Noncommutative Configuration Space. Classical and Quantum Mechanical Aspects

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In this work we examine noncommutativity of position coordinates in classical symplectic mechanics and its quantisation. In coordinates $\{q_i, p_i\}$ the canonical symplectic two-form is $\omega_0 = dq_i \wedge dp_i$. It is well known in symplectic mechanics [5, 6, 9] that the interaction of a charged particle with a magnetic field can be described in a Hamiltonian formalism without a choice of a potential. This is done by means of a modified symplectic two-form $\omega = \omega_0 - eF$, where $e$ is the charge and the (time-independent) magnetic field $F$ is closed: $dF = 0$. With this symplectic structure, the canonical momentum variables acquire non-vanishing Poisson brackets: $\{p_i, p_j\} = eF_{ij}(q)$. Similarly a closed two-form in $p$-space $G$ may be introduced. Such a dual magnetic field $G$ interacts with the particle’s dual charge $r$. A new modified symplectic two-form $\omega = \omega_0 - eF + rG$ is then defined. Now, both $p$- and $q$-variables will cease to Poisson commute and upon quantisation they become noncommuting operators. In the particular case of a linear phase space $\mathbb{R}^{2N}$, it makes sense to consider constant $F$ and $G$ fields. It is then possible to define, by a linear transformation, global Darboux coordinates: $\{q_i', p_k\} = \delta_{ik}'$. These can then be quantised in the usual way $[\hat{q}_i', \hat{p}_k] = i\hbar \delta_{ik}$. The case of a quadratic potential is examined with some detail when $N$ equals 2 and 3.

Keywords: Noncommutativity; Symplectic mechanics; Quantization

1. Introduction

The idea to consider non vanishing commutation relations between position operators $[x, y] = i\ell^2$, analogous to the canonical commutation relations between position and conjugate momentum $[x, p] = i\hbar$, is ascribed to Heisenberg, who saw there a possibility to introduce a fundamental length $\ell$ which might control the short distance singularities of quantum field theory. However, noncommutativity of coordinates appeared first nonrelativistically in the work of Peierls [2] on the diamagnetism of conduction electrons. In the limit of a strong magnetic field in the $z$-direction, the gap between Landau levels becomes large and, to leading order, one obtains $[x, y] = i\hbar e/cB$. In relativistic quantum mechanics, noncommutativity was first examined in 1947 by Snyder [3] and, in the last five years, inspired by string and brane-theory, many papers on field theory in noncommutative spaces appeared in the physics literature. The apparent unitarity problem related to time-space noncommutativity in field theory was studied and solved in [10]. Also (nonrelativistic) quantum mechanics on noncommutative twodimensional spaces has been examined more thoroughly in the recent years: [11–16]. The above mentioned unitarity problem in quantum physics is also examined in Balachandran et al. [17].

In this work we discuss noncommutativity of configuration space $Q$ in classical mechanics on the cotangent bundle $T^*(Q)$ and its canonical quantisation in the most simple case. In section II we review the classical theory of a non relativistic particle interacting with a time-independent magnetic field $F = 1/2 F_{ij}(q) dq^i \wedge dq^j$; $dF = 0$. This is done in every textbook introducing a potential in a Lagrangian formalism. The Legendre transformation defines then the Hamiltonian and the canonical symplectic two-form $dq^i \wedge dp_i$ implements the corresponding Hamiltonian vector field. We also recall the less well known procedure of avoiding the introduction of a potential using a modified symplectic structure: $\omega = dq^i \wedge dp_i - eF$. The coupling with the charge $e$ hidden in the symplectic structure and does not show up in the Hamiltonian: $H_{0[q, p]} = \delta^{ij} p_i p_j / 2m + \Psi(q)$. In section III, a closed two-form in $p$-space, the dual field: $G = 1/2 G^{ij}(p) dp_i \wedge dp_j$, is added to the symplectic structure $\omega = dq^i \wedge dp_i - eF + rG$, where $r$ is a dual charge.

Such an approach with a modified symplectic structure has been previously considered by Duval and Horvathy [11, 14] emphasizing the $N = 2$-dimensional case in connection with the quantum Hall effect. We should also mention Plyushchay’s interpretation [18] of such a dual charge $r$ which may be hidden in the symplectic structure and does not show up in the Hamiltonian: $H_{0[q, p]} = \delta^{ij} p_i p_j / 2m + \Psi(q)$. In section III, a closed two-form in $p$-space, the dual field: $G = 1/2 G^{ij}(p) dp_i \wedge dp_j$, is added to the symplectic structure $\omega = dq^i \wedge dp_i - eF + rG$, where $r$ is a dual charge.

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basic notions in symplectic geometry and in appendix B we give a brief account of the Gotay-Nester-Hinds algorithm [7] for constrained Hamiltonian systems.

II. NON RELATIVISTIC PARTICLE INTERACTING WITH A TIME-INDEPENDENT MAGNETIC FIELD

A particle of mass $m$ and charge $e$, with potential energy $\mathcal{V}$, moving in a Euclidean configuration space $Q$, with cartesian coordinates $q^i$, interacts with a (time-independent) magnetic field given by a closed two-form $F = \frac{1}{2} F_{jk}(q) dq^j \wedge dq^k$. The dynamics is given by the Laplace equation:

$$m \frac{d^2 q^i}{dt^2} = \delta_{ij} \left( \frac{d F_{kl}(q)}{dq^k} - \frac{\partial \mathcal{V}(q)}{\partial q^l} \right). \quad (II.1)$$

Assuming $Q$ to be Euclidean avoids topological subtleties, so that there exists a global potential one-form $A(q) = A_i(q) dq^i$ such that $F = dA$. A global Lagrangian formalism can then be established with a Lagrangian function on the tangent bundle $\tau : T(Q) \to Q$:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^i \dot{q}^i + e \dot{q}^i A_i(q) - \mathcal{V}(q).$$

The Euler-Lagrange equation is obtained as:

$$0 = \delta q^i \left( \frac{d \mathcal{L}}{dt} - \frac{d}{dt} \frac{d \mathcal{L}}{dq^i} \right) = \frac{\partial \mathcal{V}}{\partial q^i} + e \frac{\partial A_i(q)}{\partial q^i} - m \frac{d}{dt} \left( m \dot{q}^i \dot{q}^j + e A_i(q) \right)$$

$$= -\frac{\partial \mathcal{V}}{\partial q^i} + e q^k \frac{\partial A_k(q)}{\partial q^i} - m \frac{d}{dt} \dot{q}^j$$

$$= -\frac{\partial \mathcal{V}}{\partial q^i} + e F_{ik}(q) q^k - m \dot{q}^i,$$  

(II.2)

and coincides with the Laplace equation (II.1).

The Legendre transform

$$(q^i, \dot{q}^i) \to \left( q^i, p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = m \delta_{ij} \dot{q}^j + e A_j(q) \right),$$

defines the Hamiltonian on the cotangent bundle $T^*Q$:

$$\mathcal{H}_A(q, p) = -\mathcal{L}(q, \dot{q}) + p_i \dot{q}^i =$$

$$\frac{1}{2m} \delta_{ij} (p_k - e A_k(q)) (p_l - e A_l(q)) + \mathcal{V}(q).$$

With the canonical symplectic two-form

$$\omega_0 = dq^i \wedge dp_i,$$  

(II.3)

the Hamiltonian vector field of $\mathcal{H}_A$ is:

$$X_{\mathcal{H}_A} = \frac{\delta_{ij}}{m} (p_j - e A_j) \frac{\partial}{\partial q^i} + \left( \frac{e}{m} \delta_{ij} \frac{\partial A_k(q)}{\partial q^l} (p_l - e A_l) \right) \frac{\partial}{\partial p_i}.$$

Its integral curves are solutions of:

$$\frac{dq^i}{dt} = \frac{\delta_{ij}}{m} (p_j - e A_j), \quad \frac{dp_i}{dt} = \frac{e}{m} \delta_{ij} \frac{\partial A_k(q)}{\partial q^l} (p_l - e A_l) - \frac{\partial \mathcal{V}'}{\partial q^i},$$  

(II.4)

which is again equivalent to (II.1).

If the second de Rham cohomology were not trivial, $H^2_{dR}(Q) \neq 0$, there is no global potential $A$ and a local Lagrangian formalism is needed. This can be done enlarging the configuration space $Q$ to the total space $P$ of a principal $U(1)$ bundle over $Q$ with a connection, given locally by $A[19]$. This can be avoided using a global Hamiltonian formalism[20] in the cotangent bundle $T^*(Q)$ using a modified symplectic two-form:

$$\omega = \omega_0 - eF = dq^i \wedge dp_i + \frac{1}{2} e F_{ij}(q) dq^i \wedge dq^j,$$  

(II.5)

and a "charge-free" Hamiltonian:

$$\mathcal{H}_0(p,q) = \frac{1}{2m} \delta_{ij} p_k p_l + \mathcal{V}(q).$$

The Hamiltonian vector fields corresponding to an observable $f(q,p)$ are now defined relative to $\omega$ as $i_{X_f}^\omega = df$ and given by:

$$X_f = \frac{\delta_{ij}}{m} \frac{\partial}{\partial q^i} \left( \frac{\frac{\partial f}{\partial q^j}}{\frac{\partial f}{\partial p_l}} \right) \frac{\partial}{\partial p_i}.$$  

With the Hamiltonian $\mathcal{H}_0$, the dynamics are again given by the Laplace equation (II.1) in the form:

$$\frac{dq^i}{dt} = \frac{\delta_{ij}}{m} (p_j - e A_j), \quad \frac{dp_i}{dt} = -\delta_{ij} \left( \frac{\partial \mathcal{V}'}{\partial q^i} + \frac{e}{m} F_{ij}(q) \right).$$  

(II.6)
The Poisson brackets, relative to the symplectic structure II.5, are:

\[
\{ f, g \} = \omega (X_f, X_g) = \partial_q f \partial_p g - \partial_q g \partial_p f.
\]

In particular, the coordinates themselves have Poisson brackets:

\[
\{ q^i, q^j \} = 0, \quad \{ q^i, p^j \} = \delta^j_i, \\
\{ p_k, q^j \} = -\delta^j_k, \quad \{ p_k, p_l \} = e F_{k\ell}(q).
\]

(III.8)

Obviously, the meaning of the \( \{ q, p \} \) variables in (II.3) and (II.5) are different. However both formalisms \((\omega_0, \mathcal{H}_0)\) and \((\omega, \mathcal{H})\) lead to the same equations of motion and thus, they must be equivalent. Indeed, in each open set \( U \) homeomorphic to \( \mathbb{R}^6 \), the vanishing \( dF = 0 \) implies the existence of \( A \) such that \( F = dA \) in \( U \) and, locally:

\[
\omega = dq^i \wedge dp_i - \frac{1}{2} e F_{ij}(q) dq^i \wedge dq^j = -d[p_i + eA_i(q)]
\]

Thus there exist local Darboux coordinates:

\[
\xi^i = q^i, \quad \pi_k = p_k + e A_k(q),
\]

(III.9)

such that \( \omega = d\xi^i \wedge d\pi_i \), which is the form (II.3).

The dynamics defined by the Hamiltonian \( \mathcal{H}_0(q, p) = p^2/2m + \Psi(q) \), with symplectic two-form \( \omega \), is equivalent to the dynamics defined by the Hamiltonian \( \mathcal{H}(\xi, \pi) = (\pi - e A(\xi))^2/2m + \Psi(\xi) \) and canonical symplectic structure \( \omega = d\xi^i \wedge d\pi_i \). Equivalence is trivial since both symplectic two-forms are equal, but expressed in different coordinates \( \{ q, p \} \) and \( \{ \xi, \pi \} \), related by (II.9). It seems worthwhile to note that a gauge transformation \( A \to A' = A + \text{grad} \phi \) corresponds to a change of Darboux coordinates

\[
\{ \xi^i, \pi_k \} \to \{ \xi'^i, \pi'_k \} = \{ \pi_k + e \partial_k \phi \},
\]

i.e. a symplectic transformation.

#### III. Noncommutative Coordinates

Let us consider an affine configuration space \( Q = \mathbb{A}^N \) so that points of phase space, identified with \( \mathcal{M} = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \), may be given by linear coordinates \( (q, p) \). Together with the (usual) magnetic field \( F \), we may introduce a (dual) magnetic field \( G = 1/2 G^{ij}(p) dp_k \wedge dp_l \), a closed two-form, \( dG = 0 \), in \( \mathbb{R}^n \) space. Let \( e \) be the usual electric charge and \( r \), a dual charge, which couples the particle with \( F \) and \( G \). Consider the closed two-form:

\[
\omega = \omega_0 - eF + rG
\]

\[
= dq^i \wedge dp_i - \frac{1}{2} e F_{ij}(q) dq^i \wedge dq^j + \frac{1}{2} r G^{ij}(p) dp_k \wedge dp_l.
\]

(III.1)

In matrix notation this two-form (III.1) is represented as:

\[
\omega = \begin{pmatrix} -eF & \mathbf{1} \\ 1 - rG & \mathbf{0} \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 1 & -\Psi & 0 \\ 1 & 0 & -eF & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} eF & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}
\]

(III.2)

where \([21] \Phi = (1 - e FrG), \Psi = (1 - rG e F)\).

The fundamental Hamiltonian equation \( i_\xi \omega = df \), in (A.1), reads:

\[
(X^i - r G^{ij} X_j) dp_i - (X_k - e F_{kl} X^l) dq^k = \frac{\partial f}{\partial q^j} dq^j + \frac{\partial f}{\partial p_i} dp_i.
\]

(III.3)

This can be rewritten as

\[
\frac{\partial f}{\partial p_i} - r G^{ij} \frac{\partial f}{\partial q^j} = \Psi^j X^j, \quad \frac{\partial f}{\partial q^j} - e F_{kl} \frac{\partial f}{\partial p_i} = -\Phi_k X_l.
\]

(III.4)

Obviously, from (III.2) or (III.4), the closed two-form \( \omega \) will be non degenerate, and hence symplectic, if \( \text{det}(\omega) = \text{det}(\Psi) = \text{det}(\Phi) \neq 0 \), so that \( \omega \) has an inverse:

\[
(\omega)^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & eF \end{pmatrix}, \quad (\Psi^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\Phi^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(III.5)

Explicitly:

\[
\omega^\mathcal{A} : df \to \begin{cases} (X_f)^i = \mathcal{A}^- \mathcal{A}^- f \partial p_j - r G^{jk} \partial f / \partial q^k \\ (X_f)_k = -\mathcal{A}^- \mathcal{A}^\mathcal{A} f \partial q^j - e F_{ij} \partial f / \partial p_i \end{cases}
\]

(III.7)

The corresponding Poisson brackets are given by:

\[
\{ f, g \} = \omega(X_f, X_g) = (\partial_q f \partial_p g - \partial_q g \partial_p f) \Lambda \begin{pmatrix} \partial_q g \\ \partial_p g \end{pmatrix}
\]

(III.8)

with the matrix
\[
\{ f, g \} = -\frac{\partial f}{\partial q} (\Psi^{-1} r G) \frac{\partial g}{\partial q} + \frac{\partial f}{\partial p} (\Phi^{-1}) \frac{\partial g}{\partial p} + \frac{\partial f}{\partial q} (\Psi^{-1}) \frac{\partial g}{\partial p} + \frac{\partial f}{\partial p} (\Phi^{-1} e F) \frac{\partial g}{\partial p} .
\]  

(III.10)

In particular, for the coordinates \((q^i, p_k)\), we have:

\[
\{ q^i, q^j \} = - (\Psi^{-1})_{ik} r G^{kj} = -r G^{kj} (\Phi^{-1})_{ik},
\]

\[
\{ q^i, p_k \} = (\Psi^{-1})_{ij},
\]

\[
\{ p_k, q^j \} = - (\Phi^{-1})_{kj},
\]

\[
\{ p_k, p_l \} = (\Phi^{-1})_{kl} e F_{ij} = e F_{ij} (\Psi^{-1})_{ij}.
\]

(III.11)

With \( \mathcal{H}(q, p) = (\delta^k_i p_k p_i / 2m) + \mathcal{V}(q) \), the equations of motion read:

\[
\frac{dq}{dt} = \{ q^i, \mathcal{H} \} = (\Psi^{-1})_{ij} \left( -r G^{kj} \frac{\partial \mathcal{H}}{\partial q^j} \frac{q^j}{m} + \frac{\partial \mathcal{H}}{\partial p_j} \right),
\]

\[
\frac{dp_k}{dt} = \{ p_k, \mathcal{H} \} = (\Phi^{-1})_{ij} \left( -\frac{\partial \mathcal{H}}{\partial q^j} \frac{q^j}{m} + e F_{ij} \frac{\partial \mathcal{H}}{\partial p_j} \right) = (\Phi^{-1})_{ij} \left( -\frac{\partial \mathcal{H}}{\partial q^j} \frac{q^j}{m} + e F_{ij} \frac{p_i}{m} \right) .
\]

(III.12)

The celebrated Darboux theorem guarantees the existence of local coordinates \( (\xi_i, \pi_k) \), such that \( \omega = d\xi^i \wedge d\pi_i \). When one of the charges \((e, r)\) vanishes, such Darboux coordinates are easily obtained using the potential one-forms \( A = A_i(q) dq^i \) and \( \tilde{A} = \tilde{A}(p) dp_k \), such that \( F = dA \) and \( G = d\tilde{A} \).

Indeed, if \( r = 0 \), as in section II, Darboux coordinates are provided by \( \xi^i = q^i; \pi_k = p_k + e A_k(q) \). A modified symplectic potential and two-form are defined by:

\[
\theta = (p_k + e A_k) d\xi^k; \quad \omega = -d\theta .
\]

(III.13)

The Hamiltonian and corresponding equations of motion are:

\[
\mathcal{H}(\xi, \pi) = \frac{1}{2} \delta^k_i (\pi_k - e A_k(\xi)) (\pi_j - e A_j(\xi)) + \mathcal{V}(\xi) ,
\]

(III.14)

\[
\frac{d\xi^i}{dt} = \delta^i_j (\pi_j - e A_j(\xi)) , \quad \frac{d\pi_i}{dt} = e \delta^i_j (\pi_k - e A_k) \frac{\partial A_j}{\partial \xi^j} - \frac{\partial \mathcal{V}}{\partial \xi^j} .
\]

(III.15)

which yields the second order equation in \( \xi_i \), as in (II.1):

\[
\frac{d^2 \xi^i}{dt^2} = \delta^i_j \left( -\frac{\partial \mathcal{V}}{\partial \xi^j} + e F_{ij}(\xi) \frac{d\xi^j}{dt} \right) .
\]

(III.16)

When \( e = 0 \), Darboux variables are

\[
\xi^i = q^i + r \tilde{A}^i(p); \quad \pi_k = p_k ,
\]

and we define

\[
\theta = p_k d(q^k + r \tilde{A}^k) ; \quad \omega = -d\theta .
\]

(III.18)

The Hamiltonian and equations of motion are now given by:

\[
\mathcal{H}(\xi, \pi) = \frac{1}{2} \delta^k_i (\pi_k - e A_k(\xi)) + \mathcal{V}(\xi - r \tilde{A}(\pi)) ,
\]

(III.19)

\[
\frac{d\xi^i}{dt} = \delta^i_j (\pi_j - r \tilde{A}_k \pi_l + \mathcal{V}(\xi - r \tilde{A}(\pi)) , \quad \frac{d\pi_i}{dt} = -\frac{\partial \mathcal{V}}{\partial q^i} .
\]

(III.20)

The second order equation, obeyed by \( \pi (t) \), is given by

\[
\frac{d^2 \pi_i}{dt^2} = \delta^i_j \mathcal{V}(q) \left( -\delta^k_l \pi_k + r G^{jk}(\pi) \frac{d\pi_j}{dt} \right) .
\]

(III.21)

Here the \( q \)-variable is assumed to be solved in terms of \( \pi \) from equation \( \pi_k = -\partial \mathcal{V}(q) / \partial q^k \) and this is possible if \( \det (\delta^k_l \mathcal{V}(q)) \neq 0 \).

In the case of nonzero charges \((e, r)\) and non-constant \( F \) and \( G \) fields, there is no generic formula to define global Darboux coordinates \( (\xi_i, \pi_k) \). However, if the fields \( F \) and \( G \) are constant, the Poisson matrix (III.2) is brought in canonical Darboux form by a linear symplectic orthogonalization procedure, à la Hilbert-Schmidt. In the next section this is done explicitly for \( N = 2 \) and \( N = 3 \). Obviously such a linear transformation: \((q^i, p_k) \Rightarrow (\xi^i, \pi_k)\) is defined up to a linear symplectic map of \( Sp(2n) \). These variables \( (\xi^i, \pi_k) \in \mathbb{R}^{2n} \) can be canonically quantised as operators obeying the commutation relations

\[
[\tilde{\xi}^i, \tilde{\xi}^j] = 0 ; \quad [\tilde{\xi}^i, \tilde{\pi}_k] = i \hbar \delta^i_k ; \quad [\tilde{\pi}_k, \tilde{\pi}_l] = 0.
\]

(III.22)

As von Neumann taught us in [1], they are realised on the Hilbert space of square integrable functions of the variable \( \xi \) as

\[
(\xi^i \Psi)(\xi) = \xi^i \Psi(\xi) ; \quad (\pi_k \Psi)(\xi) = \frac{\hbar}{i} \frac{\partial \Psi(\xi)}{\partial \xi^k} .
\]

(III.23)

The original variables \((q^i, p_k)\) being linear functions of the \( (\xi^i, \pi_k) \) are then also quantised. When \( \det (\Psi) = \det (\Phi) = 0 \), the closed two-form \( \omega \) is singular. When its rank is constant, \( \omega \) defines a presymplectic structure on phase space which we call the primary constraint manifold denoted by \( M_1 \). The consistency of the resulting constrained Hamiltonian system will be examined in the \( N = 2 \) and \( N = 3 \) cases.
IV. EXAMPLES: N = 2 AND 3

In the two examples below, we consider a classical Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2m} \delta_{ij} p_i p_j + \mathcal{Q}(q) .$$  

A complete resolution will be given for a harmonic oscillator potential:

$$\mathcal{Q}(q) \equiv \frac{\kappa}{2} \delta_{ij} q^i q^j .$$

Also of interest is the case of a constant "electric field":

$$\mathcal{Q}(q) = -E_k q^k ,$$

which is exactly soluble and left to the reader.

A. Dynamics in the noncommutative plane

The magnetic fields in two dimensions, are written as:

$$e F_{ij} = B \epsilon_{ij} ; \quad e G^{ij} = C \epsilon^{ij} ,$$

where $B$ and $C$ are pseudoscalars. The closed two-form (III.11) becomes

$$\omega = dq^i \wedge dp_i - B dq^j \wedge dq^2 + C dp_1 \wedge dp_2 .$$

The equation $i_\chi \omega = df$ reads

$$X^i - C e^j X_j = \frac{df}{dp_i} ; \quad X_k - B \epsilon_{kl} X^l = -\frac{df}{dq^l} .$$

Denoting $\chi \equiv (1 + CB)$, the matrices $\Phi$ and $\Psi$ are written as $\Phi = \chi \delta_{ij}$ and $\Psi = \chi \delta_{jl}$. The matrix (III.2) is then invertible if $\chi$ does not vanish.

1. The non degenerate case

Here, we will assume $\chi$ to be strictly positive. The above equation (IV.5) can then be inverted with Hamiltonian vector fields given by:

$$X^i = \chi^{-1} \left( \frac{df}{dp_i} - C e^{ij} \frac{df}{dq^j} \right) ; \quad X_k = -\chi^{-1} \left( \frac{df}{dq^k} - B \epsilon_{kl} \frac{df}{dp^l} \right) .$$

The Poisson brackets (III.11) become:

$$\{ q^i , q^j \} = -C \chi^{-1} \delta^i_j ; \quad \{ q^i , p_j \} = \chi^{-1} \delta^i_j ,$$

$$\{ p_k , q^j \} = -\chi^{-1} \delta^i_j ; \quad \{ p_k , p_j \} = B \chi^{-1} \epsilon_{kl} .$$

Substitution of the Ansatz

$$\xi^i = \alpha q^i + \beta \frac{C}{2} p_k e^{kl} : \pi_k = \gamma \frac{B}{2} q^j e_{jk} + \delta p_k ,$$

in the canonical Poisson brackets, leads to the equations

$$\alpha^2 - \alpha \beta - \frac{CB}{4} \beta^2 = 0 , \quad \delta^2 - \delta \gamma - \frac{CB}{4} \gamma^2 = 0 ,$$

$$\alpha \delta + \frac{CB}{2} (\alpha \gamma + \beta \delta) - \frac{CB}{4} \beta \gamma = \chi .$$

We choose the solution:

$$\alpha = \delta = \sqrt{\alpha} ; \quad \beta = \gamma = \frac{1}{\sqrt{\alpha}} ; \quad u = \frac{1}{2} (1 + \sqrt{\chi}) ,$$

such that (IV.8) reduces to (II.9) when $C = 0$ or to (III.17) in case $B = 0$. The 2-form (III.1) has the canonical Darboux form $\omega = d\xi^i \wedge dp_i$ in the variables

$$\xi^i = \sqrt{\alpha} \left( q^j - \frac{C}{2u} e^{i} p_k \right) ; \quad \pi_k = \sqrt{\alpha} \left( p_k - B \epsilon_{kl} q^l \right) .$$

These have an inverse if, and only if $\chi \neq 0$:

$$\sqrt{\alpha} q^j = \sqrt{\alpha} \left( \xi^i + \frac{C}{2u} \pi_k \right) ; \quad \sqrt{\alpha} p_k = \sqrt{\alpha} \left( \pi_k + B \epsilon_{kl} \xi^l \right) .$$

With the complex variables

$$q = q^1 + i q^2 , \quad p = p_1 + i p_2 ; \quad \xi = \xi^1 + i \xi^2 , \quad \pi = \pi_1 + i \pi_2 ,$$

the above changes of variables are written as:

$$\xi = \sqrt{\alpha} \left( q + i \frac{C}{2u} p \right) ; \quad \pi = \sqrt{\alpha} \left( p + i \frac{B}{2u} q \right) .$$

The inverse transformations are:

$$q = \sqrt{\alpha} / \chi \left( \xi - i \frac{C}{2u} \pi \right) ; \quad p = \sqrt{\alpha} / \chi \left( \pi - i \frac{B}{2u} \xi \right) .$$

The Hamiltonian (IV.2) becomes

$$\mathcal{H} = \frac{1}{2m} \delta^{kl} \pi_k \pi_l + \frac{\kappa}{2} \delta_{ij} \xi^i \xi^j - \omega_i^j \Lambda$$

where $\Lambda$ is angular momentum

$$\Lambda = \frac{1}{2} \left( \epsilon_{ij} \xi^i \left( \delta^j \pi_k - e^{jk} \pi_k \right) - \omega_i^j \Lambda \right)$$

$$= \frac{1}{2} \left( \left( \xi^i \pi_k - e^{jk} \pi_k \right) - \left( \pi_k \xi^j + e^{jk} \xi^j \right) \right) .$$

The "renormalised" mass and elasticity constant are given by:

$$\frac{1}{m} = \frac{1}{2} \frac{u}{\chi} \left( 1 + \frac{c^2}{4u^2} \right) ; \quad \kappa^2 = \frac{u}{\chi} \left( 1 + \frac{b^2}{4u^2} \right) .$$

where

$$b = \frac{B}{\sqrt{m \kappa}} ; \quad c = C \sqrt{m \kappa} .$$

The corresponding frequency $\omega_0^2 = \sqrt{\chi / m}$ is given in terms of the "bare" frequency $\omega_0 = \sqrt{\kappa / m}$ by:

$$\omega_0 = \frac{\omega_0}{2 \chi} \left( (b - c)^2 + 4 \chi \right)^{1/2} .$$
and \( \omega'_L \), the induced Larmor frequency, by:
\[
\omega'_L = \frac{\omega_0}{2\chi} (b - c). \tag{IV.21}
\]
The solution of Hamiltonian’s equations with (IV.16) is standard. With[22]
\[
\text{of standard. With[22]}
\]
\[
m'\omega'_0 = \sqrt{m'k'} = \sqrt{mk} \left( \frac{1 + b'^2}{4a'^2} \right)^{1/2} \tag{IV.22}
\]
reduced variables are introduced by:
\[
Q \equiv (m'\omega'_0)^{1/2} \xi; P \equiv (m'\omega'_0)^{-1/2} \pi. \tag{IV.23}
\]
The original \((q,p)\) are expressed as:
\[
q = \sqrt{u'\chi} (m'\omega'_0)^{-1/2} \left( Q - i \frac{c'}{2a'} p \right), \tag{IV.24}
\]
\[
p = \sqrt{u'\chi} (m'\omega'_0)^{1/2} \left( P - i \frac{b'}{2a'} Q \right), \tag{IV.25}
\]
where
\[
c' = C (m'\omega'_0) = C \sqrt{m'k'}, b' = B/(m'\omega'_0) = B/\sqrt{m'k'}. \tag{IV.26}
\]
The symplectic structure and the Poisson brackets are:
\[
\omega = \frac{1}{2} \left( dQ^\dagger \wedge dP + dQ \wedge dP^\dagger \right),
\]
\[
\{ f, g \} = 2 \left( \frac{\partial f}{\partial Q} \frac{\partial g}{\partial P^\dagger} + \frac{\partial f}{\partial Q^\dagger} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q^\dagger} - \frac{\partial f}{\partial P^\dagger} \frac{\partial g}{\partial Q} \right). \tag{IV.27}
\]
The fundamental nonzero Poisson bracket is
\[
\{ Q, P^\dagger \} = 2. \tag{IV.28}
\]
In these variables, the Hamiltonian (IV.16) reads:
\[
\mathcal{H} = \frac{\omega'_0}{4} \left( (P^\dagger P + PP^\dagger) + (Q^\dagger Q + QQ^\dagger) \right) - \omega'_L \Lambda, \tag{IV.29}
\]
where
\[
\Lambda = \frac{1}{4i} \left( (Q^\dagger P - PQ^\dagger) - (PQ^\dagger + P^\dagger Q) \right). \tag{IV.30}
\]
The corresponding equations of motion are:
\[
\frac{dQ}{dt} = \{ Q, \mathcal{H} \} = 2 \frac{\partial \mathcal{H}}{\partial P^\dagger} = \omega'_0 P - i \omega'_L Q \tag{IV.31}
\]
\[
\frac{dP}{dt} = \{ P, \mathcal{H} \} = -2 \frac{\partial \mathcal{H}}{\partial Q^\dagger} = -\omega'_0 Q - i \omega'_L P \tag{IV.32}
\]
With the shift variables
\[
A_{(+)} = \frac{1}{2} (Q + iP) ; A_{(-)} = \frac{1}{2} (Q^\dagger + iP^\dagger), \tag{IV.33}
\]
the symplectic structure and the Poisson brackets are given by:
\[
\omega = -i \left( dA_{(+)}^\dagger \wedge dA_{(+)} + dA_{(-)}^\dagger \wedge dA_{(-)} \right), \tag{IV.34}
\]
with fundamental nonzero brackets:
\[
\{ A_{(+)} \wedge A_{(-)} \} = -i. \tag{IV.35}
\]
The Hamiltonian, with the (positive !) frequencies
\[
\omega_{(\pm)} = (\omega'_0 \pm \omega'_L), \tag{IV.36}
\]
reads now:
\[
\mathcal{H} = \frac{\omega_{(\pm)}}{2} \left( A_{(+)}^\dagger A_{(+)} + A_{(-)} A_{(-)}^\dagger \right) + \frac{\omega_{(\pm)}}{2} \left( A_{(-)} A_{(+)}^\dagger + A_{(+)}^\dagger A_{(-)} \right). \tag{IV.37}
\]
The corresponding equations of motion and their solutions are given by:
\[
\frac{dA_{(\pm)}}{dt} = \{ A_{(\pm)}, \mathcal{H} \} = -i \frac{\partial \mathcal{H}}{\partial A_{(\pm)}^\dagger} = -i \omega_{(\pm)} A_{(\pm)} \tag{IV.38}
\]
\[
A_{(\pm)}(t) = \exp \left\{ -i \omega_{(\pm)} t \right\} A_{(\pm)}(0). \tag{IV.39}
\]
The relations between variables are given by:

\[ A_{(+)} = \frac{1}{2} (Q + iP) = \frac{\sqrt{u}}{2} \left( (m' \omega_0')^{+1/2} (1 - \frac{b'}{2u}) q + i (m' \omega_0')^{-1/2} (1 + \frac{c'}{2u}) p \right) \]

\[ A_{(-)}^\dagger = \frac{1}{2} (Q - iP) = \frac{\sqrt{u}}{2} \left( (m' \omega_0')^{+1/2} (1 + \frac{b'}{2u}) q - i (m' \omega_0')^{-1/2} (1 - \frac{c'}{2u}) p \right). \] (IV.39)

The inverse transformations are:

\[ q = (m' \omega_0')^{-1/2} \sqrt{u/x} \left( Q - i \frac{c'}{2u} P \right), \]
\[ p = (m' \omega_0')^{+1/2} \sqrt{u/x} \left( P - i \frac{b'}{2u} Q \right), \]
\[ \omega = \frac{1}{2} \left( dq^\dagger \wedge dp + dq \wedge dp^\dagger \right) - \frac{B}{4} \left( dq^\dagger \wedge dq - dq \wedge dq^\dagger \right) + \frac{C}{4} \left( dp^\dagger \wedge dp - dp \wedge dp^\dagger \right). \] (IV.40)

Quantisation is trivial though the substitution of the fundamental Poisson brackets (IV.27),(IV.34) by operator commutators

\[ [Q, P] = 2i \hbar; \quad [A_{(\pm)}, A_{(\pm)}^\dagger] = \hbar. \] (IV.41)

Having kept the initial ordering, the quantum Hamiltonian has eigenvalues:

\[ E(n_{(+)}, n_{(-)}) = \hbar \omega_{n_{(+)}} (n_{(+)}/2 + 1/2) + \hbar \omega_{n_{(-)}} (n_{(-)}/2 + 1/2), \] (IV.42)

where \( n_{(\pm)} \) are nonnegative integers. The corresponding eigenvectors are denoted by \( |n_{(+)}, n_{(-)}\rangle \).

2. The degenerate or constraint case

The condition \( \chi \equiv 1 + BC = 0 \) determines \( \omega \) as a presymplectic structure on \( \mathcal{M} \) and shall be called the primary constraint. Again, the notation is simplified using complex variables[23]. The presymplectic two-form reads

\[ \omega = \frac{1}{2} \left( dq^\dagger \wedge dp + dq \wedge dp^\dagger \right) - \frac{B}{4} \left( dq^\dagger \wedge dq - dq \wedge dq^\dagger \right) + \frac{C}{4} \left( dp^\dagger \wedge dp - dp \wedge dp^\dagger \right). \] (IV.43)

The Hamiltonian (IV.2) becomes

\[ \mathcal{H} = \frac{1}{2m} \frac{p^\dagger p + pp^\dagger}{2} + \frac{\kappa}{2} \frac{q^\dagger q + qq^\dagger}{2}, \] (IV.44)

Writing a vector field as

\[ X = X^1 \partial/\partial q^1 + X_k \partial/\partial p_k = U \partial/\partial q + U^\dagger \partial/\partial q^1 + V \partial/\partial p + V^\dagger \partial/\partial p^1, \]

\[ r_{\mathcal{X}} \omega = \frac{1}{2} \left( (U + iCV) dq^\dagger + (U^\dagger - iCV^\dagger) dq \right) - (V + iBU) dp^\dagger + (V^\dagger - iBU^\dagger) dp \). \] (IV.45)

The homogeneous equation, \( r_{\mathcal{X}} \omega = 0 \) has nontrivial solutions. Indeed, with \( U_0 = Z^1 + iZ^2 \) and \( V_0 = Z_1 + iZ_2 \), equation (IV.45) yields the system:

\[ U_0 + iCV_0 = 0; \quad \text{or} \quad V_0 + iBU_0 = 0, \] (IV.46)
of which the determinant is \( \chi = 1 + BC = 0 \).
The inhomogeneous equation \( \kappa \omega = d \mathcal{H} \), i.e. the Hamiltonian dynamics, reads
\[
U + iCV = 2 \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} ; V + iBU = -2 \frac{\partial \mathcal{H}}{\partial q} = \kappa q . \tag{IV.47}
\]
It will have a solution if
\[
(d \mathcal{H}(Z) = 0 . \tag{IV.48}
\]
This condition, termed secondary constraint, is explicitly given by:
\[
\frac{\partial \mathcal{H}}{\partial p} - iC \frac{\partial \mathcal{H}}{\partial q} = 0 ; \text{or } \frac{\partial \mathcal{H}}{\partial q} - iB \frac{\partial \mathcal{H}}{\partial p} = 0 . \tag{IV.49}
\]
For the Hamiltonian (IV.44) this condition (IV.49) is linear:
\[
\frac{1}{m} p + iC \kappa q = 0 ; \text{or } \kappa q + iB \frac{1}{m} p = 0 . \tag{IV.50}
\]
and defines the secondary constraint manifold \( \mathcal{M}_2 \).
On \( \mathcal{M}_2 \), a particular solution of \( \kappa \omega = d \mathcal{H} \) is given by:
\[
U_p = \frac{p}{m} ; V_p = 0 . \tag{IV.51}
\]
The general solution is given by:
\[
U = \frac{p}{m} + U_0 ; V = V_0 . \tag{IV.52}
\]
where \((U_0, V_0)\) is restricted to obey (IV.46). This vector field, restricted to \( \mathcal{M}_2 \), should conserve the constraints i.e. must be tangent to \( \mathcal{M}_2 \):
\[
0 = \left( \frac{1}{m} dp + iC \kappa dq |X , \tag{IV.53}
\right.
\]
The vector fields \( U \) and \( V \) are completely defined on \( \mathcal{M}_2 \), with ensuing equations of motion:
\[
\frac{dq}{dt} = U = -\frac{i \sqrt{m \kappa}}{1 + m \kappa C^2} \omega_0 q = \frac{1}{1 + m \kappa C^2} \frac{p}{m} , \frac{dp}{dt} = V = -\frac{i \sqrt{m \kappa}}{1 + m \kappa C^2} \omega_0 p = -\frac{m \kappa C^2}{1 + m \kappa C^2} \kappa q . \tag{IV.54}
\]
In terms of the frequency:
\[
\omega_r = \frac{\sqrt{m \kappa}}{1 + m \kappa C^2} \omega_0 = \frac{B \sqrt{m \kappa}}{1 + B^2 / m \kappa} \omega_0 . \tag{IV.55}
\]
the solution is given by
\[
q(t) = \exp \{ i \omega_r t \} q_0 ; p(t) = \exp \{ i \omega_r t \} p_0 . \tag{IV.56}
\]
Obviously, if \( q_0 \) and \( p_0 \) obey the secondary constraints (IV.50), \( q(t) \) and \( p(t) \) obey them at all times.
The same result can be obtained by symplectic reduction, restricting the pre-symplectic two-form (IV.43) to \( \mathcal{M}_2 \):
\[
\omega \mid_{\mathcal{M}_2} = -\frac{1}{2C} (1 + m \kappa C^2)^2 dq \wedge dq . \tag{IV.57}
\]
\[
\{ f, g \} \mid_{\mathcal{M}_2} = \frac{2iC}{(1 + m \kappa C^2)^2} \left( \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} \right) . \tag{IV.58}
\]
The fundamental Poisson bracket is
\[
\{ q, q^\dagger \} \mid_{\mathcal{M}_2} = -\frac{2iC}{(1 + m \kappa C^2)^2} . \tag{IV.59}
\]
The dynamics are given by:
\[
\frac{dq}{dt} = -\frac{2iC}{(1 + m \kappa C^2)^2} \frac{\partial \mathcal{H}_r}{\partial q} . \tag{IV.60}
\]
And, with the reduced Hamiltonian \( \mathcal{H}_r \) given by
\[
\mathcal{H}_r = \left( 1 + m \kappa C^2 \right) \frac{\kappa}{2} q^\dagger q , \tag{IV.61}
\]
this yields equation (IV.56). When \( B > 0 \), hence \( C < 0 \), we define
\[
a = \left( \frac{1 + m \kappa C^2}{2 |C|} \right)^{1/2} q^\dagger , \tag{IV.62}
\]
such that
\[
\{ a, a^\dagger \} = -i ; \mathcal{H}_r = \frac{\omega_r}{2} (a^\dagger a + aa^\dagger) . \tag{IV.63}
\]
Quantisation is again trivial introducing operators \( a \) and \( a^\dagger \), obeying
\[
[a, a^\dagger] = i ; \mathcal{H}_r = \frac{\omega_r}{2} (a^\dagger a + aa^\dagger) . \tag{IV.64}
\]
such that the quantum Hamiltonian
\[
\mathbf{H}_r = \frac{\omega_r}{2} (a^\dagger a + aa^\dagger) . \tag{IV.65}
\]
has eigenvalues:
\[
E(n) = \hbar \omega_r (n + 1/2) . \tag{IV.66}
\]
3. The $\chi \to 0$ limit of (IV A 1).

We need the expansion of

\[
(m' \omega'_0) = (m \omega_0) \times \left( 1 + \frac{b^2}{4u^2} \right) \left( 1 + \frac{c^2}{4u^2} \right)^{-1/2},
\]

in powers of $\varepsilon = \sqrt{\chi}$, where $1 + bc = \varepsilon^2$ and $2u = 1 + \varepsilon$.

\[
(m' \omega'_0) = \frac{m \omega_0}{|c|} \left( 1 + \frac{c^2 - 1}{c^2 + 1} \varepsilon + \cdots \right)
\]

\[
= \frac{1}{|c|} \left( 1 + \frac{c^2 - 1}{c^2 + 1} \varepsilon + \cdots \right)
\]

\[
(m' \omega'_0)^{-1} = \frac{(m \omega_0)^{-1}}{|b|} \left( 1 + \frac{b^2 - 1}{b^2 + 1} \varepsilon + \cdots \right)
\]

\[
= \frac{1}{|b|} \left( 1 + \frac{b^2 - 1}{b^2 + 1} \varepsilon + \cdots \right).
\]

Also, from (IV.25), we obtain

\[
\frac{c'}{2u} = \frac{(m' \omega'_0) C}{2u} = \frac{C}{|c|} \left( 1 - \frac{2}{c^2 + 1} \varepsilon + \cdots \right)
\]

\[
\frac{b'}{2u} = \frac{B}{(m' \omega'_0) 2u} = \frac{B}{|b|} \left( 1 - \frac{2}{b^2 + 1} \varepsilon + \cdots \right). \tag{IV.69}
\]

For definiteness, we assume in the following $B > 0$ and so $C < 0$ in the limit $\varepsilon \to 0$. We obtain

\[
1 - \frac{b'}{2u} = 2 - \frac{2}{1 + b^2} \varepsilon + \cdots ;
\]

\[
1 + \frac{b'}{2u} = 2 - \frac{2}{1 + b^2} \varepsilon + \cdots ;
\]

\[
1 + \frac{c'}{2u} = 2 - \frac{2}{1 + c^2} \varepsilon + \cdots ;
\]

\[
1 - \frac{c'}{2u} = 2 - \frac{2}{1 + c^2} \varepsilon + \cdots. \tag{IV.70}
\]

Also:

\[
\omega'_0 = \frac{\omega_0}{2\varepsilon^2} (b - c) \left( 1 + \frac{2\varepsilon^2}{(b - c)^2} \right), \quad \omega'_L = \frac{\omega_0}{2\varepsilon^2} (b - c), \tag{IV.71}
\]

\[
\omega_{(+)} = \omega'_0 + \omega'_L = -\omega_0 \frac{1 + (m \omega_0)^2 C^2}{(m \omega_0) C} \frac{1}{\varepsilon^2},
\]

\[
\omega_{(-)} = \omega'_0 - \omega'_L = -\omega_0 \frac{(m \omega_0) C}{1 + (m \omega_0)^2 C^2}. \tag{IV.72}
\]

One of the frequencies $\omega_{(+)}$ diverges, while the other $\omega_{(-)}$ tends to $\omega_0$ defined in (IV.55). The relations in (IV.39) yield the initial conditions:

\[
A_{(+)}(0) \approx \sqrt{\frac{|B|}{2}} (1 + b^2)^{-1} \left( q_0 + i \frac{|B|}{(m \omega_0)^2} p_0 \right) (\varepsilon + O(\varepsilon^2))
\]

\[
A_{(-)}(0) \approx \sqrt{\frac{|B|}{2}} \left( q_0 - i \frac{1}{|B|} p_0 \right) (1 + O(\varepsilon^2)). \tag{IV.73}
\]

The solutions (IV.40), in the $\varepsilon \to 0$ limit are then written as

\[
q(t) \approx \sqrt{\frac{2}{|B|}} \left( \frac{1}{|B|} A_{(+)}(0) \exp \{-i \omega_{(+)} t\} + \frac{1}{1 + c^2} A_{(-)}(0) \exp \{i \omega_{(-)} t\} \right)
\]

\[
\approx (1 + b^2)^{-1} \left( q_0 + i \frac{|B|}{(m \omega_0)^2} p_0 \right) \exp \{-i \omega_{(+)} t\}
\]

\[
+ (1 + c^2)^{-1} \left( q_0 - i |B|^{-1} p_0 \right) \exp \{i \omega_{(-)} t\}; \tag{IV.74}
\]

\[
q(t). \text{ Similar considerations hold for } p(t) \text{ in such a way that the solution stays on } M_2.
\]
B. Noncommutative $\mathbb{R}^3$

In $\mathbb{R}^3$, the magnetic fields $\mathbf{F}$ and $\mathbf{G}$ are written in terms of pseudovectors $\mathbf{B} = \{B^k\}$ and $\mathbf{C} = \{C_k\}$ as:

$$ e F_{ij} = e_{ijk} B^k ; \quad r G^{ij} = e^{ijk} C_k . $$

(IV.75)

The closed two-form (III.1) is written as:

$$ \varpi = \mathbf{d} q^l \wedge \mathbf{d} p_i - \frac{1}{2} e_{ijk} B^j \mathbf{d} q^i \wedge \mathbf{d} q^j + \frac{1}{2} e^{klm} C_m \mathbf{d} p_k \wedge \mathbf{d} p_l . $$

(IV.76)

The fundamental equation $t_x \varpi = \mathbf{d} f$ reads

$$ X' - C_k e^{ijk} X_j = \frac{\partial f}{\partial p_i} ; \quad X_k - B^l e_{kli} X^l = - \frac{\partial f}{\partial q^k} . $$

(IV.77)

Defining $\vartheta = \mathbf{C}_1 \mathbf{B} = C_k B^k$ and $\chi = 1 + \vartheta$, this is also written as

$$ \chi X' = \left( \delta^j_1 + B^j C_j \right) \frac{\partial f}{\partial p_j} - C_k e^{ijk} \frac{\partial f}{\partial q^j} , $$

$$ \chi X_k = - \left( \delta^l_k + C_l B^l \right) \frac{\partial f}{\partial q^l} - B^l e_{kli} \frac{\partial f}{\partial p_l} . $$

(IV.78)

The Poisson brackets are given by:

$$ \{ q^1 , q^2 \} = - \chi^{-1} e^{ijk} C_k , $$

$$ \{ q^1 , p_1 \} = \chi^{-1} (\delta^1_1 + B^1 C_1) , $$

$$ \{ p_k , q^1 \} = \chi^{-1} e_{klm} B^m . $$

(IV.80)

The 3 x 3 matrices $\Phi$ and $\Psi$ read:

$$ \Phi^j_1 = \chi \delta^j_1 - C_j B^j ; \quad \Psi^j_1 = \chi \delta^j_1 - B^j C_j , $$

with $\det \Phi = \det \Psi = \chi^2$. Assuming again $\chi \neq 0$, these matrices have inverses:

$$ (\varphi^{-1})^j_1 = \frac{1}{\chi} \left( \delta^j_1 + C_j B^j \right) , \quad (\Psi^{-1})^j_1 = \frac{1}{\chi} \left( \delta^j_1 + B^j C_j \right) . $$

The Hamiltonian vector fields are obtained from (IV.78):

$$ X' = \chi^{-1} \left( \delta^j_1 + B^j C_j \right) \frac{\partial f}{\partial p_j} - C_k e^{ijk} \frac{\partial f}{\partial q^j} , $$

$$ X_k = - \chi^{-1} \left( \delta^l_k + C_l B^l \right) \frac{\partial f}{\partial q^l} - B^l e_{kli} \frac{\partial f}{\partial p_l} . $$

(IV.79)

The formulae (IV.81) are finally written as:

$$ \xi' = \sqrt{u} \left( \xi + \gamma B (C \cdot \xi) + \frac{1}{2 u} e^{ijk} p_j C_k \right) ; $$

$$ \pi_k = \sqrt{u} \left( p_k + \gamma (p \cdot B) C_k + \frac{1}{2 u} e_{klm} B^l \xi^m \right) . $$

(IV.86)

In old fashioned vector notation, this appears as:

$$ \xi = \sqrt{u} \left( \mathbf{q} + \gamma \mathbf{B} \cdot \mathbf{C} + \frac{1}{2 u} \mathbf{p} \times \mathbf{C} \right) ; $$

$$ \pi = \sqrt{u} \left( \mathbf{p} + \gamma (\mathbf{p} \cdot \mathbf{B}) \mathbf{C} + \frac{1}{2 u} \mathbf{B} \times \mathbf{C} \right) . $$

(IV.87)

The inverse formulae of (IV.86) are obtained as:

$$ q' = \frac{\sqrt{u}}{\sqrt{K}} \left( \xi' + \gamma' B' (C_k \xi_k) + \frac{1}{2 u} e^{ijk} \pi_j C_k \right) ; $$

$$ p_k' = \frac{\sqrt{u}}{\sqrt{K}} \left( p_k + \gamma' C_k (p_l B^l ) - \frac{1}{2 u} \psi_{klm} B^l \xi^m \right) . $$

(IV.88)

Or, in vector notation:

$$ \mathbf{q'} = \frac{\sqrt{u}}{\sqrt{K}} \left( \mathbf{q} + \gamma' \mathbf{B} (\mathbf{C} \cdot \mathbf{q}) + \frac{1}{2 u} \mathbf{p} \times \mathbf{C} \right) ; $$

$$ \mathbf{p'} = \frac{\sqrt{u}}{\sqrt{K}} \left( \mathbf{p} + \gamma' \mathbf{C} (\mathbf{p} \cdot \mathbf{B}) - \frac{1}{2 u} \mathbf{B} \times \mathbf{C} \right) . $$

(IV.89)
where
\[ \gamma' = \sqrt{\chi - \sqrt{u}} \] \hspace{1cm} (IV.90)

Again, for sake of simplicity, we consider a configuration space which is Euclidean \( Q = E^3 \) with metric \( < \mathbf{v}, \mathbf{w} > = \delta_{ij} v^i w^j = (\mathbf{v} \cdot \mathbf{w}) \) such that \( v_i = \delta_{ij} v^j \). Substitution of (IV.88) in a Hamiltonian of the form (IV.2), leads to a Hamiltonian quadratic in \( (\xi, \pi) \) and to a system of linear evolution equations. In the case when \( \mathbf{B} \) and \( \mathbf{C} \) point in the same direction:
\[ \mathbf{B} = B \mathbf{e}_z \; ; \; \mathbf{C} = C \mathbf{e}_z \] \hspace{1cm} (IV.91)
a particularly simple Hamiltonian is obtained. Parallel coordinates are defined by \( \xi^1, \pi_3 \) and transverse coordinate vectors
\[ \xi^1 \pm \xi^2 \mathbf{e}_z \] and \( \pi_3 = \pi - \pi_3 \mathbf{e}_z \). Indeed, eq. (IV.88) becomes
\[ q^1 = \frac{\sqrt{u}}{\sqrt{\chi}} \left( \xi^1 + \frac{1}{2u} \pi_2 C \right) \] \hspace{1cm} (IV.92)

The Hamiltonian is:
\[ \mathcal{H}(\xi, \pi) = \left( \frac{1}{2m} (\pi_1)^2 + \frac{k}{\pi} (\xi^1)^2 \right) + \left( \frac{1}{2m} (\pi_2)^2 + \frac{k}{\pi} (\xi^2)^2 \right) + \mathcal{H}_{\text{int}}(\xi, \pi) \] \hspace{1cm} (IV.93)

V. SYMMETRIES

For Euclidean configuration space \( Q = E^3 \), with metric \( \delta_{ij} \), an infinitesimal rotation is written as:
\[ \varphi: q^i \rightarrow q'^i = q^i + \frac{1}{2} \delta^{ij} \mathbf{M}_{ij} \, q^j \] \hspace{1cm} (V.98)

where \( \mathbf{M}_{ij} \) are the generators of the rotation group obeying the Lie algebra relations:
\[ [\mathbf{M}_{ij}, \mathbf{M}_{kl}] = -\delta_{kj} \mathbf{M}_{il} + \delta_{li} \mathbf{M}_{kj} - \delta_{lj} \mathbf{M}_{ik} + \delta_{ik} \mathbf{M}_{lj} \] \hspace{1cm} (V.99)

This induces the push forward in \( T^*(Q) \):
\[ \bar{\varphi}: \bar{T}^*(Q) \rightarrow T^*(Q): (q^i, p_i) \rightarrow (q'^i, \bar{p}^i) \]
\[ q'^i = q^i + \frac{1}{2} \delta^{ij} \mathbf{M}_{ij} \, q^j \] \hspace{1cm} (V.100)

In a basis[26] \( \{ \mathbf{e}_{ab} \} \) of \( L(SO(N)) \), let \( \mathbf{u} = (1/2) \mathbf{e}_{ab} u_{ab} \) denote a generic element. With \( \mathcal{R}(\mathbf{u}) = \exp \left\{ \frac{1}{2} u_{ab} \mathbf{M}_{ab} \right\} \), finite rotations are written as
\[ q^i \rightarrow q'^i = \mathcal{R}(\mathbf{u})^j_i \, q^j \; ; \; p_i \rightarrow \bar{p}^i = p_l \mathcal{R}^{-1}(\mathbf{u})^j_i \] \hspace{1cm} (V.101)

The vector field \( \mathbf{X}_u \) (see appendix A) is given by its components:
\[ (\mathbf{X}_u)^j = \frac{1}{2} u_{ab} \mathbf{M}_{ab} \, q^j ; \quad (X_u)_k = \frac{1}{2} u_{ab} p_l \mathbf{M}_{ab} \, q^j \] \hspace{1cm} (V.102)

It conserves the canonical symplectic potential and two-form:
\[ \mathcal{L}_\mathbf{X}_u q_0 = 0 ; \quad \mathcal{L}_\mathbf{X}_u q_2 = 0. \]
The action is in fact Hamiltonian for the canonical symplectic structure. With the notation of appendix A, we have
\[ X_u = \omega_f^\tau (d \Xi (u)) , \]
\[ \Xi (u) = \frac{1}{2} \alpha^{\alpha \beta} f^0_{\alpha \beta} (q, p) , \]
\[ f^0 : T^* (Q) \rightarrow L^1 (SO (N)) : (q, p) \rightarrow \frac{1}{2} f^0_{\alpha \beta} (q, p) \epsilon^{\alpha \beta} , \]
\[ f^0_{\alpha \beta} (q, p) = p_k (M_{\alpha \beta})^k_j q^j . \] (V.103)

In terms of the momenta \( f^0_{\alpha \beta} \), the rotation (V.98) reads
\[ \delta q^j = \frac{1}{2} \epsilon^{\alpha \beta} \{ q^j , f^0_{\alpha \beta} \} ; \delta p_k = \frac{1}{2} \epsilon^{\alpha \beta} \{ p_k , f^0_{\alpha \beta} \} . \] (V.104)

The Lie algebra relations (V.99) become Poisson brackets:
\[ \{ f^0_{\alpha \beta} , g^0_{\lambda \mu} \} = - \delta_{\alpha \lambda} f^0_{\beta \mu} + \delta_{\beta \mu} f^0_{\alpha \lambda} - \delta_{\alpha \lambda} g^0_{\beta \mu} + \delta_{\beta \mu} g^0_{\alpha \lambda} . \] (V.105)

Naturally, for the modified symplectic structure (III.1), the action (V.100) will be symplectic if, and only if, the magnetic fields obey:
\[ F_3 (q) = F_j (R (u) q) (R (u))^j_k (R (u))^l_j , \] (V.106)
\[ G^l (p) = (R^{-1} (u))^i_j (R^{-1} (u))^j_l G^j_i (p R^{-1} (u)) . \] (V.107)

For constant magnetic fields, this holds if \( R (u) \) belongs to the intersection of the isotropy groups of \( F \) and \( G \), which, in three dimensions, is not empty if both magnetic fields are along the same axis. A rotation along this “z-axis” is then symplectic. However, in general it will not be Hamiltonian and there will be no momentum \( J_z \) such that \( \delta q = \{ q, J_z \} \). Again the discussion simplifies when \( r = e \) vanishes. If the potentials \( A \) or \( A \) are invariant under \( R (u) \), then the action is Hamiltonian[27] with momentum defined by the symplectic potentials (III.13) or (III.18) as
\[ \{ f^0 (q, p) , u \} = \{ \theta (c, q) , X_u \} \text{ or } \{ \theta (0, r) , X_u \} . \] (V.108)

Obviously there is always an \( SO (N) \) group action on the \( (\xi, \pi) \) coordinates which is Hamiltonian with respect to (III.1) and momentum given by:
\[ J_{\alpha \beta} (\xi, \pi) = \pi_k (M_{\alpha \beta})^k_j \xi^j . \] (V.109)

However, the hamiltonian (IV.2), looking apparently \( SO (N) \) symmetric, is explicitly not seen to be so when expressed in the \( (\xi, \pi) \) variables.

VI. FINAL COMMENTS

The symplectic structure in cotangent space, \( T^* (Q) \rightarrow Q \), was modified through the introduction of a closed two-form \( F \) on \( T^* Q \), which has the geometric meaning of the pull-back of the magnetic field \( F \), a closed two-form on \( Q \): \( F = \kappa (F) \). A first caveat warns us that the other closed two-form \( G \) does not have such an intrinsic interpretation. Indeed, it is obvious that a mere change of coordinates in \( Q \) will spoil the form (III.1) of \( \omega \). This means that our approach must be restricted to configuration spaces with additional properties, which have to be conserved by coordinate changes. The most simple example is a flat linear[28] space \( Q = E^N \), when (III.1) is assumed to hold in linear coordinates. Obviously, a linear change in coordinates will then conserve this particular form. Although the restriction to constant fields \( F \) and \( G \) is a severe limitation[29], it allowed us to find explicit Darboux coordinates (IV.8) when \( N = 2 \) and (IV.8I) when \( N = 3 \).

Finally, when \( \det (1 - r G e F) = 0 \), the closed two-form \( \omega \) is degenerate with constant rank and defines a pre-symplectic structure on \( T^* (Q) \). Its null-foliation decomposes \( T^* (Q) \) in disjoint leaves and on the space of leaves, \( \omega \) projects to a unique symplectic two-form. In two dimensions, the representations of the corresponding quantum algebra in Hilbert space and its reduction in the degeneracy case were studied in [11–14, 18].

APPENDIX A: ESSENTIAL SYMPLECTIC MECHANICS

Let \{ \mathcal{M}, \omega \} be a symplectic manifold with symplectic structure defined by a two-form \( \omega \) which is closed, \( d \omega = 0 \), and nondegenerate such that the induced mapping \( \omega^*: T^\ast (\mathcal{M}) \rightarrow T^\ast (\mathcal{M}): X \rightarrow \iota_X \omega \) has an inverse \( \omega^*: T^\ast (\mathcal{M}) \rightarrow T^\ast (\mathcal{M}): \alpha \rightarrow \omega^\ast (\alpha) \). The paradigm of a (non-compact) symplectic manifold is a cotangent bundle \( T^\ast (Q) \) of a differential configuration space \( Q \). In a coordinate system \{ \text{q}' \} of \( Q \), a cotangent vector may be written as \( \alpha_q = q_i dq^i \). This defines coordinates \( z = \{ q_i, p_k \} \) of points \( z \in \mathcal{M} \equiv T^\ast (Q) \) and an associated holonomic basis \{ \text{dp}_k, dq^i \} of \( T^\ast (\mathcal{M}) \). The canonical one-form is defined as \( \theta_b = p_i dq^i \). Obviously, the exact two-form \( \omega_\alpha = -d \theta_b \wedge \text{dp}_k + \text{dp}_k \wedge \text{dp}_k \) is symplectic.

To each observable, which is a differentiable function \( f \) on \{ \mathcal{M}, \omega \}, the symplectic structure associates a Hamiltonian vector field:
\[ X_f = \omega^\ast (df) \text{ or } \iota_{X_f} \omega = df . \] (A.1)

Such a vector field generates a one-parameter (local) transformation group: \( \iota_t f : \mathcal{M} \rightarrow \mathcal{M} : z_0 \rightarrow z(t) \), solution of \( dz(t)/dt = X_f (z(t)) \). \( z(t) = z_0 \).

In particular, the Hamiltonian \( H \) generates the dynamics of the associated mechanical system. With the usual interpretation of time, \( X_H \) is assumed to be complete such that its flux is defined for all \( t \in [-\infty, +\infty] \). Transformations, induced by an Hamiltonian vector field \( X_H \), conserve the symplectic structure[30]:
\[ \iota_{X_H} \omega = \omega \text{ or, locally: } \mathcal{L}_X \omega = 0 . \] (A.2)

More generally, the transformations conserving the symplectic structure form the group \( \text{Symp}(\mathcal{M}) \) of symplectomorphisms or canonical transformations. Vector fields obeying \( \mathcal{L}_X \omega = 0 \), generate canonical transformations and are called locally Hamiltonian, since [31] \( df_k \omega = 0 \) implies that, locally in some \( U \subset \mathcal{M} \), there exists a function \( f \) such that \( df_k (x) = \iota_{X_f} \omega |_U \).
The **Darboux theorem** guarantees the existence of local charts $U \subset M$ with coordinates $\{q', p_k\}$ such that, in each $U$, $\omega$ is written as:

$$\omega_U = \text{d}q' \wedge \text{d}p_i .$$  \hspace{2cm} (A.3)

In the natural basis $\{\partial/\partial q', \partial/\partial p_k\}$ of $T_q(M)$, the Hamiltonian vector fields corresponding to $f$ reads

$$X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q'} - \frac{\partial f}{\partial q'} \frac{\partial}{\partial p_i} .$$

The **Poisson bracket** of two observables is defined by:

$$\{f, g\}_0 = \omega(X_f, X_g) ,$$

(A.4)

where $\Lambda$ is minus $\omega^{-1}$. In Darboux coordinates it reads:

$$\{f, g\}_0 = \frac{\partial f}{\partial q'} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q'} .$$  \hspace{2cm} (A.5)

The Poisson brackets of the Darboux coordinates themselves are:

$$\{q', q'\}_0 = 0 , \quad \{q', p_i\}_0 = \delta_i^l , \quad \{p_k, q'\}_0 = -\delta_k^l , \quad \{p_k, p_l\}_0 = 0 .$$  \hspace{2cm} (A.6)

The dynamical evolution of an observable is given by:

$$\frac{df}{dt} = X_{\omega(f)} = \text{d}X_{\omega} \text{d}f = \text{d}X_{\omega(\omega_f)} = \{f, \omega\} .$$  \hspace{2cm} (A.7)

A Lie group $G$ acts as a symmetery group on a symplectic manifold $M$, if there is a group homomorphism $T : G \rightarrow \text{Sympl}(M) : g \mapsto T(g)$. An infinitesimal action defined by a Lie algebra element $u$ is given by the locally Hamiltonian vector field

$$X_u(z) = \frac{d}{dt} (T(\exp(tu))z)_{t=0} .$$  \hspace{2cm} (A.8)

When each $X_u$ is Hamiltonian, the group action is said to be **almost Hamiltonian** and $\{M, \omega\}$ is called a **symplectic $G$-space**. In such a case, a linear map $\Xi : G \rightarrow \mathcal{P}(M) : u \mapsto \Xi(u)$ can always be constructed such that $X_u = \omega^\flat(\Xi(u))$.

When there is a $\Xi$ which is also a Lie algebra homomorphism: $\Xi([u, v]) = \{\Xi(u), \Xi(v)\}$, the group is said to have a **Hamiltonian action** and $\{M, \omega, \Xi\}$ is called a **Hamiltonian $G$-space**. Since $\Xi$ is linear in $G$, it defines a **momentum mapping** $\mathcal{F}$ from $M$ to the dual $G^*$ of the Lie algebra defined by:

$$\langle \mathcal{F}(z)(u) \rangle = \Xi(z, u) .$$

When $M$ is a Hamiltonian $G$-space, the momentum mapping is equivariant under the action of $G$ on $M$ and its co-adjoint action on $G^*$.

In general there may be topological obstructions to such a Lie algebra homomorphism. However, when $G$ acts on $Q$:

$$\phi : G \rightarrow \text{Diff}(Q) : g \rightarrow \phi(g) : q \rightarrow q' = \phi(g)q ,$$

the action is extended to a symplectic action in $\{ M = T^*(Q), \omega_0 \}$:

$$\phi : G \rightarrow \text{Sympl}(M) : g \rightarrow \phi(g) : (q, p) \rightarrow (q', p') ,$$

where $p'$ is defined by $p = (\phi(g))^\flat g p$. It follows that $\phi(g)^\flat \theta_0 = \theta_0 ; \phi(g)^\flat \omega_0 = \omega_0$. The infinitesimal action is given by $X_u(z) = (\text{d}\phi(exp(tu))z)/dt|_{t=0}$ and $L_{X_u} \theta_0 = 0 ; L_{X_u} \omega_0 = 0$. From $\omega_0^\flat(X_u) = \text{d}u|_{[u, v]_0}$, it follows that the action is almost Hamiltonian with $\Xi(u) = [u, v]_0$. Moreover, since $\langle \theta_0 | X_u(v) \rangle = \omega_0^\flat(X_u, v)\Xi(v)$, the action is Hamiltonian and $\{T^*(Q), \omega_0, \Xi\}$ is a Hamiltonian $G$-space.

**APPENDIX B: PRESYMPLECTIC MECHANICS**

A manifold $M$, endowed with a closed but degenerate[32] 2-form $\omega$, with constant rank, is said to be presymplectic. The mapping $\omega^\flat$ has a nonvanishing kernel, given by those nonzero vector fields $X_0$ obeying $\omega^\flat(X_0) = \iota_{X_0} \omega = 0$. The fundamental dynamical equation

$$\omega^\flat(X) = \text{d}H ,$$  \hspace{2cm} (B.1)

has then a solution if

$$\langle \text{d}H | X_0 \rangle = 0 \quad ; \forall X_0 \in \ker(\omega^\flat) .$$  \hspace{2cm} (B.2)

If this is nowhere satisfied on $M$, the Hamiltonian $H$ does not define any dynamics on $M$. When (B.2) is identically satisfied, a particular solution $X_p$ of (B.1) is defined in the entire manifold $M$ and so is the general solution obtained summing the general solution of the homogeneous equation $\iota_{X_0} \omega = 0, i.e. X_G = X_p + X_0$, which will contain arbitrary functions. When $\text{B.(2)}$ is satisfied for some points $z \in M$, we shall assume they form a submanifold, called the second constrained submanifold with injection $i_2 : M_2 \hookrightarrow M_1$. The particular solution $X_p$ of (B.1) is now defined in $M_2$ and so is the general solution $X_G$. Requiring that $X_G$ conserves the constraints amounts to ask that $X_G$ is tangent to $M_2$:

$$X_G = \iota_{X_2}(X_2) : X_2 \in \Gamma(M_2) .$$  \hspace{2cm} (B.3)

Again, when there are no points where this tangency condition is satisfied, (B.1) is meaningless. Another possibility is that some of the arbitrary functions in $X_0$ become determined and the tangency condition is obeyed on the entire $M_2$. The general solution then still contains some arbitrary functions. Finally it may happen that the conditions (B.3) are only satisfied on some $M_2$ with $i_1 : M_1 \hookrightarrow M_2$. The story then goes on until one of the first two alternatives are reached.
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[19] See e.g. [8]

[20] Well known in symplectic mechanics, see e.g. [5, 6, 9].
[21] Observe that $\Phi^\ell_j = \delta^\ell_j - c F_{jk} r G^{jk}$ and $\Psi^\ell_j = \Theta^\ell_j - r G^{jk} c F_{jk}$ are mutually transposed and that the matrices $\Psi^\ell_j r G^{jk} = r G^{kj} \Phi^\ell_j$ and $\Phi^\ell_j r G^{jk} = e F_{kj} \Psi^\ell_j$ are antisymmetric.
[22] In the limit $\chi \to 0$, we have $m^k \omega_0 = \sqrt{m^k r^k} \to |B|$.
[23] Recall that with complex variables $q = q^1 + i q^2$, the differentials $dq = dq^1 + i dq^2$ and $d\bar{q} = dq^1 - i dq^2$ have local dual vector fields $(\partial/\partial q^1 - i \partial/\partial q^2)/2 ; \partial/\partial q^2 = (\partial/\partial q^1 + i \partial/\partial q^2)/2$ and similarly for the $p = p^1 + i p_2$ variables.
[24] The $(N = 3)$ case will only be examined in the nondegenerate case $\chi > 0$.
[25] Due to $k^1 + k^2 (r C)^2 (eB)^2 + 2 x k^1 r C e B = 1$, the mass and elastic constant of the $\zeta$ degrees of freedom, as expected, are not renormalised.
[26] with dual basis $\{e^{0i}\}$ in $L^\ast (SO(N))$.
[27] Exercise 4.2A in [6], defining a (generalized) Poincaré momentum.
[28] Quantum mechanics on a noncommutative sphere $S^2$ and on general noncommutative Riemann surfaces was examined in ([12, 13]).
[29] In the case $\varepsilon = 0$, Darboux coordinates are given by (III.17) and in [16] such model was considered with the possibility of having a monopole in $p$-space!
[30] $\mathcal{T}_j (t)$ denotes the pull-back of $\mathcal{T}_j (t)$ and $\mathcal{L}$ is the Lie derivative along $\mathcal{X}_j$.
[31] We use $\mathcal{L}_X = d X_t + \mathcal{X}_t d$ on differential forms.
[32] $M_1$ is the primary constrained manifold, arising e.g. from a degenerate Lagrangian.