Braz J Phys (2015) 45:334–339
DOI 10.1007/s13538-015-0316-0

GENERAL AND APPLIED PHYSICS

Noether's Theorem of Relativistic-Electromagnetic Ideal Hydrodynamics

J. H. Gaspar Elsas[1] · **T. Koide**[1] · **T. Kodama**[1]

Received: 18 December 2014 / Published online: 24 March 2015

Abstract We present a variational approach for relativistic ideal hydrodynamics interacting with electromagnetic fields. The momentum of fluid is introduced as the canonical conjugate variable of the position of a fluid element, which coincides with the conserved quantity derived from Noether's theorem. We further show that our formulation can reproduce the usual electromagnetic hydrodynamics which is obtained so as to satisfy the conservation of the inertia of fluid motion.

Keywords Relativistic hydrodynamics · Noether theorem · Variational principle

1 introduction

Hydrodynamics has been used to describe the collective dynamics of many particle systems. For example, a relativistic hydrodynamic model is now considered as one of the basic tools for the analysis of the collective behaviors of the matter produced in relativistic heavy-ion collisions (HIC). In this example of HIC, the dynamics of microscopic degrees of freedom is described by quantum chromodynamics (QCD). Therefore, a hydrodynamic description of this produced matter should incorporate symmetries associated with the QCD Lagrangian. The relation between the microscopic and hydrodynamic symmetries is sometimes subject to debate. For example, there are several proposals which try to incorporate the effect of chiral symmetry on relativistic hydrodynamics [1–8]. In this aspect, it is desirable to study the formulation of hydrodynamics in which we can impose the symmetry in the original Lagrangian.

For an ideal fluid case, it is known that the relativistic hydrodynamic equations can be derived from the variational principle [9] where the required symmetry, in principle, should be incorporated through the Lagrangian. However, due to the strong reduction of the number of the degrees of freedom in the hydrodynamic description through a coarse-graining procedure, it is not clear whether the symmetry expressed in terms of microscopic variables can be always rewritten with only hydrodynamic variables.

Nevertheless, it is worth discussing symmetry of hydrodynamics in terms of the variational approach. For example, usually hydrodynamic equations are expressed as the conservations of energy and momentum, and the latter is identified with the conservation of the inertia of fluid elements (sometimes called kinetic momentum). This identification is, however, not reliable when there are interactions with gauge fields because the form of canonical momentum is affected by the interactions and does not coincide with inertia anymore. In the phenomenological derivation of electromagnetic hydrodynamics, no regard is given to this difference between momentum and inertia. Thus, models of electromagnetic hydrodynamics are normally constructed without referring to the momentum conservation explicitly [10].

✉ J. H. Gaspar Elsas
jhelsas@gmail.com

T. Koide
tomoikoide@gmail.com

T. Kodama
kodama.takeshi@gmail.com

1 Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, 21941-972, Rio de Janeiro, Brazil

Relativistic hydrodynamics coupled to electromagnetic fields serves to discuss, for example, relativistic plasma in the Universe, high-energy astrophysical phenomena, and HIC. There, charged fluids interact with a coherent electromagnetic fields and the boundary conditions of the fields are under control. Then we need to treat fluid and electromagnetic fields as independent dynamical variables. The purpose of this paper is to derive relativistic ideal hydrodynamics interacting with electromagnetic fields in the framework of variational approach.

This paper is organized as follows. In Section 2, we introduce our gauge-invariant Lagrangian and discuss the fluid motion in the Lagrange coordinate system. In Section 3, we further define the canonical momentum of fluid and derive the Euler-Lagrange equation. In Section 4, the energy and momentum conservations are examined through Noether's theorem. In Section 5, we summarize and discuss our results.

2 Lagrangian and Set up of Hydrodynamic Variables

A gauge invariant Lagrangian of a relativistic ideal fluid is given by

$$L = \int d^3\mathbf{x} \left[-\varepsilon^*(n_e^*, s^*) - n_e^* u_\mu A^\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right], \quad (1)$$

where

$$F^{\mu\nu}(\mathbf{x}, t) = \partial^\mu A^\nu(\mathbf{x}, t) - \partial^\nu A^\mu(\mathbf{x}, t). \quad (2)$$

Here ε^*, n_e^*, and s^* are the proper scalar densities (scalar functions) which are defined as the energy, charge, and the entropy densities in the local rest frame, respectively. The fluid velocity u^μ is normalized as $u^\mu u_\mu = 1$. The gauge filed is denoted by A^μ. This Lagrangian is written in the Euler coordinates. See also [9]. Note that there is the equation of continuity associated with the electric charge conservation, $\partial_\mu(n_e^* u_\mu) = 0$. By using this property, we can show the gauge invariance of this Lagrangian.

Note that the above Lagrangian is different from the one used in [11]. One can easily see that our Lagrangian is a natural generalization of the one-particle Lagrangian with the gauge interaction.

In order to use thermodynamic relations, we assume that ε^* is given by a function of n_e^* and s^*. Note that, in the present case, the energy flow can be chosen to be parallel to the conserved charge flow and hence there is a unique local rest frame where thermodynamic laws are applied. Thus, we can employ the following thermodynamic relation,

$$d\varepsilon^* = T ds^* - \mu dn_e^*. \quad (3)$$

It should be mentioned that even thermodynamic relations can be modified when fluids interacting with electromagnetic fields in general because interactions contribute thermodynamic works. This contribution, however, vanishes when the polarization and magnetization of the fluid are not considered as is the present case [10].

As was mentioned, there are two different approaches to describe fluids, one is in the Euler coordinates and the other in the Lagrangian coordinates. The Euler coordinates are fixed in a given referential frame. The Lagrangian coordinates are, on the other hand, fixed on a certain fluid element and changes the direction with time as is shown in Fig. 1. In the present work, we employ the variation in the Lagrangian coordinates. For this, we introduce a fluid element with a constant entropy ν which is determined by the total entropy divided by the total number of the fluid element. Note that the total entropy is conserved for ideal fluids. Then the trajectory of a fluid element is represented by $\mathbf{r}(\mathbf{R}, t)$ where $\mathbf{R}$ denotes the initial position.

We further assume that, for any t, $\mathbf{r}(\mathbf{R}, t) \neq \mathbf{r}(\mathbf{R}', t)$ with $\mathbf{R}' \neq \mathbf{R}$, and thus this argument is not applicable when turbulence appears. Let us denote the volume density of this fluid element by $\rho(\mathbf{r}(\mathbf{R}, t), t)$. If there is no chaotic motion like turbulence, the volume of each fluid element is conserved and the evolution is determined by

$$\frac{d}{dt}\rho(\mathbf{r}(\mathbf{R}, t), t) = -\rho(\mathbf{r}(\mathbf{R}, t), t)(\nabla \cdot \mathbf{v}(\mathbf{r}(\mathbf{R}, t), t)), \quad (4)$$

where $\mathbf{v}(\mathbf{r}(\mathbf{R}, t), t) = d\mathbf{r}(\mathbf{R}, t)/dt$. Note that $\rho = \gamma\rho^*$ is the density in the observational frame of reference, and $d/dt = \partial_t + \mathbf{v} \cdot \nabla$ denotes the Lagrangian time derivative. This corresponds to the equation of continuity in the Lagrangian coordinates. By using this, one can easily see the conservation of the proper scalar density associated with the entropy,

$$\partial_\mu(s^* u^\mu) = 0, \quad (5)$$

where $x^\mu = (ct, x, y, z)$ and $s^* = \nu\rho/\gamma$ with $\gamma = 1/\sqrt{1 - (\mathbf{v}/c)^2} = u^0$.

In this paper, we consider an ideal fluid, that is, there is no dissipative flow such as viscosity. Then the electric charge current also should be given by

$$\partial_\mu(n_e^* u^\mu) = 0. \quad (6)$$

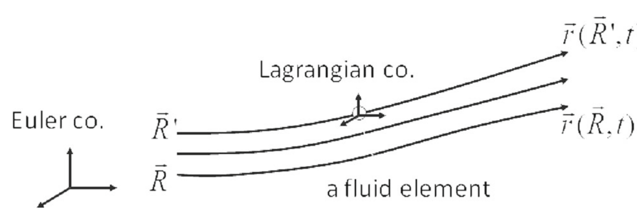

Fig. 1 The Euler coordinates are fixed in a certain frame. The Lagrangian coordinates are fixed on a certain fluid element. The trajectory of a fluid element is denoted by $\mathbf{r}(\mathbf{R}, t)$ with $\mathbf{R}$ being an initial position

To satisfy this, we need to assume that each fluid element has the same amount of electric charge e which is defined by the division of the total electric charge by the total number of the fluid elements, and $n_e^* = e\rho/\gamma$. In short, as is shown in Fig. 2, each fluid element has the same amounts of ν and e. We will discuss the physical meaning of this assumption in the concluding remarks. It is also noted that we cannot argue electromagnetic fields whose typical length scale is smaller than the size of the fluid element.

If there is no turbulent behavior, the evolution of the Lagrangian coordinates can be regarded as the variable transform from $\mathbf{R}$ to $\mathbf{r}$. We then find that the Jacobian of the variable transform is expressed as [12]

$$J(\mathbf{R}, t) = \det\left|\frac{\partial \mathbf{r}}{\partial \mathbf{R}}\right| = \frac{\rho_0(\mathbf{R})}{\rho(\mathbf{r}(\mathbf{R}, t), t)}, \tag{7}$$

where ρ_0 is the initial distribution of the fluid element, $\rho_0(\mathbf{R}) = \rho(\mathbf{r}(\mathbf{R}, t_0), t_0)$ with t_0 being the initial time. In short, the Lagrangian is obtained by the sum of the contributions from each fluid elements and thus we have

$$L = \int d^3\mathbf{R}\mathcal{L}^p(\mathbf{r}, A) + \int d^3\mathbf{x}\mathcal{L}^{em}(A), \tag{8}$$

where

$$\mathcal{L}^p(\mathbf{r}, A) = \mathcal{L}^{pf}(\mathbf{r}) - j_{e0}^{\mu} A_\mu = -\frac{\rho_0}{\rho}\varepsilon^*\left(n_e^*, s^*\right) - j_{e0}^{\mu} A_\mu, \tag{9}$$

$$\mathcal{L}^{em}(A) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{10}$$

with

$$j_0^0 = e_{\mathbf{R}}\rho_0(\mathbf{R}), \tag{11}$$

$$j_0^i = e_{\mathbf{R}}\rho_0(\mathbf{R})\frac{1}{c}v^i(\mathbf{r}, t), \tag{12}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{13}$$

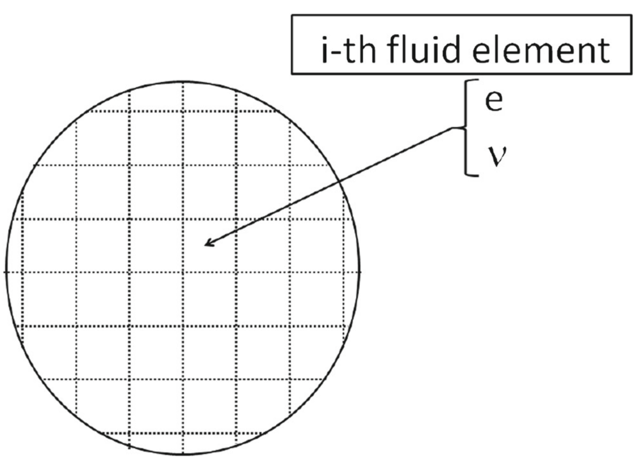

Fig. 2 Fluid is decomposed into the ensemble of fluid elements. The initially given electric charge and entropy for each fluid element are conserved during the hydrodynamic evolution

The gauge field A^μ is introduced to express electromagnetic fields. It should be emphasized that $\mathbf{r}$ denotes a trajectory of a fluid element and $\mathbf{x}$ the spatial coordinates.

3 Canonical Momentum and Euler-Lagrange Equation

We here consider the momentum of the fluid as the canonical momentum conjugated to the position of the fluid element,

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \rho_0\frac{\varepsilon^* + P^*}{\rho^*}\gamma\frac{\mathbf{v}}{c^2} + \frac{e\rho_0}{c}\mathbf{A}. \tag{14}$$

For convenience, however, we exclusively use the canonical momentum expressed in the Euler coordinates,

$$p^i(\mathbf{x}, t) = \frac{\gamma}{c}\left\{(\varepsilon^* + P^*)u^i + n_e^* A^i\right\}, \tag{15}$$

which satisfies

$$\int d^3\mathbf{R}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \int d^3\mathbf{x}\, \mathbf{p}(\mathbf{x}, t). \tag{16}$$

In the non-relativistic limit, the enthalpy density $\varepsilon^* + P^*$ is reduced to $mc^2\rho^*$ with m being the mass of the constituent particle [13]. Then the above definition coincides with the well-known expression for the canonical momentum of a non-relativistic particle interacting with electromagnetic fields by substituting Dirac's delta function to ρ^*.

The variations for the trajectory of a fluid element and the gauge fields are given by

$$\mathbf{r}(\mathbf{R}, t) \longrightarrow \mathbf{r}'(\mathbf{R}, t) = \mathbf{r}(\mathbf{R}, t) + \delta\mathbf{r}(\mathbf{R}, t), \tag{17}$$

$$A^\nu(\mathbf{x}, t) \longrightarrow A'^\nu(\mathbf{x}, t) = A^\nu(\mathbf{x}, t) + \delta A^\nu(\mathbf{x}, t), \tag{18}$$

respectively. From these, we obtain the following Euler-Lagrange equations,

$$\gamma\frac{d}{dt}\left[\frac{\varepsilon^* + P^*}{\rho^*}\mathbf{u}\right] = -\frac{1}{\rho^*}\nabla_{\mathbf{r}}P^* + e\left[\gamma\mathbf{E} + \frac{\gamma\mathbf{v}}{c}\times\mathbf{B}\right], \tag{19a}$$

$$\partial_\mu F^{\mu\nu} = n_e^* u^\nu. \tag{19b}$$

Here, the thermodynamic pressure is defined by

$$P^* = Ts^* - \varepsilon^* + \mu n_e^*. \tag{20}$$

See also the calculations in [12].

The latter equation is Maxwell's equation. The former equation can be expressed in a more familiar form by adopting the Euler coordinates as

$$(\partial_t + \mathbf{v}\cdot\nabla)\left[\frac{\varepsilon^* + P^*}{\rho^*}\mathbf{u}\right] = -\frac{1}{\rho}\nabla P^* + e\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]. \tag{21}$$

This is the well-known momentum component of the relativistic ideal hydrodynamics with the Lorentz force.

4 Noether's Theorem

A hydrodynamic model is constructed so as to satisfy the energy and momentum conservations, which are obtained as the results of the invariance of actions for the time and space translations, respectively, as is well-known in Noether's theorem.

Let us consider the following transform of the coordinate variables,

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu. \tag{22}$$

Then the gauge fields and the fluid trajectory are transformed as

$$\begin{aligned} A^\mu(x) &\longrightarrow (A')^\mu(x') = A^\mu(x) + \delta^L A^\mu(x) \\ &\quad +\epsilon^\nu \partial_\nu A^\mu, \end{aligned} \tag{23}$$

$$\begin{aligned} \mathbf{r}(\mathbf{R}, t) &\longrightarrow \mathbf{r}'(\mathbf{R}, t') = \mathbf{r}(\mathbf{R}, t) + \delta^L \mathbf{r}(\mathbf{R}, t) \\ &\quad +\epsilon^0 \frac{d}{d(ct)} \mathbf{r}(\mathbf{R}, t). \end{aligned} \tag{24}$$

Here, the Lie derivative is defined by

$$\delta^L f^i(x) = (f')^i(x) - f^i(x). \tag{25}$$

In the interaction part of the Lagrangian, we need to estimate the transform of the gauge fields on the trajectory of a fluid element as

$$\begin{aligned} A^\mu(\mathbf{r}, t) &\longrightarrow (A')^\mu(\mathbf{r}', t') = A^\mu(\mathbf{r}, t) + \delta^L A^\mu(\mathbf{r}, t) \\ &\quad +\epsilon^0 \partial_0 A^\mu(\mathbf{r}, t) \\ &\quad + \left(\delta^L \mathbf{r} + \epsilon^0 \frac{d\mathbf{r}}{d(ct)} \right) \cdot \nabla A^\mu(\mathbf{r}, t). \end{aligned} \tag{26}$$

Note that due to the transform of the trajectory of a fluid element (24), we cannot implement calculations keeping the Lorentz covariance manifestly, but the derived results have manifestly covariant forms.

Substituting these transforms into the action, we obtain

$$\begin{aligned} & I' - I \\ &= \int dt d^3\mathbf{R} \left\{ \frac{d}{dt} \left[\frac{\epsilon^0}{c} \mathcal{L}^p + \left(\frac{\partial \mathcal{L}^{pf}}{\partial \dot{\mathbf{r}}} + \frac{e\rho_0}{c} \mathbf{A} \right) \cdot \delta^L \mathbf{r} \right] \right. \\ &\quad \left. + \frac{\partial}{\partial \mathbf{R}} \cdot \left[\frac{\partial \mathcal{L}^{pf}}{\partial (\partial \mathbf{r}/\partial \mathbf{R})} \delta^L \mathbf{r} \right] \right\} \\ &\quad + \frac{1}{c} \int d^4x \, \partial_\mu \left[\epsilon^\mu \mathcal{L}^{em} + \frac{\partial \mathcal{L}^{em}}{\partial(\partial_\mu A^\nu)} \delta^L A^\nu \right], \end{aligned} \tag{27}$$

where I and I' denote the action before and after the transform, respectively. We used the Euler-Lagrange equations (19a) in the derivation of this expression. In the following subsections, we discuss the momentum and energy conservations, separately.

4.1 Momentum Conservation

The momentum conservation is observed by employing the invariance of the action for the spatial translation, which is expressed as

$$\epsilon^0 = 0, \tag{28}$$

$$\delta^L \mathbf{r} = \vec{\epsilon}, \tag{29}$$

$$\delta t = 0, \tag{30}$$

$$\delta^L A^\mu = -(\vec{\epsilon} \cdot \nabla) A^\mu. \tag{31}$$

By substituting these into (27) and re-expressing in the Euler coordinates, we have

$$\partial_\mu \left[(T_{fluid})^{\mu i} + (T_{em})^{\mu i} \right] = 0, \tag{32}$$

where

$$T^{\mu i}_{fluid} = (\varepsilon^* + P^*) u^\mu u^i - P^* g^{\mu i} + \frac{n_e^*}{c} u^\mu A^i, \tag{33}$$

$$T^{\mu i}_{em} = \frac{g^{\mu i}}{2} \partial_\alpha A_\nu (\partial^\alpha A^\nu - \partial^\nu A^\alpha) - \partial^i A^\nu (\partial^\mu A_\nu - \partial_\nu A^\mu). \tag{34}$$

By using the definition given by (15), the time component $(T_{fluid})^{0i}$ is expressed as

$$(T_{fluid})^{0i} = c p^i. \tag{35}$$

In short, (32) represents the conservation of the canonical momentum which is defined by the Legendre transform (15). In other words, the definition of the canonical momentum of fluid introduced here is consistent with Noether's theorem, and the hydrodynamics constructed in our formulation conserves the canonical momentum appropriately.

On the other hand, the momentum of electromagnetic fields in the present case, from the definition (34), is given by

$$(T_{em})^{0i} = (\mathbf{E} \times \mathbf{B})^i - \frac{1}{c} \gamma n_e^* A^i - \nabla \cdot (A^i \mathbf{E}). \tag{36}$$

The first term on the right-hand side is the well-known momentum density of electromagnetic fields, while the second term represents the modification of this momentum density by the interaction with the fluid. Furthermore, the corresponding stress tensor is calculated as:

$$\begin{aligned} (T_{em})^{ji} &= -E^j E^i - B^j B^i + \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) \delta^{ji} \\ &\quad - \frac{n_e^*}{c} u^j A^i - \sum_{\alpha=0}^{3} \partial_\alpha (F^{j\alpha} A^i). \end{aligned} \tag{37}$$

Note that the term $\nabla \cdot (A^i \mathbf{E})$ appearing in (36) is cancelled by the corresponding term in $\partial_\alpha (F^{j\alpha} A^i)$ when these are substituted into the equation of continuity.

The momenta of fluid and electromagnetic fields are modified as they interact as shown by (35) and (36). Note, however, that the modifications have the same form with the opposite sign. In short, the sum of the two momenta is reduced to the sum of the inertia of the fluid and of the free electromagnetic momentum density:

$$(T_{fluid})^{0i} + (T_{em})^{0i} = \gamma(\varepsilon^* + P^*)u^i + (\mathbf{E} \times \mathbf{B})^i. \tag{38}$$

Therefore, the conservation of momentum is equivalent to that of inertia as is expected in usual phenomenological argument. Note that this does not imply that the momentum density of fluid is given by $(T_{fluid})^{0i}$. The evolution equation is given by (21) and the deviation of the momentum from the inertia leads to the Lorentz force.

4.2 Energy Conservation

The energy conservation is discussed in a similar fashion. The time translation is expressed by

$$\begin{aligned} \epsilon^i &= 0, &(39)\\ \delta^L \mathbf{r} &= -\frac{\epsilon^0}{c}\dot{\mathbf{r}}, &(40)\\ \delta t &= \frac{\epsilon^0}{c}, &(41)\\ \delta^L A^\mu &= -\epsilon^0 \partial_0 A^\mu. &(42)\end{aligned}$$

Then (27) is reduced to

$$\frac{1}{c}\frac{\partial}{\partial t}\left(T^{00}_{fluid} + T^{00}_{em}\right) + \sum_i \partial_i \left(T^{i0}_{fluid} + T^{i0}_{em}\right) = 0, \tag{43}$$

where

$$\begin{aligned} T^{\mu 0}_{fluid} &= (\varepsilon^* + P^*)u^\mu u^0 - P^* g^{\mu 0} + \frac{n_e^*}{c} u^\mu A^0, &(44)\\ T^{00}_{em} &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \frac{1}{c}\gamma n_e^* A^0 - \nabla \cdot (A^0 \mathbf{E}), &(45)\\ T^{i0}_{em} &= (\mathbf{E} \times \mathbf{B})^i - n_e^* A^0 u^i - \sum_{\alpha=0}^{3} \partial_\alpha (A^0 F^{i\alpha}). &(46)\end{aligned}$$

We observe that the energy density of the fluid is given by the sum of the non-interacting ideal fluid part $(\varepsilon^* + P^*)\gamma^2 - P^*$ and the contribution from the Coulomb energy $\gamma n_e^* A^0$. The same Coulomb contribution appears in the energy density of electromagnetic fields with the opposite sign. The third term in (45) cancels with a term in (46) in substituting into the equation of continuity as is the case of the momentum conservation. In short, the total energy density which is conserved is expressed as

$$T^{00}_{fluid} + T^{00}_{em} = (\varepsilon^* + P^*)\gamma^2 - P^* + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2). \tag{47}$$

Again, the sum of the two energy densities is reduced to that of the non-interacting ideal fluid and electromagnetic fields.

In short, the derived energy and momentum conservations can be cast into the traditional covariant forms as

$$\begin{aligned} \partial_\mu (T^{\mu\nu}_{ideal} + T^{\mu\nu}_E) &= 0, &(48)\\ \partial_\mu N^\mu &= 0, &(49)\end{aligned}$$

where

$$\begin{aligned} T^{\mu\nu}_{ideal} &= (\varepsilon^* + P^*)u^\mu u^\nu - P^* g^{\mu\nu}, &(50)\\ T^{\mu\nu}_E &= F^{\mu\lambda}F^\nu_\lambda + \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}, &(51)\\ N^\mu &= n_E^* u^\mu. &(52)\end{aligned}$$

These are well-known phenomenological results of relativistic ideal hydrodynamics interacting with electromagnetic field.

5 Concluding Remarks

In this paper, we reformulated relativistic ideal hydrodynamics interacting with electromagnetic field in the framework of the variational approach. We interpreted the canonical momentum of fluid as the canonical conjugate variable of the position of a fluid element. This canonical momentum is affected by the interaction with gauge fields as is well-known in the case of charged particle systems. To confirm our assumption for the definition of the canonical momentum of fluid, the conservation law associated with the the spatial translational invariance of the action was derived by applying Noether's theorem and it was confirmed that this Noether charge is equivalent to the above canonical momentum. Such a modification of momentum by the gauge interaction appears even for the electromagnetic momentum density. However, the sum of the two momenta is simply given by that of fluid inertia and free electromagnetic fields.

This is true even for the definitions of energy densities. The fluid energy density defined by the time translation invariance contains the contribution from the Coulomb energy. This contribution is, however, canceled with the corresponding term in the electromagnetic energy density and hence the total energy density is again given by the sum of the non-interacting ideal fluid and electromagnetic fields.

In summary, our formulation reproduces the usual phenomenologically derived electromagnetic hydrodynamics, and we confirmed that the usual derivation of electromagnetic hydrodynamics using the conservation of inertia is still consistent with the momentum conservation. The results suggest that the symmetry of fluid is possible to be investigated in terms of the Lagrangian approach. The potential application is the modeling of hydrodynamics including the effect of the chiral symmetry, which has been discussed by several groups [1]. For this, the chiral symmetry should be re-expressed in terms of hydrodynamic variables. To see the relation between this symmetry and hydrodynamic variables, it will be useful to express the Dirac equation with hydrodynamic variable [14].

The variational formulation of relativistic ideal hydrodynamic is already discussed in [9] in terms of the Euler coordinates. Noether's theorem of the relativistic magneto hydrodynamics is discussed in [11]. However, the derived hydrodynamic equation (given by (30)) does not coincide with the usual hydrodynamic theory. Moreover, special conditions associated with the magneto hydrodynamics (given by (3a)) is employed in the derivation. Our result is consistent with the usual relativistic ideal hydrodynamics and both of electric and magnetic fields are treat on an equal footing.

In our derivation, we assumed the existence of fluid elements which have a constant entropy ν and charge e and these are simultaneously homogeneous with respect to the Lagrangian coordinates $\partial \nu/\partial \mathbf{R} = \partial e/\partial \mathbf{R} = 0$. This is a strong assumption and will be satisfied only in an idealized situation because this is equivalent to require the uniqueness of the local rest frame even though we can have two independent flows. This is not likely in realistic hydrodynamic evolution. Thus, the argument developed here seems to be valid for the case where the chemical potential of a conserved charge is extremely large and the contribution from anti-particles are negligibly small compared to that of particles. The same idealization is used for the kinetic derivation in [15].

In the present argument, we have ignored the effect of dissipations, which is necessary to describe more realistic fluids. Usually, such an effect is taken into account by introducing the so-called Rayleigh dissipation function. In this case, however, Noether's theorem associated with the invariance of actions cannot be obtained. Recently, it was shown that the Navier-Stokes-Fourier Equation can be formulated in the framework of the stochastic variational method [12] and the stochastic generalization of Noether's theorem is still applicable in this formulation. Thus, it is interesting to investigate whether this method is still useful to derive relativistic dissipative hydrodynamics.

Acknowledgments The authors acknowledge financial supports by CNPq, FAPERJ and PRONEX.

References

1. I.N. Mishustin, O. Scavenius, Phys. Rev. Lett. **83**, 3134 (1999)
2. D.T. Son, Phys. Rev. Lett. **84**, 3771 (2000)
3. Y. Lallouet, D. Davesne, C. Pujol, Phys. Rev. **C67**, 057901 (2003)
4. K. Paech, S. Pratt, Phys. Rev. C **74**, 014901 (2006)
5. C.E. Aguiar, T. Kodama, J. Phys. G **32**, 179 (2006), Fraga, E.S.
6. S. Pluman, et al. Phys. Lett. B **689**, 18 (2010),
7. J. Peralta-Ramos, G. Krein, Phys. Rev. C **84**, 044904 (2011)
8. C. Herold, et al., Phys. Rev. **C87**, 014907 (2013). and references therein
9. H.-T. Elze, et al. J. Phys. **G25**, 1935 (1999),
10. See, for example, Koide, T. Phys. Rev. C87, 034902 (2013), and references therein
11. A. Achterberg, Phys. Rev. **A28**, 2449 (1983)
12. For the possible modification of the variational principle to include the effect of dissipation, see, for example, Koide, T. and Kodama, T. J. Phys. A45, 255204 (2012) and references therein
13. For the non-relativistic limit of relativistic hydrodynamics, see, Koide, T., Ramos, R. O. and Vicente, Braz, G. S. J. Phys. 45, 102 (2015)
14. T. Takabayasi, Prog. Theor. Phys. **13**(2), 222 (1955)
15. G. Gatoff, A.K. Kerman, T. Matsui, Phys. Rev. **D36**, 114 (1987)