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GENERAL AND APPLIED PHYSICS

Particle Transmission Through a Barrier of Time-Dependent Opacity

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Abstract We discuss the one-dimensional scattering of an arbitrary incident wave packet by a localized potential subjected to a time-dependent perturbation. It is shown that an exact solution can be obtained for an ample class of time dependencies provided that they are smaller than the unperturbed potential. Simpler solutions can also be obtained (under particular conditions) by looking for the incident wave which produces a specified transmitted wave packet.

Keywords Time-dependent potentials · Scattering theory

1 Introduction

Transmission or reflection of a particle going through a time-dependent potential barrier is hardly mentioned in quantum-mechanics textbooks. This is not surprising, since those problems are difficult to treat in the framework of basic quantum physics since they do not lend themselves to exact analytical investigation. Moreover, examples included in those books must be in some way connected to the physicist's everyday life, there being little room for subjects of scant practical applications or mere academic interest.

This situation has been gradually changed in the last 30 or 40 years, there being now a sizable amount of practical problems in which, in some way or another, time dependencies play an important role [1–8]. Unfortunately, those problems are difficult to treat and tend to produce approximate relations or involve a good deal of computational work. However, as shown in this paper, there are time-dependent problems with the solutions that can be worked out by purely analytical methods. Although these potentials may not be realistic enough, their extreme generality (in time dependencies and incident waves) may still be useful in presenting features that are common to other problems of this kind.

2 Amplitude of the Transmitted Waves

In this paper, we shall consider the potential barrier

$$V(x,t) = K\,(1+\alpha f(t))\,\psi(0,t)\delta(x), \tag{1}$$

where K is a positive constant, α is a real number, and $f(t)$ is a transitory perturbation [$f(t) \to 0$ for $t \to \pm\infty$]. Due to the nature of the problem, we can safely assume the existence of a wavefunction satisfying the condition

$$\lim_{t\to\pm\infty} \psi(x,t) \to 0, \tag{2}$$

for all fixed values of x.

The Schrödinger equation for the potential energy (1) is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + K\,(1+\alpha f(t))\,\psi(0,t)\delta(x) = i\hbar\frac{\partial\psi(x,t)}{\partial t}. \tag{3}$$

Introducing the Fourier transforms

$$\psi(x,t) = \int_{-\infty}^{\infty} \phi(x,\omega)e^{-i\omega t}\,\mathrm{d}\omega, \tag{4}$$

$$f(t) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}\,\mathrm{d}\omega, \tag{5}$$

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multiplying (3) by $e^{i\omega t}/(2\pi)$ and integrating its second member from $-\infty < t < \infty$, we get

$$\frac{i\hbar}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\psi(x,t)e^{i\omega t}\,\mathrm{d}t = \frac{i\hbar}{2\pi}\psi(x,t)e^{i\omega t}\Big|_{-\infty}^{\infty} + \frac{i\hbar}{2\pi}\int_{-\infty}^{\infty}-i\omega\psi(x,t)e^{i\omega t}\,\mathrm{d}t = \hbar\omega\phi(x,\omega), \tag{6}$$

where we have used (2) and the inverse Fourier transform of (4).

Using (6) and the convolution theorem in the first member of (3), we are left with

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2} + K\left(\phi(0,\omega) + \alpha\int_{-\infty}^{\infty}G(\omega-\omega')\phi(0,\omega')\right) \times\delta(x) = \hbar\omega\phi(x,\omega). \tag{7}$$

For $x \neq 0$, (7) gives

$$\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2} + \hbar\omega\phi(x,\omega) = 0, \tag{8}$$

with the result

$$\phi(x,\omega) = \begin{cases} A_+e^{ikx} + B_+e^{-ikx} & (x>0,\omega>0) \\ C_+e^{-kx} & (x>0,\omega<0) \\ A_-e^{ikx} + B_-e^{-ikx} & (x<0,\omega>0) \\ C_-e^{kx} & (x<0,\omega<0) \end{cases}, \tag{9}$$

where

$$k(\omega) = \sqrt{\frac{2m|\omega|}{\hbar}} \qquad (-\infty<\omega<\infty), \tag{10}$$

and $A_\pm$, $B_\pm$ and $C_\pm$ are arbitrary functions of ω.

Let us now consider the boundary conditions at $x = 0$: from the continuity of $\psi(x,t)$, we have $\phi(0^+,\omega) = \phi(0^-,\omega) = \phi(0,\omega)$, or from (9),

$$\phi(0,\omega) = \begin{cases} A_+ + B_+ = A_- + B_- & (\omega>0) \\ C_+ = C_- \equiv C & (\omega<0). \end{cases} \tag{11}$$

Integrating now (7) from $x = -\epsilon$ to $x = \epsilon$ and taking $\epsilon \to 0$, we obtain

$$\frac{\partial}{\partial x}\phi(0^+,\omega) - \frac{\partial}{\partial x}\phi(0^-,\omega) = \frac{2mK}{\hbar^2} \times\left(\phi(0,\omega) + \alpha\int_{-\infty}^{\infty}G(\omega-\omega')\phi(0,\omega')\,\mathrm{d}\omega'\right). \tag{12}$$

The first member of (12) is obtained from (9). For $\omega > 0$, we have

$$ik(A_+ + B_- - B_+ - A_-) = 2ik\left(\phi(0,\omega) - (B_+ + A_-)\right), \tag{13}$$

whereas for $\omega < 0$ the result is

$$-2k\phi(0,\omega). \tag{14}$$

Summing up, we have

$$\frac{\partial}{\partial x}\phi(0^+,\omega) - \frac{\partial}{\partial x}\phi(0^-,\omega) = \begin{cases} 2ik\left(\phi(0,\omega) - (B_+ + A_-)\right) & (\omega>0) \\ -2k\phi(0,\omega) & (\omega<0) \end{cases}. \tag{15}$$

Let us now introduce the following definitions:

$$\begin{cases} y(\omega) & = \phi(0,\omega) \\ X(\omega) & = \begin{cases} B_+ + A_- & (\omega>0) \\ 0 & (\omega<0) \end{cases} \\ \text{where} & B_+ + A_- \text{ is the sum of the incident amplitudes at } x = 0, \\ g(\omega) = & \begin{cases} 2ik(\omega) & (\omega>0) \\ -2k(\omega) \end{cases} \\ \text{or, in} & \text{other words, } g(\omega) = 2i\sqrt{\dfrac{2m\omega}{\hbar}} \\ & (-\infty<\omega<\infty), \text{ with } \sqrt{\omega} = i\sqrt{|\omega|} \text{ for } \omega<0. \end{cases} \tag{16}$$

Using (12), (15), and (16), we can write

$$\left(g(\omega) - \frac{2mK}{\hbar^2}\right)y(\omega) - \frac{2mK}{\hbar^2}\alpha\int_{-\infty}^{\infty}G(\omega-\omega')y(\omega')\,\mathrm{d}\omega' = g(\omega)X(\omega). \tag{17}$$

Calling $\omega_0 = \dfrac{mK^2}{2\hbar 3}$ [$-\hbar\omega_0$ is the energy of the bound state of the potential $-K\delta(x)$], we have $2mK/\hbar^2 = 2\sqrt{2m\omega_0/\hbar}$, canceling the term $2\sqrt{2m/\hbar}$ which appears in both members of (17). Dividing (17) by $i\sqrt{\omega} - \sqrt{\omega_0}$, we obtain at last

$$y(\omega) = \alpha r(\omega)\int_{-\infty}^{\infty}G(\omega-\omega')y(\omega')\,\mathrm{d}\omega' + t(\omega)X(\omega), \tag{18}$$

where

$$t(\omega) \equiv \frac{\sqrt{\omega}}{\sqrt{\omega} + i\sqrt{\omega_0}}; \tag{19}$$

and

$$r(\omega) \equiv \frac{-i\sqrt{\omega_0}}{\sqrt{\omega} + i\sqrt{\omega_0}} \tag{20}$$

are respectively the well-known transmission and reflection amplitudes for the "unperturbed" potential $K\delta(x)$. Equation (18) is the main result of this work.

3 Calculation of the Wavefunctions

Let us consider for the sake of simplicity an incident wave packet coming form $x < 0$ which corresponds to setting $B_+ = 0$ in (9). From the definition (16), we have

$$y(\omega) = \begin{cases} A_+(\omega) = A_-(\omega) + B_-(\omega) & (\omega>0) \\ C(\omega) & (\omega<0) \end{cases}$$

$$X(\omega) = \begin{cases} A_-(\omega) & (\omega>0) \\ 0 & (\omega<0). \end{cases}.$$

Once we have solved the integral equation (18), we can express A_+, A_-, B_-, and C in terms of y and X. The result is $A_+ = y(\omega > 0)$, $A_- = X(\omega > 0)$, $B_- = y - X(\omega > 0)$, and $C = y(\omega < 0)$. From (9), it follows that

$$\phi(x,\omega) = \begin{cases} y(\omega)e^{ikx} & (x > 0, \omega > 0) \\ y(\omega)e^{-k|x|} & (\text{for all } x, \omega < 0) \\ Xe^{ikx} + (y - X)\, e^{-ikx} & (x < 0, \omega > 0) \end{cases} .$$

Finally, from (15), we have the wavefunction $\psi(x,t)$ expressed in terms of $X(\omega)$ and $y(\omega)$:

$$\psi(x,t) = \int_{-\infty}^{0} y(\omega)e^{-k|x|}e^{-i\omega t}\,\mathrm{d}\omega + \begin{cases} \int_0^\infty y(\omega)e^{i(kx-\omega t)}\,\mathrm{d}\omega & (x > 0) \\ \int_0^\infty X(\omega)e^{i(kx-\omega t)}\,\mathrm{d}\omega \\ \quad + \int_0^\infty (y - X)\, e^{-i(kx+\omega t)}\,\mathrm{d}\omega & (x < 0) \end{cases} . \tag{21}$$

The first thing we notice in (21) [see also (18)] is that we can have in $y(\omega)$ energies that are absent in $X(\omega)$. This is a consequence of the lack of energy conservation expressed by the kernel $G(\omega - \omega')$. Only when $\alpha G(\omega - \omega') = 0$, we will have the same energies in the incident and transmitted waves. This spectral change may include negative energies [see (21)] corresponding to transient states temporarily bounded at $x = 0$. Although those states do not appear in the shape ($t \to \infty$) of the transmitted or reflected wave packets, their contributions are essential to the correct solution of (18).

4 Solution of the Integral Equation

Let us make two preliminary considerations. In the first place, it is convenient, before proceeding with the calculations, to normalize the time dependent part of the potential by requiring that

$$\int_{-\infty}^{\infty} |G(\omega)|\,\mathrm{d}\omega = 1. \tag{22}$$

With this choice (which involves no loss in generality), we will always have $|f(t)| \le 1$ [see (4)], so that the strength of the potential perturbation in (1) will be controlled by the constant α alone.

The other point deserving our interest is the behavior of the reflected amplitude $r(\omega)$. From (19), we obtain

$$|r(\omega)| = \begin{cases} \dfrac{\sqrt{\omega_0}}{\sqrt{\omega + \omega_0}} & (\omega > 0) \\ \dfrac{\sqrt{\omega_0}}{\sqrt{|\omega|} + \sqrt{\omega_0}} & (\omega < 0) \end{cases} ,$$

so that

$$|r(\omega)| \le 1 \qquad (-\infty < \omega < \infty). \tag{23}$$

We can now turn our attention to the solution of (18). Taking for $y(\omega)$, the power series

$$y(\omega) = \sum_{n=0}^{\infty} \alpha^n y_n(\omega) \tag{24}$$

and substituting (24) into (18), we obtain

$$\begin{cases} y_0(\omega) &= t(\omega)X(\omega) \\ y_{n+1}(\omega) &= r(\omega)\int_{-\infty}^{\infty} G(\omega-\omega')y_n(\omega')\,\mathrm{d}\omega' \quad (n = 0, 1, \ldots). \end{cases} \tag{25}$$

The convergence of (24) can be investigated with the help of (22) and (23). In fact, assuming

$$|y_0(\omega)| \le |X(\omega)| \le M \qquad (0 < \omega < \infty),$$

where M is a suitable constant, we have from (22), (23), and (25)

$$|y_{n+1}(\omega)| \le |r(\omega)| \int_{-\infty}^{\infty} |G(\omega-\omega')||y_n(\omega')|\,\mathrm{d}\omega' \le | \int_{-\infty}^{\infty} |G(\omega-\omega')||Y_n(\omega')|\,\mathrm{d}\omega'. \tag{26}$$

Using now (26), we can conclude that

$$|y_n(\omega)| \le M \qquad (n = 1, 2, \ldots), \tag{27}$$

with the result

$$\sum_{n=0}^{\infty} |\alpha^n y_n(\omega)| \le M \sum_{n=0}^{\infty} |\alpha|^n \qquad (-\infty < \omega < \infty),$$

which guarantees the absolute and uniform convergence of (24) for $|\alpha| < 1$.

Since we are assuming $|f(t)| < 1$, this condition for α implies in $1 + \alpha f(t) > 0$, that is, the potential must be repulsive for all values of time.

One last observation: The case of an attractive unperturbed potential cannot be treated by simply substituting K by $-K$ in (8), since in this case the function $r(\omega)$ will have a pole at $\omega = -\omega_0$. This fact suggests that for $K < 0$ we have a particle capture in which the wavefunction tends to $e^{-\lambda t}e^{i\omega_0 t}$ ($\lambda = m|K|/\hbar$) for $t \to \infty$, which contradicts the hypothesis (2).

5 Inverse Scattering and Simpler Exact Solutions

In the absence of the perturbing potential, we have $y(\omega) = t(\omega)X(\omega)$ which gives $y(\omega)$ in terms of $X(\omega)$ or the other way around [provided $\lim_{\omega\to 0} y(\omega)/\sqrt{\omega}$ is a finite number]. Looking at (18), it may appear that it is possible to obtain $X(\omega)$ from $y(\omega)$ through a simple integration. However, this is not true since we must have $X(\omega) = 0$ for $\omega < 0$ [see (16)], which restricts the possible choices of $y(\omega)$. Even then there are ample choices of $y(\omega)$ for which both $y(\omega)$ and $\int_{-\infty}^{\infty} G(\omega-\omega')y(\omega')\,\mathrm{d}\omega'$ vanishes for $\omega < 0$. For example,

$$\begin{cases} G(\omega) &= 0 \text{ for all } |\omega| \geq \omega_1, \text{[naturally } G^*(\omega) = G(-\omega)] \\ y(\omega) &= 0 \text{ for } \omega < \omega_2, \omega_2 > \omega_1 \end{cases}. \tag{28}$$

To give an example, we can consider the case of an oscillating potential. Since the shape of $G(\omega)$ is arbitrary, we can consider the limit case in which

$$f(t) \to \cos(\omega_1 t), \quad G(\omega) \to \frac{1}{2}\left(\delta(\omega-\omega_1)+\delta(\omega+\omega_1)\right), \tag{29}$$

with the result

$$X(\omega) = \frac{1}{t(\omega)}\left(y(\omega) - \frac{\alpha}{2}\left(y(\omega-\omega_1)+y(\omega+\omega_1)\right)\right). \tag{30}$$

From (30), we see that in order to obtain the transmitted wave $y(\omega)$ we need, besides the "normal" incident wave $y(\omega)/t(\omega)$, two sidebands $[\alpha r(\omega)/2t(\omega)]y(\omega \pm \omega_1)$, which although totally reflected are essential to obtain the desired result. These solutions hold for all values of α.

6 The Question of the Energy

This scattering is obviously inelastic since the particle an exchange energy with the system responsible for the time-dependent potential. It is interesting then to calculate the energy change between the initial, $\psi(x, -\infty)$ and the final $\psi(x, \infty)$ states where $\psi(x, -\infty)$ contains the incident wave and $\psi(x, +\infty)$ the reflected and transmitted ones. Using the average energy

$$E(t) = \frac{\int_{-\infty}^{\infty} \psi^* \left(i\hbar \frac{\partial \psi}{\partial t}\right) \mathrm{d}x}{\int_{-\infty}^{\infty} |\psi|^2 \,\mathrm{d}x}$$

we obtain from (21) after somewhat lengthy but straightforward calculations

$$\Delta E = \langle E(\infty) - E(-\infty)\rangle = \frac{\int_0^{\infty}(2|y|^2 - Xy^* - X * y)\,\mathrm{d}\omega}{\int_0^{\infty} k|X(\omega)|^2\,\mathrm{d}\omega} \tag{31}$$

To give a simple example, we can consider the case of the unperturbed potential. Letting $y = tX$ in the numerator of (31), we obtain

$$2|t|^2|X|^2 - |X|^2(t+t^*) = 2|X|^2(|t|^2 - \Re t),$$

Using now (19), we obtain

$$t = \frac{\sqrt{\omega}}{\sqrt{\omega} + i\sqrt{\omega_0}} = \sqrt{\omega}\frac{\sqrt{\omega} - i\sqrt{\omega_0}}{\omega + \omega_0} \quad \text{and}$$
$$|t|^2 = \frac{\omega}{\omega+\omega_0} = \Re t$$

and the expected result $\Delta E = 0$.

7 Summary and Final Comments

The physics of one-dimensional scattering by a localized potential $V(x) = K\delta(x)$ has been understood since the early days of quantum mechanics. Here, we have extended the analysis to let the opacity of the potential be time dependent, with the extension $K \to K(1 + \alpha f(t))$. The time-dependent perturbation $K\alpha f(t)$ is restricted only by the conditions $|\alpha f(t)| < 1$ $(-\infty > t > \infty)$ and $K > 0$. The more complicated case of an attractive potential, with $K < 0$, will be considered in a future study.

As (21) shows, to determine the time-dependent wave function $\psi(x, t)$, we have to solve the integral (18). The solution is provided by the recursive (25) and (26), which generate an absolutely, uniformly convergent series. The scattering process can be shown conserve the total probability and even the local density. In contrast, as expected, energy is not conserved, its variation being related to the solution of the integral equation (18) by the explicit relation (31).

Although the localized potential we have studied may be insufficiently realistic to afford practical applications, we expect time-dependent potentials with longer spatial range to share the central features of the scattering properties stemming from (1). For this reason, we expect the general character of the solution obtained in this paper to provide insight into the behavior of particles subject to time-dependent scattering potentials.

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