Braz J Phys (2015) 45:265–271
DOI 10.1007/s13538-015-0308-0

ATOMIC PHYSICS

Minimum Uncertainty Coherent States Attached to Nondegenerate Parametric Amplifiers

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Received: 27 October 2014 / Published online: 9 April 2015

Abstract Exact analytical solutions for the two-mode nondegenerate parametric amplifier have been obtained by using the transformation from the two-dimensional harmonic oscillator Hamiltonian. Some important physical properties such as quantum statistics and quadrature squeezing of the corresponding states are investigated. In addition, these states carry classical features such as Poissonian statistics and minimize the Heisenberg uncertainty relation of a pair of the coordinate and the momentum operators.

Keywords Parametric amplifier · Quadrature squeezing · Poissonian statistics · Uncertainty relation

1 Introduction

The uncertainty principle, as an impossibility of the simultaneous measurement of the coordinate q and the momentum p, was introduced by Heisenberg [1] and proved by Kennard [2] in the form of the inequality $\langle\Delta p\rangle^2\langle\Delta q\rangle^2 \geq \frac{\hbar^2}{4}$, where $\langle\Delta X\rangle^2$ is the dispersion of the observable X and $\hbar$ is the Planck constant. It is a basic feature of quantum physics and indicates that the product of the uncertainties Δq and Δp of the measurements of two quantum observables in one and the same state is not less than one half of the absolute mean value of $-i[q, p]$. Therefore, this uncertainty relation provides a protocol of the preparation of a quantum state. Uncertainty relations are formal expressions of the uncertainty principle of quantum physics [1–4] and impose naturally fundamental limitations on the accuracy of measurements. Indeed, the main problem is how to optimize the intrinsic quantum fluctuations in the measurement process. Significant progress has been achieved in this direction in the last two decades of using the squeezed state technique. The concept of squeezed state [5] came from the observation of the equality in the Heisenberg uncertainty relation of the canonical observables q and p which can be maintained if the fluctuations of one of the two observables are reduced at the expense of the other. So, the uncertainty relations play a dual role, they cause limitations on the measurement precision, and at the same time, indicate ways to improve the accuracy of the measurement devices.

Analysis of two coupled harmonic oscillators, from the quantum-mechanical point of view, pervades many fields of physics, including the design of detectors for gravitational radiation, the dynamics of a harmonic oscillator, and for a transducer which is of great interest [6–12]. The coupling of free quantum fields with classical external fields can also be thought of as a system of harmonic oscillators, in general, with space- and time-dependent couplings. In the field of quantum optics, similar systems of coupled harmonic oscillator are important, such as in the description of parametric down-conversion or parametric amplification of photons [13–19]. Especially, there has been much attraction in studying the quantum properties displayed by the output fields of a nondegenerate parametric amplifier. In such a device, a pump photon is destroyed and a signal and idler photon pair is created (further details have been given in refs.

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[20–32]). The equivalence of a nondegenerate parametric amplifier with two parallel degenerate parametric amplifiers has been also exploited experimentally to obtain a matched local oscillator for the detection of quadrature squeezing [33] and has been extended to all the stages of the communication channel [34]. The nondegenerate parametric amplifier has attracted much attention because of its important role in quantum optics. For instance, squeezed states can be generated in nonlinear optical process such as nondegenerate parametric amplification, in which one photon of a large frequency is turned in the medium into two photons of equal or nonequal frequencies [35–38]. A method for producing light field Fock state based on nondegenerate parametric amplifier model was proposed by Björk [39]. By using Lie algebra representation theory, Z-Jie Wang [40] has solved the master equation for the nondegenerate parametric amplifier in a thermal reservoir. Applying a series of transformations, they show that the master equation has multiple commuted $SU(1,1)$ Lie algebra structures. The explicit solution to the master equation has been obtained. Also, the EPR paradox was demonstrated via quadrature phase measurements performed on the two output beams of a nondegenerate parametric amplifier [41]. Entangled states of light field and entanglement swapping have been well studied using nondegenerate parametric amplifier method [42, 43].

Major efforts in these documents, are focused on the study of nonclassical properties of the parametric amplifier [44–48]. Hence, our main motivation in this paper will be concentrated on the classical one. For this reason, we will consider a new technique of preparation of the minimum-uncertainty states corresponding to a nondegenerate parametric amplifier. A method of characterizing these wave packets, which are based on dynamical symmetry of the Hamiltonian is analyzed. The basic idea of the construction of these states is closely related to that of coherent states introduced by the pioneering works of Glauber, Klauder, and Sudarshan [49–56]. Such quantum states which fulfill this requirement will be constructed as solutions of an eigenvalue problem of some symmetry operators lowering the energy, as well as an orbit of states generated by a chosen group element from a fixed state and as minimum-uncertainty states for some physically significant operators.

This paper is organized as follows: taking advantage of quantum analysis of the two coupled harmonic oscillators, then by using a similarity transformation, we obtain the nondegenerate parametric amplifier Hamiltonian in Section 2. Accurate analysis of nondegenerate parametric amplifier, its eigenvalues, and eigenvectors are given in Section 2.1. Section 3, is devoted to introduce an approach which results in a new kind of minimum-uncertainty states, corresponding to the parametric amplifier. Also, basic statistical properties of these states are studied in more details. Finally, we conclude in Section 4.

2 Nondegenerate Parametric Amplifier and it's Connection with 2D Harmonic Oscillator

Here, we make brief reviews of two-mode nondegenerate and stationary parametric amplifier Hamiltonian

$$H = H_a + H_b + H_{ab} \tag{1}$$

where

$$H_a = \omega\left(a^\dagger a + \frac{1}{2}\right) \tag{2}$$

$$H_b = \omega\left(b^\dagger b + \frac{1}{2}\right) \tag{3}$$

in natural units ($\hbar = c = 1$). The interaction term between the modes a and b is chosen to be of the form

$$H_{ab} = i\left(\mathfrak{g}ab - \mathfrak{g}^* a^\dagger b^\dagger\right), \quad \mathfrak{g}\left(= |\mathfrak{g}|e^{i\Phi}\right) \in \mathbb{C} \tag{4}$$

which describes a two-mode nondegenerate parametric down-conversion process. The time-independent pump parameter $|\mathfrak{g}|$ denotes an arbitrary stationary and classical pump field. Here, $|\mathfrak{g}|$ is proportional to the second-order susceptibility of the medium and amplitude of the pump, Φ is the phase of the pump field, and ω is frequency [57–60]. It is straightforward that the Hamiltonian (1) can be rewritten in the following form

$$H = 2\omega K_0 + i\mathfrak{g}K_- - i\mathfrak{g}^* K_+, \tag{5}$$

where we have used the two-mode representation [61] of the Lie algebra $su(1,1)$[1]:

$$K_+ = a^\dagger b^\dagger,\ K_- = ab,\ K_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1), \tag{6}$$

$$[K_+, K_-] = -2K_0,\ [K_0, K_\pm] = \pm K_\pm. \tag{7}$$

Along with the application of a similarity transformation

$D(\xi)HD^\dagger(\xi)$,

with $D(\xi)$ as Klauder's displacement operator, we have

$$D(\xi) = e^{\xi K_+ - \bar{\xi} K_-},$$
$$\xi := |\xi|e^{i\varphi},\ 0 \le \varphi \le 2\pi \tag{8}$$

then, by choosing

$$\varphi = -\Phi + \left(n + \frac{1}{2}\right)\pi, \quad n \in \{1, 2, 3, \ldots\} \tag{9}$$

$$\tanh(|\xi|) = \left(\gamma - \sqrt{\gamma^2 - 1}\right), \ \gamma\left(= \frac{\omega}{|\mathfrak{g}|}\right) \ge 1. \tag{10}$$

[1]Here, $a(a^\dagger)$ and $b(b^\dagger)$ are the annihilation (creation) operators corresponding to the two hermitian harmonic oscillators, respectively.

An effective Hamiltonian representing a two-dimensional harmonic oscillator can be obtained[2]

$$D(\xi)HD^{\dagger}(\xi) = 2|\mathfrak{g}|f(\gamma)K_0 = |\mathfrak{g}|f(\gamma)(a^{\dagger}a + b^{\dagger}b + 1), \quad (13)$$

where $f(\gamma)$ can be calculated to be the following positive definite function

$$f(\gamma) = \frac{\gamma^2 - 1}{1 - \gamma^2 + \gamma\sqrt{\gamma^2 - 1}}\left(\gamma - \sqrt{\gamma^2 - 1}\right). \quad (14)$$

Obviously, (13) shows that the Hamiltonian $D(\xi)HD^{\dagger}(\xi)$ is "really" proportional to the two-dimensional harmonic oscillator Hamiltonian H_{ho}, i.e.,

$$D(\xi)HD^{\dagger}(\xi) = \frac{f(\gamma)}{\gamma}H_{\text{ho}}, \quad (15)$$

and

$$H_{\text{ho}} := \omega(a^{\dagger}a + b^{\dagger}b + 1). \quad (16)$$

Then, the eigenfunctions of the Hamiltonian $D(\xi)HD^{\dagger}(\xi)$ are those of the Hamiltonian H_{ho} [62]. Along with the energy eigenvalues and the eigenstates of the transferred Hamiltonian H_{ho}, we try to get the analogous ones for the original Hamiltonian H.

Because of the fact that the 2D Harmonic Oscillator includes $su(2)$ Lie algebra as dynamical symmetry [63], so it enables us to introduce a new class of eigenstates corresponding to the Hamiltonian H_{ho}. Indeed, they are associated with the "$su(2)$"-Perelomov coherent states and provide new eigenvectors for the Hamiltonian H, that will be introduced later as the displaced $su(2)$-Perelomov coherent states.

2.1 Eigenvalue Equation for the Nondegenerate Parametric Amplifier H Based on the $su(2)$-Perelomov Coherent States Attached to the Hamiltonian, H_{ho}

It is well known that using the representation of two Hermitian harmonic oscillators (a and b), i.e.,

$$[a, a^{\dagger}] = 1, \quad [b, b^{\dagger}] = 1, \quad [a, b] = 0, \quad (17)$$

which act on the complete and ortho-normal Fock space states, $\mathcal{H}_j = \{|j, m\rangle \mid -j \leq m \leq j\}$, with the following laddering relations:

$$a|j, m\rangle = \sqrt{\frac{j-m}{2}}|j-1, m+1\rangle,$$

$$a^{\dagger}|j-1, m+1\rangle = \sqrt{\frac{j-m}{2}}|j, m\rangle, \quad (18)$$

and

$$b|j, m\rangle = \sqrt{\frac{j+m}{2}}|j-1, m-1\rangle,$$

$$b^{\dagger}|j-1, m-1\rangle = \sqrt{\frac{j+m}{2}}|j, m\rangle, \quad (19)$$

the unitary and irreducible j representation of $su(2)$ can be attained [64]

$$[J_+, J_-] = 2J_0, \quad [J_0, J_{\pm}] = \pm J_{\pm}. \quad (20)$$

It is spanned by the generators

$$J_+ = ab^{\dagger}, \quad J_- = a^{\dagger}b, \quad J_0 = \frac{bb^{\dagger} - aa^{\dagger}}{2}, \quad (21)$$

and realize the following positive j integer irreducible representation of $su(2)$ Lie algebra on the Hilbert subspaces $\mathcal{H}_j$, as [65]

$$J_{\pm}|j, m\rangle = \sqrt{\left(\frac{j \mp m}{2}\right)\left(\frac{j \pm m}{2} + 1\right)}|j, m \pm 2\rangle,$$

$$J_0|j, m\rangle = \frac{m}{2}|j, m\rangle. \quad (22)$$

The so-called Klauder-Perelomov coherent states for a degenerate Hamiltonian H_{ho}, are defined as the action of a displacement operator on the normalized lowest (or highest) weight vectors:

$$|\alpha\rangle_j := e^{\alpha J_+ - \bar{\alpha}J_-}|j, -j\rangle, = (1 + |\eta|^2)^{\frac{-j}{2}} \sum_{m=0}^{j} \sqrt{\frac{\Gamma(j+1)}{\Gamma(m+1)\Gamma(j-m+1)}}\eta^m |j, -j + 2m\rangle, \quad (23)$$

where $\eta \left(= \alpha\frac{\tan|\alpha|}{|\alpha|}\right)$ is an arbitrary complex variable with the polar form $\eta = \varrho e^{i\theta}$ so that $0 \leq \varrho < \infty$ and $0 \leq \theta < 2\pi$. Clearly, the positive definite and non-oscillating measure is,

$$d\mu(\alpha) = 2\frac{j+1}{(1+\varrho^2)^2}\frac{d\varrho^2}{2}d\theta, \quad (24)$$

which satisfies the resolution of the identity condition on the whole of the complex plane for the coherent states $|\alpha\rangle_j$ in the Hilbert sub-spaces $\mathcal{H}_j$, i.e.,

$$\oint_{\mathbb{C}} |\alpha\rangle_j\, {}_j\langle\alpha| d\mu(\alpha) = I_j. \quad (25)$$

[2]It should be noticed that, to achieve the (13) we put

$$D(\xi)aD^{\dagger}(\xi) = \left(D(\xi)a^{\dagger}D^{\dagger}(\xi)\right)^{\dagger} = \cosh(\xi)a - \frac{\xi\sinh(\xi)}{|\xi|}b^{\dagger} \quad (11)$$

$$D(\xi)bD^{\dagger}(\xi) = \left(D(\xi)b^{\dagger}D^{\dagger}(\xi)\right)^{\dagger} = \cosh(\xi)b - \frac{\xi\sinh(\xi)}{|\xi|}a^{\dagger} \quad (12)$$

Also, they satisfy the completeness as well as the orthogonality relationships as follows

$$\sum_{j=0}^{\infty} |\alpha\rangle_{j\ j}\langle\alpha| = \sum_{j=0}^{\infty} |j,-j\rangle\ \langle j,-j| = I \tag{26}$$

$$_{j'}\langle\alpha\,|\alpha\rangle_j = \langle j', -j'\,|j,-j\rangle = \delta_{j'j}, \tag{27}$$

which is inherited because of the fact that the displacement operator $D(\xi) = e^{\alpha J_+ - \bar{\alpha} J_-}$ is a unitary action.

Using (16), (17), and (21), one can show that the Hamiltonian H_{ho} commutes[3] with all of the generators $J_\pm$ and J_0, therefore the following eigenvalue equation is achieved

$$H_{\text{ho}}\,|\alpha\rangle_j = \omega\,(j+1)\,|\alpha\rangle_j\,, \tag{28}$$

and provides us with the following eigenvalue equation corresponding to the Hamiltonian H, in terms of the displaced $su(2)$-coherent states, $|\xi,\alpha\rangle_j := D^\dagger(\xi)\,|\alpha\rangle_j$, as

$$H\left(D^\dagger(\xi)\,|\alpha\rangle_j\right) = |\mathfrak{g}|\,f(\gamma)\,(j+1)\left(D^\dagger(\xi)\,|\alpha\rangle_j\right). \tag{29}$$

They form a complete and orthonormal set of eigenbasis vectors, which is inherited from the completeness and orthogonality relations (26) and (27), respectively; in other words

$$\sum_{j=0}^{\infty} |\xi,\alpha\rangle_{j\ j}\langle\xi,\alpha| = \sum_{j=0}^{\infty} |\alpha\rangle_{j\ j}\langle\alpha| = \sum_{j=0}^{\infty} |j,-j\rangle\langle j,-j| = I, \tag{30}$$

$$_{j'}\langle\xi,\alpha\,|\xi,\alpha\rangle_j =_{j'} \langle\alpha\,|\alpha\rangle_j = \delta_{j'j}. \tag{31}$$

3 Representation of the Heisenberg Algebras Through the States $|\alpha\rangle_j$ and $|\xi,\alpha\rangle_j$

On one hand, the $su(2)$-Perelomov coherent states $|\alpha\rangle_j$ represent the Hamiltonian H_{ho} as well as the Hamiltonian H, and both of them are factorized in terms of two harmonic oscillator modes $a(a^\dagger)$ and $b(b^\dagger)$ (see (5) and (16)). Then a natural question arises here: is it possible for these states to be used for showing harmonic oscillator algebra? The reason comes from (18), (19), and (23) that the operators a and b act on the vectors $|\alpha\rangle_j$ as the lowering operators, which means

$$\underbrace{a\,|\alpha\rangle_j = \frac{\sqrt{j}}{\sqrt{1+|\eta|^2}}\,|\alpha\rangle_{j-1}\,,\quad b\,|\alpha\rangle_j = \frac{\eta\sqrt{j}}{\sqrt{1+|\eta|^2}}\,|\alpha\rangle_{j-1}}$$

$$\Downarrow$$

$$\frac{a+\bar{\eta}b}{\sqrt{1+|\eta|^2}}\,|\alpha\rangle_j = \sqrt{j}\,|\alpha\rangle_{j-1}\,, \tag{32}$$

and

$$\frac{a^\dagger+\eta b^\dagger}{\sqrt{1+|\eta|^2}}\,|\alpha\rangle_j = \sqrt{j+1}\,|\alpha\rangle_{j+1}\,, \tag{33}$$

where we have used, here, the formulas

$$e^{-\alpha J_+ + \bar{\alpha} J_-}\, a\, e^{\alpha J_+ - \bar{\alpha} J_-} = \frac{a+\bar{\eta}b}{\sqrt{1+|\eta|^2}}, \tag{34}$$

$$e^{-\alpha J_+ + \bar{\alpha} J_-}\, b\, e^{\alpha J_+ - \bar{\alpha} J_-} = \frac{b+\eta a}{\sqrt{1+|\eta|^2}}. \tag{35}$$

Now, taking into account (32), (33), and (17), we find the following harmonic oscillator algebra:

$$\left[\frac{a+\bar{\eta}b}{\sqrt{1+|\eta|^2}}, \frac{a^\dagger+\eta b^\dagger}{\sqrt{1+|\eta|^2}}\right] = 1. \tag{36}$$

Note that the above algebraic structure remains invariant under the unitary transformation, $D(\xi)$. Then, by using the displaced $su(2)$-coherent states $|\xi,\alpha\rangle_j$, other irreps (irreducible representations) of Weyl-Heisenberg algebra are obtainable

$$\left[A, A^\dagger\right] = 1, \tag{37}$$

$$A|\xi,\alpha\rangle_j = \sqrt{j}\,|\xi,\alpha\rangle_{j-1},\qquad A^\dagger|\xi,\alpha\rangle_j = \sqrt{j+1}\,|\xi,\alpha\rangle_{j+1}, \tag{38}$$

through the operators

$$A := D^\dagger(\xi)\left(\frac{a+\bar{\eta}b}{\sqrt{1+|\eta|^2}}\right)D(\xi) = \frac{\cosh|\xi|}{\sqrt{1+|\eta|^2}}\,(a+\bar{\eta}b) + \frac{\xi\sinh|\xi|}{|\xi|\sqrt{1+|\eta|^2}}(b^\dagger+\bar{\eta}a^\dagger), \tag{39}$$

$$A^\dagger := D^\dagger(\xi)\left(\frac{a^\dagger+\eta b^\dagger}{\sqrt{1+|\eta|^2}}\right)D(\xi) = \frac{\cosh|\xi|}{\sqrt{1+|\eta|^2}}\left(a^\dagger+\eta b^\dagger\right) + \frac{\bar{\xi}\sinh|\xi|}{|\xi|\sqrt{1+|\eta|^2}}(b+\eta a). \tag{40}$$

They will be utilized to generate a new kind of minimum uncertainty states associated with the Hamiltonian H_{ho} and H, respectively.

3.1 Minimum Uncertainty Quantum States Attached to the 2D Harmonic Oscillator H_{ho}

Let us define following new normalized states

$$|\beta,\alpha\rangle := e^{\frac{\beta}{\sqrt{1+|\eta|^2}}(a^\dagger+\eta b^\dagger) - \frac{\bar{\beta}}{\sqrt{1+|\eta|^2}}(a+\bar{\eta}b)}\,|\alpha\rangle_0 = e^{\frac{\beta}{\sqrt{1+|\eta|^2}}a^\dagger}\,e^{\frac{\beta\eta}{\sqrt{1+|\eta|^2}}b^\dagger}\,|0,0\rangle\,, \tag{41}$$

which can be considered as eigenvectors of annihilation operator $\frac{a+\bar{\eta}b}{\sqrt{1+|\eta|^2}}$ that will be written in a series form as

[3] In other words, the Lie algebra symmetry of $su(2)$ is dynamical symmetry of the Hamiltonian H_{ho}.

follows:

$$
\begin{aligned}
|\beta,\alpha\rangle &= e^{-\frac{|\beta|^2}{2}}\sum_{j=0}^{\infty}\frac{\beta^j}{\sqrt{\Gamma(j+1)}}\,|\alpha\rangle_j\,,\\
&= e^{-\frac{|\beta|^2}{2}}\sum_{j=0}^{\infty}\left(\frac{\beta}{\sqrt{1+|\eta|^2}}\right)^j\\
&\quad\times\sum_{m=0}^{j}\frac{\eta^m}{\sqrt{\Gamma(m+1)\Gamma(j-m+1)}}|j,-j+2m\rangle. \qquad (42)
\end{aligned}
$$

Resolution of the identity condition is realized for such two-variable coherent states on complex plane $\mathbb{C}^2$ by the measure $\frac{1}{\pi}d^2\alpha d^2\beta$.

Finally, these states are temporally stable, i.e.,

$$
\begin{aligned}
e^{-itH_{\text{ho}}}\,|\beta,\alpha\rangle &= e^{-i\omega t-\frac{|\beta|^2}{2}}\sum_{j=0}^{\infty}\frac{\left(\beta e^{-i\omega t}\right)^j}{\sqrt{\Gamma(j+1)}}\,|\alpha\rangle_j\\
&= e^{-i\omega t}\left|\beta e^{-i\omega t},\alpha\right\rangle. \qquad (43)
\end{aligned}
$$

The time-dependent coherent states of the generalized time-dependent parametric oscillator [66] will be useful for future studies in quantum optics as well as in atomic and molecular physics.

3.1.1 Classical Properties of the States $|\beta,\alpha\rangle$

In order to clarify why these states can be called minimum-uncertainty states, we now introduce two hermitian operators

$$q = \frac{1}{2}\left(b+b^\dagger-a-a^\dagger\right), \qquad (44)$$

$$p = \frac{-i}{2}\left(b-b^\dagger-a+a^\dagger\right), \qquad (45)$$

such that $[q,p] = i$, which leads to the following uncertainty relation:

$$\triangle(=\sigma_{qq}\sigma_{pp}-\sigma_{qp}^2)\geq\frac{1}{4}, \qquad (46)$$

where $\sigma_{ab} = \frac{1}{2}\langle ab + ba\rangle - \langle ab\rangle$ and the angular brackets denote averaging over an arbitrary normalizable state for which the mean values are well defined, $\langle a\rangle = \langle\beta,\alpha|\,a\,|\beta,\alpha\rangle$. Averaging over the classical-quantum states $|\beta,\alpha\rangle$, one finds: $\langle q^2\rangle = \frac{1}{2} + \langle q\rangle^2$, $\langle p^2\rangle = \frac{1}{2} + \langle p\rangle^2$, and $\langle pq + qp\rangle = 2\langle q\rangle\langle p\rangle$, where

$$
\begin{aligned}
\langle q\rangle &= \frac{\beta\eta+\overline{\beta\eta}-\beta-\bar\beta}{2\sqrt{1+|\eta|^2}},\\
\langle p\rangle &= -i\frac{\beta\eta-\overline{\beta\eta}-\beta+\bar\beta}{2\sqrt{1+|\eta|^2}},\\
\sigma_{qq} &= \sigma_{pp} = \frac{1}{2},\\
\sigma_{qp} &= 0. \qquad (47)
\end{aligned}
$$

They lead to $\Delta = \frac{1}{4}$, which is the lower bound of the Heisenberg uncertainty relation as well. In other words, the states $|\beta,\alpha\rangle$ meet the minimal requirement of the Heisenberg uncertainty relation.

As specific criteria to illustrate the inherited statistical properties of these states, it is necessary to analyze the behavior of Mandel's $Q(|\beta|)$[4] parameter, which is defined with respect to the expectation values of the number operator[5] ,$\hat J$, and its square in the basis of the states $|\beta,\alpha\rangle$:

$$Q(|\beta|) = \langle\hat J\rangle\left[\frac{\langle\hat J^2\rangle-\langle\hat J\rangle}{\langle\hat J\rangle^2}-1\right]. \qquad (48)$$

It is worth noting that for all accessible frequencies, the states $|\beta,\alpha\rangle$ follow the Poissonian statistics, i.e., $Q(|\beta|) = 0$, which represents classical effects.

3.2 Minimum Uncertainty Quantum States Attached to the Hamiltonian H

As it was shown in the above relations (37)–(40), the wave functions $|\xi,\alpha\rangle_j$ reproduce an irreps corresponding to the Heisenberg Lie algebra through the two ladder operators $A, A^\dagger$. Which, in turn, leads to derivation of their corresponding three variable coherence,

$$
\begin{aligned}
|\beta,\xi,\alpha\rangle &:= e^{\beta A^\dagger-\overline{\beta}A}\,|\xi,\alpha\rangle_0\,,\\
&= e^{-\frac{|\beta|^2}{2}}\sum_{j=0}^{\infty}\frac{\beta^j}{\sqrt{\Gamma(j+1)}}\,|\xi,\alpha\rangle_j\,,\\
&= D^\dagger(\xi)\,|\beta,\alpha\rangle\,. \qquad (49)
\end{aligned}
$$

They clearly satisfy the following eigenvalue equation

$$A\,|\beta,\xi,\alpha\rangle = \beta\,|\beta,\xi,\alpha\rangle\,. \qquad (50)$$

They include all of the features of true coherent states. For instance, they admit a resolution of the identity through positive definite measures. Also, they are temporally stable.

3.2.1 Classical Properties of the States $|\beta,\xi,\alpha\rangle$

Here, the uncertainty condition for the variances of the quadratures q and p, over the states $|\beta,\xi,\alpha\rangle$ will be exam-

[4] A state for which $Q(|\beta|) > 0$ is called super-Poissonian (bunching effect), if $Q(|\beta|) = 0$ the state is called Poissonian, while a state for which $Q(|\beta|) < 0$ is called sub-Poissonian (antibunching effect).

[5] It is well known that the number operator $\hat J$ is defined as the operator which diagonalizes the basis $|\alpha\rangle_j$. Using (27), we obtain

$${}_{j'}\langle\alpha|\hat J\,|\alpha\rangle_j = j\delta_{j'j}.$$

ined. For instance, the following relations can be calculated (easily)

$$\langle a\rangle = \beta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\eta\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}, \tag{51a}$$

$$\left\langle a^2\right\rangle = \left(\beta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\eta\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right)^2 = \langle a\rangle^2, \tag{51b}$$

$$\langle b\rangle = \beta\eta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}, \tag{51c}$$

$$\left\langle b^2\right\rangle = \left(\beta\eta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right)^2 = \langle b\rangle^2, \tag{51d}$$

$$\langle ab\rangle = \left(\beta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\eta\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right) \times\left(\beta\eta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right) = \langle a\rangle\langle b\rangle \tag{51e}$$

$$\left\langle ab^\dagger\right\rangle = \left(\beta\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \overline{\eta\beta}\frac{\xi\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right) \times\left(\overline{\beta\eta}\frac{\cosh(\xi)}{\sqrt{1+|\eta|^2}} - \beta\frac{\overline{\xi}\sinh(\xi)}{|\xi|\sqrt{1+|\eta|^2}}\right) = \langle a\rangle\left\langle b^\dagger\right\rangle. \tag{51f}$$

Our final step is to reveal that measurements of the states $|\beta,\xi,\alpha\rangle$ come with minimum uncertainty of the field quadrature operators q and p. In this case, uncertainty factors σ_{qq}, σ_{pp}, and σ_{qp} can be evaluated to be taken as, respectively,

$$\sigma_{qq} = \sigma_{pp} = \frac{1}{2} \tag{52a}$$

$$\sigma_{qp} = 0, \tag{52b}$$

consequently

$$\triangle = \sigma_{qq}\sigma_{pp} - {\sigma_{qp}}^2 = \frac{1}{4}. \tag{52c}$$

Calculation of other statistical quantities including the second-order correlation function and Mandel's parameter indicates that these states follow the classical regime.

4 Conclusions

We have constructed minimum-uncertainty wave packets of the two-dimensional harmonic oscillators as well as their analogous corresponding to the nondegenerate parametric amplifiers. Remarkably, these states are nonspreading wave packets that minimize the uncertainty of the measurement of the position and the momentum operators. We would like to emphasize the minimum-uncertainty quantum states which play important roles in quantum optics and mathematical physics; hence, this algebraic process can be applied to perform minimum-uncertainty coherent, squeezed and intelligent states associated with the other physical systems.

A precise analysis of Mandel's parameter confirms that Poissonian statistics is achievable. Finally, we have shown that these states are temporally stable and may be useful for future studies in the time-dependent parametric amplifiers [45] too.

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