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Off-Diagonal Mass Generation for Yang-Mills Theories in the Maximal Abelian Gauge

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We investigate a dynamical mass generation mechanism for the off-diagonal gluons and ghosts in $SU(N)$ Yang-Mills theories, quantized in the maximal Abelian gauge. Such a mass can be seen as evidence for the Abelian dominance in that gauge. It originates from the condensation of a mixed gluon-ghost operator of mass dimension two, which lowers the vacuum energy. We construct an effective potential for this operator by a combined use of the local composite operators technique with algebraic renormalization and we discuss the gauge parameter independence of the results. We also show that it is possible to connect the vacuum energy, due to the mass dimension two condensate discussed here, with the non-trivial vacuum energy originating from the condensate $\langle A_\mu^2 \rangle$, which has attracted much attention in the Landau gauge.

Keywords: Gauge theory; BRST quantization; Renormalization group; Nonperturbative effects

I. INTRODUCTION.

An unresolved problem of $SU(N)$ Yang-Mills theory is color confinement. A physical picture that might explain confinement is based on the mechanism of the dual superconductivity [1, 2], according to which the low energy regime of *QCD* should be described by an effective Abelian theory in the presence of magnetic monopoles. These monopoles should condense, giving rise to the formation of flux tubes which confine the chromoelectric charges.

Let us provide a very short overview of the concept of Abelian gauges, which are useful in the search for magnetic monopoles, a crucial ingredient in the dual superconductivity picture.

Abelian gauges

We recall that $SU(N)$ has a $U(1)^{N-1}$ subgroup, consisting of the diagonal generators. In [2], 't Hooft proposed the idea of the Abelian gauges. Consider a quantity $X(x)$, transforming in the adjoint representation of $SU(N)$.

$$X(x) \to U(x)X(x)U^{+}(x) \text{ with } U(x) \in SU(N) \,. \tag{1}$$

The transformation $U(x)$ which diagonalizes $X(x)$ is the one that defines the gauge. If $X(x)$ is already diagonal, then clearly $X(x)$ remains diagonal under the action of the $U(1)^{N-1}$ subgroup. Hence, the gauge is only partially fixed because there is a residual Abelian gauge freedom.

In certain space time points x_i, the eigenvalues of $X(x)$ can coincide, so that $U(x_i)$ becomes singular. These possible singularities give rise to the concept of (Abelian) magnetic monopoles. They have a topological meaning since $\pi_2\left(SU(N)/U(1)^{N-1}\right) \neq 0$ and we refer to [3, 4] for all the necessary details.

The dual superconductor as a mechanism behind confinement

Let us give a simplified picture of the dual superconductor to explain the idea. If the *QCD* vacuum contains monopoles and if these monopoles condense, there will be a dual Meissner effect which squeezes the chromoelectric field into a thin flux tube. This results in a linearly rising potential, $V(r) = \sigma r$, between static charges, as can be guessed from Gauss' law, $\int E dS = cte$ or, since the main contribution is coming from the flux tube, one finds $E\Delta S \approx cte$, hence $V = -\int E dr \approx cte \times r$

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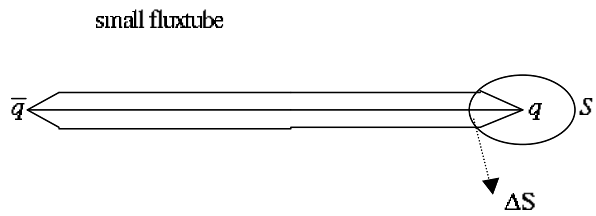

FIG. 1: A chromoelectric flux tube between a static quark-antiquark pair.

An example of an Abelian gauge: the maximal Abelian gauge (MAG)

Let A_μ be the Lie algebra valued connection for the gauge group $SU(N)$, whose generators T^A, satisfying $\left[T^A, T^B\right] = f^{ABC}T^C$, are chosen to be antihermitean and to obey the orthonormality condition $\mathrm{Tr}\left(T^A T^B\right) = -T_F \delta^{AB}$, with $A,B,C = 1,\ldots,\left(N^2-1\right)$. In the case of $SU(N)$, one has $T_F = \frac{1}{2}$. We decompose the gauge field into its off-diagonal and diagonal parts, namely

$$A_\mu = A_\mu^A T^A = A_\mu^a T^a + A_\mu^i T^i, \tag{2}$$

where the indices $i, j, \ldots$ label the $N-1$ generators of the Cartan subalgebra. The remaining $N(N-1)$ off-diagonal generators will be labeled by the indices a, b, ... The field strength decomposes as

$$F_{\mu\nu} = F_{\mu\nu}^A T^A = F_{\mu\nu}^a T^a + F_{\mu\nu}^i T^i , \tag{3}$$

with the off-diagonal and diagonal parts given respectively by

$$\begin{aligned} F_{\mu\nu}^a &= D_\mu^{ab} A_\nu^b - D_\nu^{ab} A_\mu^b + g f^{abc} A_\mu^b A_\nu^c , \\ F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{abi} A_\mu^a A_\nu^b , \end{aligned} \tag{4}$$

where the covariant derivative D_μ^{ab} is defined with respect to the diagonal components A_μ^i

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abi} A_\mu^i \ . \tag{5}$$

For the Yang-Mills action one obtains

$$S_{\mathrm{YM}} = -\frac{1}{4}\int d^4x \left(F_{\mu\nu}^a F^{\mu\nu a} + F_{\mu\nu}^i F^{\mu\nu i}\right) . \tag{6}$$

The maximal Abelian gauge (MAG), introduced in [2–4], corresponds to minimizing the functional

$$\mathcal{R}[A] = \int d^4x \left[A_\mu^a A^{\mu a}\right] \tag{7}$$

One checks that $\mathcal{R}[A]$ does exhibit a residual $U(1)^{N-1}$ invariance.

The MAG can be recast into a differential form

$$D_\mu^{ab} A^{\mu b} = 0 \tag{8}$$

Although we have introduced the MAG here in a functional way, it is worth mentioning that the MAG does correspond to the diagonalization of a certain adjoint operator, see e.g. [5].

The renormalizability in the continuum of the MAG was proven in [6, 7], at the cost of introducing a quartic ghost interaction. The corresponding gauge fixing term turns out to be [6, 7]

$$S_{\mathrm{MAG}} = s\int d^4x \left(\overline{c}^a\left(D_\mu^{ab}A^{b\mu} + \frac{\alpha}{2}b^a\right) - \frac{\alpha}{2} g f^{abi}\overline{c}^a\overline{c}^b c^i - \frac{\alpha}{4} g f^{abc} c^a \overline{c}^b \overline{c}^c\right) , \tag{9}$$

where α is the MAG gauge parameter and s denotes the nilpotent BRST operator, acting as

$$\begin{aligned} sA_\mu^a &= -\left(D_\mu^{ab}c^b + g f^{abc}A_\mu^b c^c + g f^{abi}A_\mu^b c^i\right), & sA_\mu^i &= -\left(\partial_\mu c^i + g f^{iab}A_\mu^a c^b\right) , \\ sc^a &= g f^{abi}c^b c^i + \frac{g}{2} f^{abc}c^b c^c, & sc^i &= \frac{g}{2} f^{iab}c^a c^b, \\ s\overline{c}^a &= b^a , & s\overline{c}^i &= b^i , \\ sb^a &= 0 , & sb^i &= 0 . \end{aligned} \tag{10}$$

Here c^a, c^i are the off-diagonal and the diagonal components of the Faddeev-Popov ghost field, while $\overline{c}^a, b^a$ are the off-diagonal antighost and Lagrange multiplier. We also observe that the BRST transformations (10) have been obtained by their standard form upon projection on the off-diagonal and diagonal components of the fields. We remark that the MAG (9) can be written in the form

$$S_{\mathrm{MAG}} = s\overline{s}\int d^4x \left(\frac{1}{2}A_\mu^a A^{\mu a} - \frac{\alpha}{2}c^a\overline{c}^a\right) , \tag{11}$$

with $\bar{s}$ being the nilpotent anti-BRST transformation, acting as

$$
\begin{aligned}
\bar{s}A^a_\mu &= -\left(D^{ab}_\mu \bar{c}^b + g f^{abc} A^b_\mu \bar{c}^c + g f^{abi} A^b_\mu \bar{c}^i\right), \quad \bar{s}A^i_\mu = -\left(\partial_\mu \bar{c}^i + g f^{iab} A^a_\mu \bar{c}^b\right) , \\
\bar{s}\bar{c}^a &= g f^{abi} \bar{c}^b \bar{c}^i + \frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c, \qquad\qquad \bar{s}\bar{c}^i = \frac{g}{2} f^{iab} \bar{c}^a \bar{c}^b, \\
\bar{s}c^a &= -b^a + g f^{abc} c^b \bar{c}^c + g f^{abi} c^b \bar{c}^i + g f^{abi} \bar{c}^b c^i , \ \bar{s}c^i = -b^i + g f^{ibc} c^b \bar{c}^c , \\
\bar{s}b^a &= -g f^{abc} b^b \bar{c}^c - g f^{abi} b^b \bar{c}^i + g f^{abi} \bar{c}^b b^i \qquad \bar{s}b^i = -g f^{ibc} b^b \bar{c}^c .
\end{aligned}
\tag{12}
$$

It can be checked that s and $\bar{s}$ anticommute.

Expression (9) is easily worked out and yields

$$
\begin{aligned}
S_{\rm MAG} &= \int d^4x \left(b^a \left(D^{ab}_\mu A^{\mu b} + \frac{\alpha}{2} b^a \right) + \bar{c}^a D^{ab}_\mu D^{\mu bc} c^c + g \bar{c}^a f^{abi} \left(D^{bc}_\mu A^{\mu c}\right) c^i + g \bar{c}^a D^{ab}_\mu \left(f^{bcd} A^{\mu c} c^d \right) \right. \\
&- \alpha g f^{abi} b^a \bar{c}^b c^i - g^2 f^{abi} f^{cdi} \bar{c}^a c^d A^b_\mu A^{\mu c} - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c - \frac{\alpha}{4} g^2 f^{abi} f^{cdi} \bar{c}^a \bar{c}^b c^c c^d \\
&- \left. \frac{\alpha}{4} g^2 f^{abc} f^{adi} \bar{c}^b \bar{c}^c c^d c^i - \frac{\alpha}{8} g^2 f^{abc} f^{ade} \bar{c}^b \bar{c}^c c^d c^e \right) .
\end{aligned}
\tag{13}
$$

We note that $\alpha = 0$ does in fact correspond to the "real" MAG condition, given by eq.(8). However, one cannot set $\alpha = 0$ from the beginning since this would lead to a nonrenormalizable gauge. Some of the terms proportional to α would reappear due to radiative corrections, even if $\alpha = 0$. See, for example, [30]. For our purposes, this means that we have to keep α general throughout and leave to the end the analysis of the limit $\alpha \to 0$, to recover condition (8).

In order to have a complete quantization of the theory, one has to fix the residual Abelian gauge freedom by means of a suitable further gauge condition on the diagonal components A^i_μ of the gauge field. A common choice for the Abelian gauge fixing, also adopted in the lattice papers [5, 8], is the Landau gauge, given by

$$
S_{\rm diag} = s \int d^4x \ \bar{c}^i \partial_\mu A^{\mu i} = \int d^4x \ \left(b^i \partial_\mu A^{\mu i} + \bar{c}^i \partial^\mu \left(\partial_\mu c^i + g f^{iab} A^a_\mu c^b \right) \right) ,
\tag{14}
$$

where $\bar{c}^i, b^i$ are the diagonal antighost and Lagrange multiplier.

Abelian dominance

According to the concept of Abelian dominance, the low energy regime of *QCD* can be expressed solely in terms of Abelian degrees of freedom [9]. Lattice confirmations of the Abelian dominance can be found in [10, 11]. To our knowledge, there is no analytic proof of the Abelian dominance. Nevertheless, an argument that can be interpreted as evidence of it, is the fact that the off-diagonal gluons would attain a dynamical mass. At energies below the scale set by this mass, the off-diagonal gluons should decouple, and in this way one should end up with an Abelian theory at low energies.

A lattice study of such an off-diagonal gluon mass reported a value of approximately 1.2GeV [5]. More recently, the off-diagonal gluon propagator was investigated numerically in [8], reporting a similar result.

There have been several efforts to give an analytic description of the mechanism responsible for the dynamical generation of the off-diagonal gluon mass. In [12, 13], a certain ghost condensate was used to construct an effective, off-diagonal mass. However, in [14] it was shown that the obtained mass was a tachyonic one, a fact confirmed later in [15]. Another condensation, namely that of the mixed gluon-ghost operator $(\frac{1}{2}A^a_\mu A^{\mu a} + \alpha \bar{c}^a c^a)$ [39], that could be responsible for the off-diagonal mass, was proposed in [16]. That this operator should condense can be expected on the basis of a close analogy existing between the MAG and the renormalizable nonlinear Curci-Ferrari gauge [17, 18]. In fact, it turns out that the mixed gluon-ghost operator can be introduced also in the Curci-Ferrari gauge. A detailed analysis of its condensation and of the ensuing dynamical mass generation can be found in [19, 20].

Here, we shall report on the results of [21]. It was investigated explicitly if the mass dimension two operator $(\frac{1}{2}A^a_\mu A^{\mu a} + \alpha \bar{c}^a c^a)$ condenses, so that a dynamical off-diagonal mass is generated in the MAG. The pathway we intend to follow is based on previous research in this direction in other gauges. In [22], the local composite operator (LCO) technique was used to construct a renormalizable effective potential for the operator $A^A_\mu A^{\mu A}$ in the Landau gauge. As a consequence of $\left\langle A^A_\mu A^{\mu A} \right\rangle \neq 0$, a dynamical mass parameter is generated [22]. The condensate $\left\langle A^A_\mu A^{\mu A} \right\rangle$ has attracted attention from theoretical [23, 24] as well as from the lattice side [25]. It was shown by means of the algebraic renormalization technique [26] that the LCO formalism for the condensate $\left\langle A_\mu^A A^{\mu A}\right\rangle$ is renormalizable to all orders of perturbation theory [27]. The same formalism was successfully employed to study the condensation of $(\frac{1}{2}A_\mu^A A^{\mu A}+\alpha\overline{c}^A c^A)$ in the Curci-Ferrari gauge [19, 20]. We would like to note that the Landau gauge corresponds to $\alpha=0$. Later on, the condensation of $A_\mu^A A^{\mu A}$ was confirmed in the linear covariant gauges [28, 29], which also possess the Landau gauge as a special case. It was proven formally that the vacuum energy does not depend on the gauge parameter in these gauges. As such, the linear, Curci-Ferrari and Landau gauges are all connected to each other. We managed to connect also the MAG with the Landau gauge, and as such with the linear and Curci-Ferrari gauges [21].

II. RENORMALIZABILITY OF $SU(N)$ YANG-MILLS THEORIES IN THE MAG IN THE PRESENCE OF THE LOCAL COMPOSITE OPERATOR $(\frac{1}{2}A_\mu^a A^{\mu a}+\alpha\overline{c}^a c^a)$

To prove the renormalizability to all orders of perturbation theory, we shall rely on the algebraic renormalization formalism [26]. In order to write down a suitable set of Ward identities, we first introduce external fields $\Omega^{\mu i}$, $\Omega^{\mu a}$, L^i, L^a coupled to the BRST nonlinear variations of the fields, namely

$$S_{\rm ext}=\int d^4x\left(-\Omega^{\mu a}\left(D_\mu^{ab}c^b+gf^{abc}A_\mu^b c^c+gf^{abi}A_\mu^b c^i\right)-\Omega^{\mu i}\left(\partial_\mu c^i+gf^{iab}A_\mu^a c^b\right)+L^a\left(gf^{abi}c^b c^i+\frac{g}{2}f^{abc}c^b c^c\right)+L^i\frac{g}{2}f^{iab}c^a c^b\right)\,, \tag{15}$$

with

$$\begin{aligned} s\Omega^{\mu a} &= s\Omega^{\mu i}=0\,, \\ sL^a &= sL^i=0\,. \end{aligned} \tag{16}$$

Moreover, in order to discuss the renormalizability of the gluon-ghost operator

$$O_{\rm MAG}=\frac{1}{2}A_\mu^a A^{\mu a}+\alpha\overline{c}^a c^a\,, \tag{17}$$

we introduce it in the starting action by means of a BRST doublet of external sources (J,λ)

$$s\lambda=J\,,\quad sJ=0\,, \tag{18}$$

so that

$$\begin{aligned} S_{\rm LCO} &= s\int d^4x\left(\lambda\left(\frac{1}{2}A_\mu^a A^{\mu a}+\alpha\overline{c}^a c^a\right)+\zeta\frac{\lambda J}{2}\right) \\ &= \int d^4x\left(J\left(\frac{1}{2}A_\mu^a A^{\mu a}+\alpha\overline{c}^a c^a\right)+\zeta\frac{J^2}{2}-\alpha\lambda b^a c^a\right. \\ &+ \left.\lambda A^{\mu a}\left(D_\mu^{ab}c^b+gf^{abi}A_\mu^b c^i\right)+\alpha\lambda\overline{c}^a\left(gf^{abi}c^b c^i+\frac{g}{2}f^{abc}c^b c^c\right)\right)\,, \end{aligned} \tag{19}$$

where ζ is the LCO parameter accounting for the divergences present in the vacuum correlator $\langle O_{\rm MAG}(x)O_{\rm MAG}(y)\rangle$, which are proportional to J^2. Therefore, the complete action

$$\Sigma=S_{\rm YM}+S_{\rm MAG}+S_{\rm diag}+S_{\rm ext}+S_{\rm LCO}\,, \tag{20}$$

is BRST invariant

$$s\Sigma=0\,. \tag{21}$$

It is worth mentioning that the mixed gluon-ghost mass operator, defined in eq.(17), is built using off-diagonal components only. As noticed in [16, 31], the operator (17) is also BRST invariant on-shell. We have written down in [21] all the Ward identities, which are sufficient to prove that the most general local counterterm, compatible with the symmetries of the model, can always be reabsorbed by means of multiplicative renormalization. As an interesting by-product, we have been able to establish a relation between the anomalous dimension of the gluon-ghost operator $O_{\rm MAG}$ and other, more elementary, renormalization group functions. Explicitly, it holds to all orders of perturbation theory that

$$\gamma_{O_{\rm MAG}}(g^2)=-2\left(\frac{\beta(g^2)}{2g^2}-\gamma_{c^i}(g^2)\right)\,, \tag{22}$$

where $\beta(g^2)=\mu\frac{\partial g^2}{\partial\mu}$ and $\gamma_{c^i}(g^2)$ denotes the anomalous dimension of the diagonal ghost field.

A. The effective potential via the LCO method.

We present here the main steps in the construction of the effective potential for a local composite operator. A more detailed account of the LCO formalism can be found in [32, 33].

To obtain the effective potential for the condensate $\langle O_{\rm MAG}\rangle$, we set the sources Ω^i_μ, Ω^a_μ, L^a, L^i and λ to zero and consider the renormalized generating functional

$$\begin{aligned}
\exp(-i\mathcal{W}(J)) &= \int [D\varphi]\exp iS(J)\,,\\
S(J) &= S_{\rm YM}+S_{\rm MAG}+S_{\rm diag}+S_{\rm count}+\int d^4x\left(Z_J J\left(\frac{1}{2}\widetilde{Z}_A A^a_\mu A^{\mu a}+Z_\alpha\widetilde{Z}_c\alpha\overline{c}^a c^a\right)+(\zeta+\delta\zeta)\frac{J^2}{2}\right)\,,
\end{aligned} \tag{23}$$

where φ denotes the relevant fields and $S_{\rm count}$ is the usual counterterm contribution, i.e. the part without the composite operator $O_{\rm MAG}$. The quantity $\delta\zeta$ is the counterterm accounting for the divergences proportional to J^2. Using dimensional regularization throughout with the convention that $d=4-\varepsilon$, one has the following identification

$$\zeta_0 J_0^2=\mu^{-\varepsilon}(\zeta+\delta\zeta)J^2\,. \tag{24}$$

where the subscript "0" denotes bare quantities. The functional $\mathcal{W}(J)$ obeys the renormalization group equation (RGE)

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g^2)\frac{\partial}{\partial g^2}+\alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha}-\gamma_{O_{\rm MAG}}(g^2)\int d^4x J\frac{\delta}{\delta J}+\eta(g^2,\zeta)\frac{\partial}{\partial\zeta}\right)\mathcal{W}(J)=0\,, \tag{25}$$

where

$$\begin{aligned}
\gamma_\alpha(g^2) &= \mu\frac{\partial}{\partial\mu}\ln\alpha\,,\\
\eta(g^2,\zeta) &= \mu\frac{\partial}{\partial\mu}\zeta\,.
\end{aligned} \tag{26}$$

Acting with $\mu\frac{\partial}{\partial\mu}$ on eq.(24) and keeping in mind that bare quantities do not depend on the renormalization scale μ, one finds

$$\eta(g^2,\zeta)=2\gamma_{O_{\rm MAG}}(g^2)\zeta+\delta(g^2,\alpha)\,, \tag{27}$$

with

$$\delta(g^2,\alpha)=\left(\varepsilon+2\gamma_{O_{\rm MAG}}(g^2)-\beta(g^2)\frac{\partial}{\partial g^2}-\alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha}\right)\delta\zeta\,. \tag{28}$$

Up to now, the LCO parameter ζ is still an arbitrary coupling. As explained in [32, 33], simply setting $\zeta=0$ would give rise to an inhomogeneous RGE for $\mathcal{W}(J)$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g^2)\frac{\partial}{\partial g^2}+\alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha}-\gamma_{O_{\rm MAG}}(g^2)\int d^4x J\frac{\delta}{\delta J}\right)\mathcal{W}(J)=\delta(g^2,\alpha)\int d^4x\frac{J^2}{2}\,, \tag{29}$$

and a non-linear RGE for the associated effective action Γ for the composite operator $O_{\rm MAG}$. Furthermore, multiplicative renormalizability is lost and by varying the value of $\delta\zeta$, minima of the effective action can change into maxima or can get lost. However, ζ can be made such a function of g^2 and α so that, if g^2 runs according to $\beta(g^2)$ and α according to $\gamma_\alpha(g^2)$, $\zeta(g^2,\alpha)$ will run according to its RGE (27). This is accomplished by setting ζ equal to the solution of the differential equation

$$\left(\beta(g^2)\frac{\partial}{\partial g^2}+\alpha\gamma_\alpha(g^2,\alpha)\frac{\partial}{\partial\alpha}\right)\zeta(g^2,\alpha)=2\gamma_{O_{\rm MAG}}(g^2)\zeta(g^2,\alpha)+\delta(g^2,\alpha)\,. \tag{30}$$

Doing so, $\mathcal{W}(J)$ obeys the homogeneous renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g^2)\frac{\partial}{\partial g^2}+\alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha}-\gamma_{O_{\rm MAG}}(g^2)\int d^4x J\frac{\delta}{\delta J}\right)\mathcal{W}(J)=0\,. \tag{31}$$

To lighten the notation, we will drop the renormalization factors $Z_J,\widetilde{Z}_A$, etc. from now on. One will notice that there are terms quadratic in the source J present in $\mathcal{W}(J)$, obscuring the usual energy interpretation. This can be cured by removing the terms proportional to J^2 in the action to get a generating functional that is linear in the source, a goal easily achieved by inserting the following unity,

$$1=\frac{1}{N}\int[D\sigma]\exp\left[i\int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\sigma}{g}-\mathcal{O}_{\mathrm{MAG}}-\zeta J\right)^2\right)\right], \tag{32}$$

with N the appropriate normalization factor, in eq.(23) to arrive at the Lagrangian

$$\mathcal{L}(A_\mu,\sigma)=-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}-\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i}+\mathcal{L}_{\mathrm{MAG}}+\mathcal{L}_{\mathrm{diag}}-\frac{\sigma^2}{2g^2\zeta}+\frac{1}{g^2\zeta}g\sigma\mathcal{O}_{\mathrm{MAG}}-\frac{1}{2\zeta}\left(\mathcal{O}_{\mathrm{MAG}}\right)^2, \tag{33}$$

while

$$\exp(-i\mathcal{W}(J)) = \int[D\varphi]\exp iS_\sigma(J), \tag{34}$$

$$S_\sigma(J) = \int d^4x\left(\mathcal{L}(A_\mu,\sigma)+J\frac{\sigma}{g}\right). \tag{35}$$

From eqs.(23) and (34), one has the following simple relation

$$\left.\frac{\delta\mathcal{W}(J)}{\delta J}\right|_{J=0}=-\langle\mathcal{O}_{\mathrm{MAG}}\rangle=-\left\langle\frac{\sigma}{g}\right\rangle, \tag{36}$$

meaning that the condensate $\langle\mathcal{O}_{\mathrm{MAG}}\rangle$ is directly related to the expectation value of the field σ, evaluated with the action $S_\sigma=\int d^4x\mathcal{L}(A_\mu,\sigma)$. As it is obvious from eq.(33), $\langle\sigma\rangle\neq0$ is sufficient to have a tree level dynamical mass for the off-diagonal fields. At lowest order (i.e. tree level), one finds

$$m^{\mathrm{off-diag.}}_{\mathrm{gluon}}=\sqrt{\frac{g\sigma}{\zeta_0}}, \qquad m^{\mathrm{off-diag.}}_{\mathrm{ghost}}=\sqrt{\alpha\frac{g\sigma}{\zeta_0}}. \tag{37}$$

Meanwhile, the diagonal degrees of freedom remain massless.

III. GAUGE PARAMETER INDEPENDENCE OF THE VACUUM ENERGY.

We begin this section with a few remarks on the determination of $\zeta(g^2,\alpha)$. From explicit calculations in perturbation theory, it will become clear [40] that the RGE functions showing up in the differential equation (30) look like

$$\begin{aligned}
\beta(g^2) &= -\varepsilon g^2-2\left(\beta_0g^2+\beta_1g^2+\cdots\right),\\
\gamma_{\mathcal{O}_{\mathrm{MAG}}}(g^2) &= \gamma_0(\alpha)g^2+\gamma_1(\alpha)g^4+\cdots,\\
\gamma_\alpha(g^2) &= a_0(\alpha)g^2+a_1(\alpha)g^4+\cdots,\\
\delta(g^2,\alpha) &= \delta_0(\alpha)+\delta_1(\alpha)g^2+\cdots.
\end{aligned} \tag{38}$$

As such, eq.(30) can be solved by expanding $\zeta(g^2,\alpha)$ in a Laurent series in g^2,

$$\zeta(g^2,\alpha)=\frac{\zeta_0(\alpha)}{g^2}+\zeta_1(\alpha)+\zeta_2(\alpha)g^2+\cdots. \tag{39}$$

More precisely, for the first coefficients ζ_0, ζ_1 of the expression (39), one obtains

$$\begin{aligned}
2\beta_0\zeta_0+\alpha a_0\frac{\partial\zeta_0}{\partial\alpha} &= 2\gamma_0\zeta_0+\delta_0,\\
2\beta_1\zeta_0+\alpha a_0\frac{\partial\zeta_1}{\partial\alpha}+\alpha a_1\frac{\partial\zeta_0}{\partial\alpha} &= 2\gamma_0\zeta_1+2\gamma_1\zeta_0+\delta_1.
\end{aligned} \tag{40}$$

Notice that, in order to construct the n-loop effective potential, knowledge of the $(n+1)$-loop RGE functions is needed.

The effective potential calculated with the Lagrangian (33) will explicitly depend on the gauge parameter α. The question arises concerning the vacuum energy E_{vac}, (i.e. the effective potential evaluated at its minimum); will it be independent of the choice of α? Also, as it can be seen from the equations (40), each $\zeta_i(\alpha)$ is determined through a first order differential equation in α. Firstly, one has to solve for $\zeta_0(\alpha)$. This will introduce one arbitrary integration constant C_0. Using the obtained value for $\zeta_0(\alpha)$, one can consequently solve the first order differential equation for $\zeta_1(\alpha)$. This will introduce a second integration constant C_1, etc. In principle, it is possible that these arbitrary constants influence the vacuum energy, which would represent an unpleasant feature. Notice that the differential equations in α for the ζ_i are due to the running of α in eq.(30), encoded in the renormalization group function $\gamma_\alpha(g^2)$. Assume that we would have already shown that E_{vac} does not depend on the choice of α. If we then set $\alpha=\alpha^*$, with α^* a fixed point of the RGE for α at the considered order of perturbation theory, then equation (30) determining ζ simplifies to

$$\beta(g^2)\frac{\partial}{\partial g^2}\zeta(g^2,\alpha^*)=2\gamma_{\mathcal{O}_{\mathrm{MAG}}}(g^2)\zeta(g^2,\alpha^*)+\delta(g^2,\alpha^*), \tag{41}$$

since

$$\left.\gamma_\alpha(g^2)\alpha\right|_{\alpha=\alpha^*}=0. \tag{42}$$

This will lead to simple algebraic equations for the $\zeta_i(\alpha^*)$. Hence, no integration constants will enter the final result for the vacuum energy for $\alpha=\alpha^*$, and since E_{vac} does not depend on α, E_{vac} will never depend on the integration constants, even when calculated for a general α. Hence, we can put them equal to zero from the beginning for simplicity.

Summarizing, two questions remain. Firstly, we should prove that the value of α will not influence the obtained value for $E_{\rm vac}$. Secondly, we should show that there exists a fixed point α^*. We postpone the discussion concerning the second question to the next section, giving a positive answer to the first one. In order to do so, let us reconsider the generating functional (34). We have the following identification, ignoring the overall normalization factors

$$\exp(-i\mathcal{W}(J)) = \int [D\varphi] \exp iS_\sigma(J) = \frac{1}{N}\int [D\varphi D\sigma] \exp i\left[S(J) + \int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\sigma}{g} - O_{\rm MAG} - \zeta J\right)^2\right)\right] , \tag{43}$$

where $S(J)$ and $S_\sigma(J)$ are given respectively by eq.(23), and eq.(35). Obviously,

$$\frac{d}{d\alpha}\frac{1}{N}\int [D\sigma] \exp\left[i\int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\sigma}{g} - O_{\rm MAG} - \zeta J\right)^2\right)\right] = \frac{d}{d\alpha}1 = 0 , \tag{44}$$

so that

$$\frac{d\mathcal{W}(J)}{d\alpha} = -\left\langle s\int d^4x\bar{s}\left(\frac{1}{2}c^a\overline{c}^a\right)\right\rangle\Big|_{J=0} + \text{terms} \propto J , \tag{45}$$

which follows directly from

$$\frac{dS(J)}{d\alpha} = s\bar{s}\int d^4x\left(\frac{1}{2}c^a\overline{c}^a\right) + \text{terms} \propto J . \tag{46}$$

The terms proportional to the source J are originating from the term $\frac{1}{2}\zeta J^2$ present in eq.(23).

We see that the first term in the right hand side of (46) is an exact BRST variation. As such, its vacuum expectation value vanishes. This is the usual argument to prove the gauge parameter independence in the BRST framework [26]. Note that no local operator $\hat{O}$, with $s\hat{O} = O_{\rm MAG}$, exists. Furthermore, extending the action of the BRST transformation on the σ-field by

$$s\sigma = gsO_{\rm MAG} =$$

$$-A^{\mu a}D^{ab}_\mu c^b + \alpha b^a c^a - \alpha g f^{abi}\overline{c}^a\overline{c}^b c^i - \frac{\alpha}{2}g f^{abc}\overline{c}^a c^b c^c \tag{47}$$

one can easily check that

$$s\int d^4x \mathcal{L}(A_\mu, \sigma) = 0 , \tag{48}$$

so that we have a BRST invariant σ-action. Thus, when we consider the vacuum, corresponding to $J = 0$, only the BRST exact term in eq.(45) survives. The effective action Γ is related to $\mathcal{W}(J)$ through a Legendre transformation $\Gamma\left(\frac{\sigma}{g}\right) = -\mathcal{W}(J) - \int d^4y J(y)\frac{\sigma(y)}{g}$, while the effective potential $V(\sigma)$ is defined as

$$-V(\sigma)\int d^4x = \Gamma\left(\frac{\sigma}{g}\right) . \tag{49}$$

If $\sigma_{\rm min}$ is the solution of $\frac{dV(\sigma)}{d\sigma} = 0$, then it follows from

$$\frac{\delta}{\delta\left(\frac{\sigma}{g}\right)}\Gamma = -J, \tag{50}$$

that

$$\sigma = \sigma_{\rm min} \Rightarrow J = 0 , \tag{51}$$

and hence,

$$\frac{d}{d\alpha}V(\sigma)\Big|_{\sigma=\sigma_{\rm min}}\int d^4x = \frac{d}{d\alpha}\mathcal{W}(J)\Big|_{J=0} , \tag{52}$$

or, due to eq.(45),

$$\frac{d}{d\alpha}V(\sigma)\Big|_{\sigma=\sigma_{\rm min}} = 0 . \tag{53}$$

We conclude that the vacuum energy $E_{\rm vac}$ should be independent from the gauge parameter α.

A completely analogous derivation was performed in the case of the linear gauge [29]. Nevertheless, in spite of the previous argument, explicit results in that case showed that $E_{\rm vac}$ did depend on α. In [29] it was argued that this apparent disagreement was due to a mixing of different orders of perturbation theory. We explain this with a simple example. A key argument in the previous analysis is that the source $J = 0$ vanishes at the end of the calculations. In practice, $J = 0$ is achieved by solving the gap equation $\frac{dV}{d\sigma} = 0$. Perturbation theory corresponds to a power series expansion in the coupling constant. The derivative of the effective potential with respect to σ will hence look like

$$\left(v_0 + v_1 g^2 + O(g^4)\right)\sigma , \tag{54}$$

where we assume that we work up to order g^2. The corresponding gap equation reads

$$v_0 + v_1 g^2 + O(g^4) = 0 . \tag{55}$$

Due to eqs.(49) and (50), one also has

$$J = g\left(v_0 + v_1 g^2 + O(g^4)\right)\sigma \, . \tag{56}$$

Imposing the gap equation (55) leads to

$$J = g\left(0 + O(g^4)\right)\sigma \, . \tag{57}$$

However, as it can be immediately checked from expression (43), there are several terms proportional to J in the right-hand side of eq.(45). For instance, one of them is given by $\frac{\partial\zeta}{\partial\alpha}J^2$. Since

$$\frac{\partial\zeta}{\partial\alpha} = \frac{\partial\zeta_0}{\partial\alpha}\frac{1}{g^2} + \frac{\partial\zeta_1}{\partial\alpha} + O(g^2) \, , \tag{58}$$

we find

$$\frac{\partial\zeta}{\partial\alpha}J^2 = \left(\frac{\partial\zeta_0}{\partial\alpha}v_0^2 + \left(\frac{\partial\zeta_0}{\partial\alpha}2v_0v_1 + \frac{\partial\zeta_1}{\partial\alpha}v_0^2\right)g^2 + O(g^4)\right)\sigma^2 \, . \tag{59}$$

Squaring the gap equation (55),

$$v_0^2 + 2v_1v_0g^2 + O(g^4) = 0 \, , \tag{60}$$

leads to

$$\frac{\partial\zeta}{\partial\alpha}J^2 = \left(\frac{\partial\zeta_1}{\partial\alpha}v_0^2 g^2 + O(g^4)\right)\sigma^2 \, . \tag{61}$$

We see that, if one consistently works to the first order, terms such as $\frac{\partial\zeta}{\partial\alpha}J^2$ do not equal zero, although $J = 0$ to that order. Terms like those on the right-hand side of eq.(61) are canceled by terms which are formally of higher order, requiring thus a mixing of different orders of perturbation theory. Of course, this problem would not have occurred if we were be able to compute the effective potential up to infinite order. We proposed a modification of the LCO formalism suitable circumventing this problem and obtaining a well defined gauge independent vacuum energy E_{vac}, without the need of working at infinite order [29]. Instead of the action (23), let us consider the following action

$$\widetilde{S}(\widetilde{J}) \;=\; S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}} + \int d^4x \left[\widetilde{J}\mathcal{F}(g^2,\alpha)\mathcal{O}_{\text{MAG}} + \frac{\zeta}{2}\mathcal{F}^2(g^2,\alpha)\widetilde{J}^2\right] , \tag{62}$$

where, for the moment, $\mathcal{F}(g^2,\alpha)$ is an arbitrary function of α of the form

$$\mathcal{F}(g^2,\alpha) = 1 + f_0(\alpha)g^2 + f_1(\alpha)g^4 + O(g^6) \, , \tag{63}$$

and $\widetilde{J}$ is now the source. The generating functional becomes

$$\exp(-i\widetilde{\mathcal{W}}(\widetilde{J})) = \int [D\phi] \exp i\widetilde{S}(\widetilde{J}) \, . \tag{64}$$

Taking the functional derivative of $\widetilde{\mathcal{W}}(\widetilde{J})$ with respect to $\widetilde{J}$, we obtain

$$\left.\frac{\delta\widetilde{\mathcal{W}}(\widetilde{J})}{\delta\widetilde{J}}\right|_{\widetilde{J}=0} = -\mathcal{F}(g^2,\alpha)\left\langle\mathcal{O}_{\text{MAG}}\right\rangle \, . \tag{65}$$

Once more, we insert unity via

$$1 = \frac{1}{N}\int [D\widetilde{\sigma}] \exp\left[i\int d^4x \left(-\frac{1}{2\zeta}\left(\frac{\widetilde{\sigma}}{g\mathcal{F}(g^2,\alpha)} - \mathcal{O}_{\text{MAG}} - \zeta\widetilde{J}\mathcal{F}(g^2,\alpha)\right)^2\right)\right] , \tag{66}$$

to arrive at the following Lagrangian

$$\widetilde{\mathcal{L}}(A_\mu,\widetilde{\sigma}) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} + \mathcal{L}_{\text{MAG}} + \mathcal{L}_{\text{diag}} - \frac{\widetilde{\sigma}^2}{2g^2\mathcal{F}^2(g^2,\alpha)\zeta} + \frac{1}{g^2\mathcal{F}(g^2,\alpha)\zeta}g\widetilde{\sigma}\mathcal{O}_{\text{MAG}} - \frac{1}{2\zeta}\left(\mathcal{O}_{\text{MAG}}\right)^2 . \tag{67}$$

From the generating functional

$$\exp(-i\widetilde{\mathcal{W}}(\widetilde{J})) \;=\; \int [D\phi] \exp iS_{\widetilde{\sigma}}(\widetilde{J}) \, , \tag{68}$$

$$S_{\widetilde{\sigma}}(\widetilde{J}) \;=\; \int d^4x \left(\mathcal{L}(A_\mu,\widetilde{\sigma}) + \widetilde{J}\frac{\widetilde{\sigma}}{g}\right) . \tag{69}$$

it follows that

$$\left.\frac{\delta\widetilde{\mathcal{W}}(\widetilde{J})}{\delta\widetilde{J}}\right|_{\widetilde{J}=0} = -\left\langle\frac{\widetilde{\sigma}}{g}\right\rangle \Rightarrow \left\langle\widetilde{\sigma}\right\rangle = g\mathcal{F}(g^2,\alpha)\left\langle\mathcal{O}_{\text{MAG}}\right\rangle \, , \tag{70}$$

The renormalizability of the action (35) implies that the action (69) will be renormalizable too. Notice indeed that both actions are connected through the transformation

$$\widetilde{J} = \frac{J}{\mathcal{F}(g^2,\alpha)} . \tag{71}$$

The tree level off-diagonal masses are now provided by

$$m_{\rm gluon}^{\rm off-diag.} = \sqrt{\frac{g\widetilde{\sigma}}{\zeta_0}} , \qquad m_{\rm ghost}^{\rm off-diag.} = \sqrt{\alpha\frac{g\widetilde{\sigma}}{\zeta_0}} , \tag{72}$$

while the vacuum configuration is determined by solving the gap equation

$$\frac{d\widetilde{V}(\widetilde{\sigma})}{d\widetilde{\sigma}} = 0 , \tag{73}$$

with $\widetilde{V}(\widetilde{\sigma})$ the effective potential. Minimizing $\widetilde{V}(\widetilde{\sigma})$ will lead to a vacuum energy $E_{\rm vac}(\alpha)$ which will depend on α and the hitherto undetermined functions $f_i(\alpha)$ [41]. We will determine those functions $f_i(\alpha)$ by requiring that $E_{\rm vac}(\alpha)$ is α-independent. More precisely, one has

$$\frac{dE_{\rm vac}}{d\alpha} = 0 \Rightarrow \text{first order differential equations in } \alpha \text{ for } f_i(\alpha) . \tag{74}$$

Of course, in order to be able to determine the $f_i(\alpha)$, we need an initial value for the vacuum energy $E_{\rm vac}$. This corresponds to initial conditions for the $f_i(\alpha)$. In the case of the linear gauges, to fix the initial condition we employed the Landau gauge [29], a choice which would also be possible in case of the Curci-Ferrari gauges, since the Landau gauge belongs to these classes of gauges. This choice of the Landau gauge can be motivated by observing that the integrated operator $\int d^4x A_\mu^A A^{\mu A}$ has a gauge invariant meaning in the Landau gauge, due to the transversality condition $\partial_\mu A^{\mu A} = 0$, namely

$$(VT)^{-1} \min_{U\varepsilon SU(N)} \int d^4x \left[\left(A_\mu^A\right)^U \left(A^{\mu A}\right)^U\right]$$

$$= \int d^4x (A_\mu^A A^{\mu A}) \quad \text{in the Landau gauge} , \tag{75}$$

with the operator on the left hand side of eq.(75) being gauge invariant. Moreover, the Landau gauge is also an all-order fixed point of the RGE for the gauge parameter in case of the linear and Curci-Ferrari gauges. At first glance, it could seem that it is not possible anymore to make use of the Landau gauge as initial condition in the case of the MAG, since the Landau gauge does not belong to the class of gauges we are currently considering. Fortunately, we shall be able to prove that we can use the Landau gauge as initial condition for the MAG too. This will be the content of the next section.

Before turning our attention to this task, it is worth noticing that, if one would work up to infinite order, the expressions (62) and (69) can be transformed *exactly* into those of (23), respectively (35) by means of eq.(71) and its associated transformation

$$\widetilde{\sigma} = \mathcal{F}(g^2,\alpha)\sigma , \tag{76}$$

so that the effective potentials $\widetilde{V}(\widetilde{\sigma})$ and $V(\sigma)$ are *exactly* the same at infinite order, and as such will give rise to the same, gauge parameter independent, vacuum energy.

IV. INTERPOLATING BETWEEN THE MAG AND THE LANDAU GAUGE

In this section we shall introduce a generalized renormalizable gauge which interpolates between the MAG and the Landau gauge. This will provide a connection between these two gauges, allowing us to use the Landau gauge as initial condition. An example of such a generalized gauge, interpolating between the Landau and the Coulomb gauge was already presented in [34]. Moreover, we must realize that in the present case, we must also interpolate between the composite operator $\frac{1}{2}A_\mu^A A^{\mu A}$ of the Landau gauge and the gluon-ghost operator $O_{\rm MAG}$ of the MAG. Although this seems to be a highly complicated assignment, there is an elegant way to treat it.

Consider again the $SU(N)$ Yang-Mills action with the MAG gauge fixing (11). For the residual Abelian gauge freedom, we impose

$$\begin{aligned} S'_{\rm diag} = \int d^4x \Big(& b^i\partial_\mu A^{\mu i} + \overline{c}^i\partial^2 c^i + \overline{c}^i\partial_\mu \left(g f^{iab} A^{\mu a} c^b\right) \\ & + \kappa g f^{iab} A_\mu^a \left(\partial_\mu c^i\right)\overline{c}^b + \kappa g^2 f^{iab} f^{icd} \overline{c}^a c^d A_\mu^b A^{\mu c} \\ & - \kappa g f^{iab} A_\mu^i A^{\mu a}(b^b - g f^{jbc}\overline{c}^c c^j) + \kappa g f^{iab} A^{\mu i}(D_\mu^{ac} c^c)\overline{c}^b \\ & + \kappa g^2 f^{abi} f^{acd} A_\mu^i A^{\mu c} c^d \overline{c}^b \Big) , \end{aligned} \tag{77}$$

where κ is an additional gauge parameter. The gauge fixing (77) can be rewritten as a BRST exact expression

$$S'_{\rm diag} = \int d^4x \left[(1-\kappa)\, s\left(\overline{c}^i\partial_\mu A^{\mu i}\right) + \kappa s\overline{s}\left(\frac{1}{2}A_\mu^i A^{\mu i}\right)\right] \tag{78}$$

Next, we will introduce the following generalized mass dimension two operator,

$$O = \frac{1}{2}A_\mu^a A^{\mu a} + \frac{\kappa}{2}A_\mu^i A^{\mu i} + \alpha\overline{c}^a c^a , \tag{79}$$

by means of

$$S'_{\rm LCO} = s\int d^4x \left(\lambda O + \zeta\frac{\lambda J}{2}\right)$$

$$= \int d^4x \left(JO + \zeta\frac{J^2}{2} - \alpha\lambda b^a c^a + \lambda A^{\mu a} D_\mu^{ab} c^b\right.$$

$$+\alpha\lambda\overline{c}^a\left(gf^{abi}c^bc^i+\frac{g}{2}f^{abc}c^bc^c\right)$$

$$-\kappa\lambda c^i\partial_\mu A^{\mu i}+\kappa g f^{iab}\lambda A^a_\mu A^{\mu i}c^b\Big) \ , \tag{80}$$

with (J,λ) a BRST doublet of external sources,

$$s\lambda = J \ , \quad sJ=0 \ . \tag{81}$$

As in the case of the gluon-ghost operator (17), the generalized operator of eq.(79) turns out to be BRST invariant on-shell.

Let us take a closer look at the action

$$\Sigma' = S_{\rm YM}+S_{\rm MAG}+S'_{\rm diag}+S'_{\rm LCO}+S_{\rm ext} \ . \tag{82}$$

The external source part of the action, $S_{\rm ext}$, is the same as given in eq.(15).

Also, it can be noticed that, for $\kappa\to 0$, the generalized local composite operator O of eq.(79) reduces to the composite operator $O_{\rm MAG}$ of the MAG, while the diagonal gauge fixing (78) reduces to the Abelian Landau gauge (14). Said otherwise, for $\kappa\to 0$, the action Σ' of eq.(82) reduces to the one we are actually interested in and which we have discussed in the previous sections.

Another special case is $\kappa\to 1$, $\alpha\to 0$. Then the gauge fixing terms of Σ' are

$$S_{\rm MAG}+S'_{\rm diag} = \int d^4x s\left(-A^A_\mu\partial^\mu\overline{c}^A\right) =$$

$$\int d^4x\left(\overline{c}^A\partial^\mu D^{AB}_\mu c^B + b^A\partial^\mu A^A_\mu\right) \ , \tag{83}$$

which is nothing else than the Landau gauge. At the same time, we also have

$$\lim_{(\alpha,\kappa)\to(0,1)} O = \frac{1}{2}A^A_\mu A^{\mu A} \ , \tag{84}$$

which is the pure gluon mass operator of the Landau gauge [22, 27].

From [27], we already know that the Landau gauge with the inclusion of the operator $A^A_\mu A^{\mu A}$ is renormalizable to all orders of perturbation theory. On the other hand, we have already proven the renormalizability for $\kappa=0$. The complete action Σ', as given in eq.(82), is BRST invariant

$$s\Sigma' = 0 \ . \tag{85}$$

In [21], we have written down the Ward identities of this model for $\kappa\neq 0$ and general α, and we have proven the renormalizability to all orders of perturbation theory. It was found that the additional gauge parameter κ does not renormalize in an independent way, while also a generalized version of the relation (22) emerges

$$\gamma_O(g^2) = -2\left(\frac{\beta(g^2)}{2g^2}-\gamma_{c^i}(g^2)\right) \ . \tag{86}$$

Summarizing, we have constructed a renormalizable gauge that is labeled by a couple of parameters (α,κ). It allows us to introduce a generalized composite operator O, given by eq.(79), which embodies the local operator $A^A_\mu A^{\mu A}$ of the Landau gauge as well as the operator $O_{\rm MAG}$ of the MAG. To construct the effective potential, one sets all sources equal to zero, except J, and introduces unity to remove the J^2 terms. A completely analogous argument as the one given in section III allows to conclude that the minimum value of $V(\sigma)$, thus $E_{\rm vac}$, will be independent of α and κ, essentially because the derivative with respect to α as well as with respect to κ is BRST exact, up to terms in the source J. This independence of α and κ is again only assured at infinite order in perturbation theory, so we can generalize the construction, proposed in section III, by making the function $\mathcal{F}$ of eq.(63) also dependent on κ. The foregoing analysis is sufficient to make sure that we can use the Landau gauge result for $E_{\rm vac}$ as the initial condition for the vacuum energy of the MAG. Moreover, we are now even in the position to answer the question about the existence of a fixed point of the RGE for the gauge parameter α, which was necessary to certify that no arbitrary constants would enter the results for $E_{\rm vac}$. We already mentioned that the Landau gauge, i.e. the case $(\alpha,\kappa)=(0,1)$, is a renormalizable model [27], i.e. the Landau gauge is stable against radiative corrections. This can be reexpressed by saying that $(\alpha,\kappa)=(0,1)$ is a fixed point of the RGE for the gauge parameters, and this to all orders of perturbation theory.

V. NUMERICAL RESULTS FOR $SU(2)$

After a quite lengthy formal construction of the LCO formalism in the case of the MAG, we are now ready to present explicit results. In this paper, we will restrict ourselves to the evaluation of the one-loop effective potential in the case of $SU(2)$. As renormalization scheme, we adopt the modified minimal substraction scheme ($\overline{\rm MS}$). Let us give here, for further use, the values of the one-loop anomalous dimensions of the relevant fields and couplings in the case of $SU(2)$. In our conventions, one has [35–37]

$$\gamma_{c^i}(g^2) \;=\; (-3-\alpha)\frac{g^2}{16\pi^2}+O(g^4) \ , \tag{87}$$

$$\gamma_\alpha(g^2) \;=\; \left(-2\alpha+\frac{8}{3}-\frac{6}{\alpha}\right)\frac{g^2}{16\pi^2}+O(g^4) \ , \tag{88}$$

while

$$\beta(g^2) = -\varepsilon g^2 - 2\left(\frac{22}{3}\frac{g^4}{16\pi^2}\right)+O(g^6) \ , \tag{89}$$

and exploiting the relation (22)

$$\gamma_{O_{\rm MAG}}(g^2) = \left(\frac{26}{3}-2\alpha\right)\frac{g^2}{16\pi^2}+O(g^4) \ , \tag{90}$$

a result consistent with that of [36].

The reader will notice that we have given only the 1-loop values of the anomalous dimensions, despite the fact that we have announced that one needs $(n+1)$-loop knowledge of the RGE functions to determine the n-loop potential. As we shall see soon, the introduction of the function $\mathcal{F}(g^2,\alpha)$ and the use of the Landau gauge as initial condition allow us to determine the 1-loop results we are interested in, from the 1-loop RGE functions only.

Let us first determine the counterterm $\delta\zeta$. For the generating functional $\mathcal{W}(J)$, we find at 1-loop [42]

$$\mathcal{W}(J) = \int d^dx\left(-(\zeta+\delta\zeta)\frac{J^2}{2}\right)+i\ln\det\left[\delta^{ab}\left(\partial^2+\alpha J\right)\right]-\frac{i}{2}\ln\det\left[\delta^{ab}\left(\left(\partial^2+J\right)g_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)\right] , \tag{91}$$

and employing

$$\ln\det\left[\delta^{ab}\left(\left(\partial^2+J\right)g_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)\right]=\delta^{aa}\left[(d-1)\mathrm{tr}\ln\left(\partial^2+J\right)+\mathrm{tr}\ln\left(\partial^2+\alpha J\right)\right] , \tag{92}$$

with

$$\delta^{aa}=N(N-1)=2 \text{ for } N=2 , \tag{93}$$

one can calculate the divergent part of eq.(91),

$$\mathcal{W}(J) = \int d^4x\left[-\delta\zeta\frac{J^2}{2}-\frac{3}{16\pi^2}J^2\frac{1}{\varepsilon}-\frac{1}{16\pi^2}\alpha^2J^2\frac{1}{\varepsilon}+\frac{1}{8\pi^2}\alpha^2J^2\frac{1}{\varepsilon}\right] . \tag{94}$$

Consequently,

$$\delta\zeta=\frac{1}{8\pi^2}\left(\alpha^2-3\right)\frac{1}{\varepsilon}+O(g^2) . \tag{95}$$

Next, we can compute the RGE function $\delta(g^2,\alpha)$ from eq.(28), obtaining

$$\delta(g^2,\alpha)=\frac{\alpha^2-3}{8\pi^2}+O(g^2) . \tag{96}$$

Having determined this, we are ready to calculate ζ_0. The differential equation (40) is solved by

$$\zeta_0(\alpha)=\alpha+\left(9-4\alpha+3\alpha^2\right)C_0 , \tag{97}$$

with C_0 an integration constant. As already explained in the previous sections, we can consistently put $C_0=0$. Here, we have written it explicitly to illustrate that, if α would coincide with the 1-loop fixed point of the RGE for the gauge parameter, the part proportional to C_0 in eq.(97) would drop. Indeed, the equations $9-4\alpha+3\alpha^2=0$ and $-2\alpha+\frac{8}{3}-\frac{6}{\alpha}=0$, stemming from eq.(88), are the same. Moreover, we also notice that this equation has only complex valued solutions. Therefore, it is even more important to have made the connection between the MAG and the Landau gauge by embedding them in a bigger class of gauges, since then we have the fixed point, even at all orders. In what follows, it is understood that $\zeta_0=\alpha$.

We now have all the ingredients to construct the 1-loop effective potential $\widetilde{V}_1(\widetilde{\sigma})$. One obtains

$$\widetilde{V}_1(\widetilde{\sigma}) = \frac{\widetilde{\sigma}^2}{2\zeta_0}\left(1-\left(2f_0+\frac{\zeta_1}{\zeta_0}\right)g^2\right)+\frac{3}{32\pi^2}\frac{g^2\widetilde{\sigma}^2}{\zeta_0^2}\left(\ln\frac{g\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}-\frac{5}{6}\right)-\frac{1}{32\pi^2}\frac{g^2\alpha^2\widetilde{\sigma}^2}{\zeta_0^2}\left(\ln\frac{g\alpha\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}-\frac{3}{2}\right) . \tag{98}$$

It can be checked explicitly that $\widetilde{V}_1(\widetilde{\sigma})$ obeys the renormalization group

$$\mu\frac{d}{d\mu}\widetilde{V}_1(\widetilde{\sigma})=0+\text{terms of higher order} , \tag{99}$$

by using the RGE functions (87)-(90) and the fact that the anomalous dimension of $\widetilde{\sigma}$ is given by

$$\gamma_{\widetilde{\sigma}}(g^2)=\frac{\beta(g^2)}{2g^2}+\gamma_{\mathcal{O}_{\mathrm{MAG}}}(g^2)+\mu\frac{\partial\ln\mathcal{F}(g^2,\alpha)}{\partial\mu} , \tag{100}$$

which is immediately verifiable from eq.(70).

We now search for the vacuum configuration by minimizing $\widetilde{V}_1(\widetilde{\sigma})$ with respect to $\widetilde{\sigma}$. We will put $\overline{\mu}^2=\frac{g\widetilde{\sigma}}{\zeta_0}$ to exclude possibly large logarithms, and find two solutions of the gap equation $\left.\frac{d\widetilde{V}_1}{d\sigma}\right|_{\overline{\mu}^2=\frac{g\widetilde{\sigma}}{\zeta_0}} = 0$, namely

$$\begin{aligned} \widetilde{\sigma} &= 0\,, && (101)\\ y &\equiv \left.\frac{g^2N}{16\pi^2}\right|_{N=2} = \frac{2\zeta_0}{16\pi^2\left(2f_0\zeta_0+\zeta_1\right)+\alpha^2\ln\alpha-\alpha^2+1}\,. && (102)\end{aligned}$$

The quantity y is the relevant expansion parameter, and should be sufficiently small to have a sensible expansion.The value for $\langle\widetilde{\sigma}\rangle$ corresponding to eq.(102) can be extracted from the 1-loop coupling constant

$$g^2(\overline{\mu}) = \frac{1}{\beta_0\ln\frac{\overline{\mu}^2}{\Lambda^2_{\overline{MS}}}}\,. \qquad (103)$$

The first solution (101) corresponds to the usual, perturbative vacuum ($E_{\text{vac}}=0$), while eq.(102) gives rise to a dynamically favoured vacuum with energy

$$\begin{aligned} E_{\text{vac}} &= -\frac{1}{64\pi^2}\left(3-\alpha^2\right)\left(m_{\text{gluon}}^{\text{off-diag}}\right)^4\,, && (104)\\ m_{\text{gluon}}^{\text{off-diag}} &= e^{\frac{3}{22y}}\Lambda_{\overline{MS}}\,. && (105)\end{aligned}$$

From eq.(104), we notice that at the 1-loop approximation, $\alpha^2\leq 3$ must be fulfilled in order to have $E_{\text{vac}}\leq 0$. In principle, the unknown function $f_0(\alpha)$ can be determined by solving the differential equation

$$\begin{aligned} \frac{dE_{\text{vac}}}{d\alpha} = 0 &\Leftrightarrow 2\alpha\left(m_{\text{gluon}}^{\text{off-diag}}\right)^4 + 4\left(\alpha^2-3\right)\left(m_{\text{gluon}}^{\text{off-diag}}\right)^3\frac{dm_{\text{gluon}}^{\text{off-diag}}}{d\alpha} = 0\\ &\Leftrightarrow \alpha + \frac{3-\alpha^2}{y^2}\left(\frac{\partial y}{\partial\alpha}+\frac{\partial y}{\partial\zeta_0}\frac{\partial\zeta_0}{\partial\alpha}+\frac{\partial y}{\partial\zeta_1}\frac{\partial\zeta_1}{\partial\alpha}+\frac{\partial y}{\partial f_0}\frac{\partial f_0}{\partial\alpha}\right)=0 \qquad (106)\end{aligned}$$

with initial condition $E_{\text{vac}}(\alpha)=E_{\text{vac}}^{\text{Landau}}$. However, to solve eq.(106) knowledge of ζ_1 is needed. Since we are not interested in $f_0(\alpha)$ itself, but rather in the value of the vacuum energy E_{vac}, the off-diagonal mass $m_{\text{gluon}}^{\text{off-diag}}$ and the expansion parameter y, there is a more direct way to proceed, without having to solve the eq.(106). Let us first give the Landau gauge value for E_{vac} in the case $N=2$, which can be easily obtained from [22, 38],

$$E_{\text{vac}}^{\text{Landau}} = -\frac{9}{128\pi^2}e^{\frac{17}{6}}\Lambda^4_{\overline{MS}}\,. \qquad (107)$$

Since the construction is such that $E_{\text{vac}}(\alpha)=E_{\text{vac}}^{\text{Landau}}$, we can equally well solve

$$-\frac{9}{128\pi^2}e^{\frac{17}{6}}\Lambda^4_{\overline{MS}} = -\frac{1}{64\pi^2}\left(3-\alpha^2\right)\left(m_{\text{gluon}}^{\text{off-diag}}\right)^4\,, \qquad (108)$$

which gives the lowest order masses

$$m_{\text{gluon}}^{\text{off-diag}} = \left(\frac{9}{2}\frac{e^{\frac{17}{6}}}{3-\alpha^2}\right)^{\frac{1}{4}}\Lambda_{\overline{MS}}\,, \qquad m_{\text{ghost}}^{\text{off-diag}} = \sqrt{\alpha}\left(\frac{9}{2}\frac{e^{\frac{17}{6}}}{3-\alpha^2}\right)^{\frac{1}{4}}\Lambda_{\overline{MS}}\,, \qquad (109)$$

The result (109) can be used to determine y. From eq.(105) one easily finds

$$y = \frac{36}{187+66\ln\frac{9}{2\left(3-\alpha^2\right)}}\,. \qquad (110)$$

We see thus that, for the information we are currently interested in, we do not need explicit knowledge of ζ_1 and f_0. We want to remark that, if ζ_1 were known, the value for y obtained in eq.(110) can be used to determine f_0 from eq.(102). This is a nice feature, since the possibly difficult differential equation (106) never needs to be solved in this fashion. Before we come to the conclusions, let us consider the limit $\alpha\to 0$, corresponding to the "real" MAG $D_\mu^{ab}A^{\mu b}=0$. One finds

$$\begin{aligned} m_{\text{gluon}}^{\text{off-diag}} &= \left(\frac{3}{2}e^{\frac{17}{6}}\right)^{\frac{1}{4}}\Lambda_{\overline{MS}} \approx 2.25\Lambda_{\overline{MS}}\,,\\ y &= \frac{36}{187+66\ln\frac{3}{2}} \approx 0.168\,. && (111)\end{aligned}$$

The relative smallness of y means that our perturbative analysis should give qualitatively meaningful results.

VI. DISCUSSION AND CONCLUSION

The aim of this paper was to give analytic evidence, as expressed by eq.(111), of the dynamical mass generation for off-diagonal gluons in Yang-Mills theory quantized in the maximal Abelian gauge. This mass can be seen as support for the Abelian dominance [9–11] in that gauge. This result is in qualitative agreement with the lattice version of the MAG, were such a mass was also reported [5, 8]. The off-diagonal lattice gluon propagator could be fitted by $\frac{1}{p^2+m^2}$, which is in correspondence with the tree level propagator we find. We have been able to prove the existence of the off-diagonal mass by investigating the condensation of a mass dimension two operator, namel $\left(\frac{1}{2}A^a_\mu A^{\mu a}+\alpha\bar{c}^a c^a\right)$. It was shown how a meaningful, renormalizable effective potential for this local composite operator can be constructed. By evaluating this potential explicitly at 1-loop order in the case of $SU(2)$, the formation of the condensate is favoured since it lowers the vacuum energy. The latter does not depend on the choice of the gauge parameter α, at least if one would work to infinite order in perturbation theory. We have explained in short how to overcome the problem at finite order and gave a way to overcome it. Moreover, we have been able to interpolate between the Landau gauge and the MAG by unifying them in a larger class of renormalizable gauges. This observation was used to prove that the vacuum energy of Yang-Mills theory in the MAG due to its mass dimension two condensate should be the same as the vacuum energy of Yang-Mills theory in the Landau gauge with the much explored condensate $\left\langle A^A_\mu A^{\mu A}\right\rangle$. It is worth noticing that all the gauges, where a dimension two condensate provides a dynamical gluon mass parameter, such as the Landau gauge [22], the Curci-Ferrari gauges [20], the linear gauges [29] and the MAG, can be connected to each other, either directly (e.g. Landau-MAG) or via the Landau gauge (e.g. MAG and linear gauges). This also implies that, if $\left\langle A^A_\mu A^{\mu A}\right\rangle \neq 0$ in the Landau gauge, the analogous condensates in the other gauges cannot vanish either.

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[1] Y. Nambu, Phys. Rev. D **10**, 4262 (1974);
G. 't Hooft, *High Energy Physics EPS Int. Conference,* Palermo 1975, ed. A. Zichichi;
S. Mandelstam, Phys. Rept. **23**, 245 (1976).
[2] G. 't Hooft, Nucl. Phys. B **190**, 455 (1981).
[3] A. S. Kronfeld, G. Schierholz, and U. J. Wiese, Nucl. Phys. B **293**, 461 (1987).
[4] A. S. Kronfeld, M. L. Laursen, G. Schierholz, and U. J. Wiese, Phys. Lett. B **198**, 516 (1987).
[5] K. Amemiya and H. Suganuma, Phys. Rev. D **60**, 114509 (1999).
[6] H. Min, T. Lee, and P. Y. Pac, Phys. Rev. D **32**, 440 (1985).
[7] A. R. Fazio, V. E. R. Lemes, M. S. Sarandy, and S. P. Sorella, Phys. Rev. D **64**, 085003 (2001).
[8] V. G. Bornyakov, M. N. Chernodub, F. V. Gubarev, S. M. Morozov, and M. I. Polikarpov, Phys. Lett. B **559**, 214 (2003).
[9] Z. F. Ezawa and A. Iwazaki, Phys. Rev. D **25**, 2681 (1982).
[10] T. Suzuki and I. Yotsuyanagi, Phys. Rev. D **42**, 4257 (1990).
[11] S. Hioki, S. Kitahara, S. Kiura, Y. Matsubara, O. Miyamura, S. Ohno, and T. Suzuki, Phys. Lett. B **272**, 326 (1991) [Erratum-ibid. B **281**, 416 (1992)].
[12] M. Schaden, hep-th/9909011.
[13] K. I. Kondo and T. Shinohara, Phys. Lett. B **491**, 263 (2000).
[14] D. Dudal and H. Verschelde, J. Phys. A **36**, 8507 (2003).
[15] H. Sawayanagi, Phys. Rev. D **67**, 045002 (2003).
[16] K. I. Kondo, Phys. Lett. B **514**, 335 (2001).
[17] G. Curci and R. Ferrari, Nuovo Cim. A **32**, 151 (1976).
[18] G. Curci and R. Ferrari, Phys. Lett. B **63**, 91 (1976).
[19] D. Dudal, H. Verschelde, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, S. P. Sorella, M. Picariello, and J.A. Gracey, Phys. Lett. B **569**, 57 (2003).
[20] D. Dudal, H. Verschelde, V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, and M. Picariello, Annals Phys. **308**, 62 (2003).
[21] D. Dudal, J. A. Gracey, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, S. P. Sorella, and H. Verschelde, Phys. Rev. D **70**, 114038 (2004).
[22] H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen, Phys. Lett. B **516**, 307 (2001).
[23] F. V. Gubarev, L. Stodolsky, and V. I. Zakharov, Phys. Rev. Lett. **86**, 2220 (2001).
[24] F. V. Gubarev and V. I. Zakharov, Phys. Lett. B **501**, (2001).
[25] P. Boucaud, A. Le Yaouanc, J. P. Leroy, J. Micheli, O. Pène and J. Rodriguez-Quintero, Phys. Rev. D **63**, 114003 (2001).
[26] O. Piguet and S. P. Sorella, Lect. Notes Phys. **M28**, 1 (1995).
[27] D. Dudal, H. Verschelde, and S. P. Sorella, Phys. Lett. B **555**, 126 (2003).
[28] D. Dudal, H. Verschelde, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, S. P. Sorella, and J. A. Gracey, Phys. Lett. B **574**, 325 (2003).
[29] D. Dudal, H. Verschelde, J. A. Gracey, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, and S. P. Sorella, JHEP **0401** 044 (2004).
[30] K. I. Kondo, Phys. Rev. D **57**, 7467 (1998).
[31] K. I. Kondo, T. Murakami, T. Shinohara, and T. Imai, Phys. Rev. D **65**, 085034 (2002).
[32] H. Verschelde, Phys. Lett. B **351**, 242 (1995).
[33] K. Knecht and H. Verschelde, Phys. Rev. D **64**, 085006 (2001).
[34] L. Baulieu and D. Zwanziger, Nucl. Phys. B **548**, 527 (1999).
[35] T. Shinohara, T. Imai, and K. I. Kondo, Int. J. Mod. Phys. A **18**, 5733 (2003).
[36] U. Ellwanger and N. Wschebor, Int. J. Mod. Phys. A **18**, 1595 (2003).
[37] K. I. Kondo, hep-th/0303251.
[38] R. E. Browne and J. A. Gracey, JHEP **0311**, 029 (2003).
[39] The index a runs only over the $N(N-1)$ off-diagonal generators.
[40] See section V.
[41] At first order, $E_{\rm vac}$ will depend on $f_0(\alpha)$, at second order on $f_0(\alpha)$ and $f_1(\alpha)$, etc.
[42] We will do the transformation of $\mathcal{W}(J)$ to $\mathcal{W}(\widetilde{J})$ only at the end.