$ obeying $\omega^\flat(\mathbf{X}_0) \doteq \imath_{\mathbf{X}_0}\omega = 0$. The fundamental dynamical equation

$$\omega^\flat(\mathbf{X}) = \mathbf{d}\mathcal{H} \,, \tag{B.1}$$

has then a solution if

$$\langle \mathbf{d}\mathcal{H}|\mathbf{X}_0\rangle = 0 \quad ; \forall \mathbf{X}_0 \in \mathcal{K}er(\omega^\flat) \,. \tag{B.2}$$

If this is nowhere satisfied on $\mathcal{M}_1$, the hamiltonian $\mathcal{H}$ does not define any dynamics on $\mathcal{M}_1$. When **(B.2)** is identically satisfied, a particular solution $\mathbf{X}_P$ of **(B.1)** is defined in the entire manifold $\mathcal{M}_1$ and so is the general solution obtained summing the general solution of the homogeneous equation $\imath_{\mathbf{X}_0}\omega = 0$, i.e $\mathbf{X}_G = \mathbf{X}_P + \mathbf{X}_0$, which will contain arbitrary functions. When **(B.2)** is satisfied for some points $z \in \mathcal{M}_1$, we shall asssume they form a submanifold, called the secondary constrained submanifold with injection $\imath_2 : \mathcal{M}_2 \hookrightarrow \mathcal{M}_1$. The particular solution $\mathbf{X}_P$ of **(B.1)** is now defined in $\mathcal{M}_2$ and so is the general solution $\mathbf{X}_G$. Requiring that $\mathbf{X}_G$ conserves the constraints amounts to ask that $\mathbf{X}_G$ is tangent to $\mathcal{M}_2$:

$$\mathbf{X}_G = \imath_{2\star}(\mathbf{X}_2) \,;\ \mathbf{X}_2 \in \Gamma(\mathcal{M}_2, T\mathcal{M}_2) \,. \tag{B.3}$$

Again, when there are no points where this tangency condition is satisfied, **(B.1)** is meaningless. Another possibility is that some of the arbitrary functions in $\mathbf{X}_0$ become determined and the tangency condition is obeyed on the entire $\mathcal{M}_2$. The general solution then still contains some arbitrary functions. Finally it may happen that the conditions **(B.3)** are only satisfied on some $\mathcal{M}_3$ with $\imath_3 : \mathcal{M}_3 \hookrightarrow \mathcal{M}_2$. The story then goes on until one of the first two alternatives are reached.

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[19] See e.g. [8]
[20] Well known in symplectic mechanics, see e.g.[5, 6, 9].
[21] Observe that ${\Phi_k}^\ell = {\delta_k}^\ell - e\mathbf{F}_{kj}\, r\mathbf{G}^{j\ell}$ and ${\Psi^i}_j = {\delta^i}_j - r\mathbf{G}^{i\ell}\, e\mathbf{F}_{\ell j}$ are mutually transposed and that the matrices ${\Psi^k}_j\, r\,\mathbf{G}^{j\ell} = r\,\mathbf{G}^{kj}\,{\Phi_j}^\ell$ and ${\Phi_k}^j\, e\,\mathbf{F}_{j\ell} = e\,\mathbf{F}_{kj}\,{\Psi^j}_\ell$ are antisymmetric.
[22] In the limit $\chi \to 0$, we have $m'\omega_0' = \sqrt{m'\kappa'} \to |B|$.
[23] Recall that with complex variables $q = q^1 + \mathbf{i}q^2$, the diferentials $dq = dq^1 + \mathbf{i}\,dq^2$ and $dq^\dagger = dq^1 - \mathbf{i}\,dq^2$ have local dual vector fields $\{\partial/\partial q = (\partial/\partial q^1 - \mathbf{i}\,\partial/\partial q^2)/2\ ;\ \partial/\partial q^\dagger = (\partial/\partial q^1 + \mathbf{i}\,\partial/\partial q^2)/2$ and similarly for the $p = p_1 + \mathbf{i}\,p_2$ variables.
[24] The $(N = 3)$ case wil only be examined in the nondegenerate case $\chi > 0$.
[25] Due to $\kappa^2 + \kappa'^2 (rC)^2\,(eB)^2 + 2\kappa\kappa'\, rCeB = 1$, the mass and elastic constant of the z degrees of freedom, as expected, are not renormalised.
[26] with dual basis $\{\mathbf{e}^{\alpha\beta}\}$ in $\mathcal{L}^*(SO(N))$.
[27] Exercise 4.2A in [6], defining a (generalized) Poincaré momentum.
[28] Quantum mechanics on a noncommutative shere S^2 and on general noncommutative Riemann surfaces was examined in ([12, 13].
[29] In the case $e = 0$, Darboux coordinates are given by **(III.17)** and in [16] such model was considered with the possibility of having a monopole in p-space!
[30] $\mathcal{T}_f(t)^*$ denotes the pull-back of $\mathcal{T}_f(t)$ and $\mathcal{L}$ is the Lie derivative along $\mathbf{X}_f$.
[31] We use $\mathcal{L}_\mathbf{X} = \mathbf{d}\,\imath_\mathbf{X} + \imath_\mathbf{X}\,\mathbf{d}$ on differential forms.
[32] $\mathcal{M}_1$ is the primary constrained manifold, arising e.g. from a degenerate Lagrangian.