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GENERAL AND APPLIED PHYSICS

Self-Similar Asymptotic Optical Waves in Quintic Nonlinear Media with Distributed Coefficients

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Abstract The self-similar asymptotic optical waves propagating in quintic nonlinear media with distributed coefficients are investigated. These optical waves are predicted to exist in (i) normal dispersion and self-focusing quintic nonlinear media and (ii) anomalous dispersion and self-defocusing quintic nonlinear media. The possibility of controlling the shape of output asymptotic optical waves is demonstrated. The analytical results are confirmed by numerical simulations.

Keywords Asymptotic optical wave · Numerical simulations · Quintic nonlinear media

1 Introduction

In the past decades, the optical similaritons in gain amplifier systems have been studied extensively due to their potential applications in nonlinearity and dispersion management systems [1–3], both in exact analytical and asymptotic forms. These similaritons possess many attractive features that make them potentially useful for various applications in fiber-optic telecommunications and photonics, since they can maintain their overall shapes but allow their amplitudes and widths to change with the modulation of the system's parameters such as dispersion, nonlinearity, gain, and inhomogeneity. The dynamics of optical self-similar wave solutions is usually governed by the nonlinear Schrödinger equation (NLSE), where the self-similarity of the solutions has been allowed by reducing the governing equations to ordinary differential equations or even algebraic equations. Generally speaking, optical similaritons can be divided into two categories. The first category is the exact optical similaritons, which are mainly described by the exact solitary-wave solutions, including the bright and dark soliton solutions, the quasisoliton solutions, the nonlinear Bloch waves, and the solitons on the continuous-wave background [4–11]. However, the existence of these similaritons requires a delicate balance between the system parameters such as dispersion, nonlinearity, gain, and inhomogeneity. These requirements are, sometimes, difficult to realize in real applications. The second category is the asymptotic optical similaritons, which are mainly described by the parabolic, Hermite-Gaussian, and hybrid functions [12–17] and exist in a wide range of gain amplifier when the strict balance of system parameters is broken. Among these optical similaritons, the asymptotic parabolic similariton has attracted more attention due to the following reasons: (i) they can be easily generated from arbitrary input optical waves; (ii) they have strict linear chirps, which are important to the effective compression of optical waves; (iii) their stabilities are guaranteed even with the high power input; and so on. These advantages usually do not belong to the exact optical similaritons. Similaritons also exist in other fields of physics such as Bose-Einstein condensates and plasmas [18–20].

It should be noted that, to date, most of the previous theoretical and experimental studies are focused on optical pulses in the media with cubic Kerr nonlinearity, i.e., the refractive index is $n=n_0+n_2I$, where n_0, n_2, and I are the linear refractive index coefficient, the cubic nonlinearity coefficient, and the pulse's intensity, respectively, usually, positive n_2 for self-focusing nonlinearity and negative n_2 for self-defocusing nonlinearity. However, when the pulse's peak intensity is sufficiently large, the field-induced change of the refractive index is no longer described by the above usual Kerr-type

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nonlinearity, and a higher-order nonlinear effect such as the quintic nonlinearity, in this situation, should be taken into account [1, 21]. The refractive index is $n=n_0+n_2I-n_4I^2$, where n_4 is the quintic coefficient of the Kerr-type nonlinearity which may assume positive or negative values. In experiments, the cubic-quintic nonlinearities can be obtained by doping a fiber with two appropriate semiconductor materials [22, 23]. In many fields of nonlinear science, one can deal with a pure quintic nonlinear Schrödinger equation (QNLSE) model, for example, in nonlinear optics under power-law nonlinearity (which is known in various materials, such as semiconductors); in Bose-Einstein condensates with the three-body interaction. Recently, a generic model for the quintic nonlinearity has been realized in a centrosymmetric nonlinear medium doped with resonant impurities in the limit of a large light carrier frequency detuning from the impurity resonance [24].

On the other hand, in the realistic fiber optics, there will always be some nonuniformity factors. They may arise from the variation of the fiber geometry, e.g., diameter fluctuation, which would influence various effects such as loss (gain) and phase modulation. These effects can be modeled by making dispersion, nonlinearity, and gain to be space-dependent, i.e., inhomogeneous along the propagation distance. By considering these, one should study the so-called generalized nonautonomous NLSE, which was proposed by Serkin et al. [4–6]. Furthermore, due to the unavoidable experimental fluctuations in the coefficients, it is not possible to derive an analytical solution of the optical wave equation. Therefore, one may need to search for some other forms of solutions.

In this paper, we study the self-similar asymptotic optical waves propagating in quintic nonlinear media with distributed coefficients. Our results show that the asymptotic self-similar waves can exist in (i) normal dispersion and self-focusing quintic nonlinear media and (ii) anomalous dispersion and self-defocusing quintic nonlinear media. The possibility of controlling the shape of output asymptotic optical waves is demonstrated. The analytical results are confirmed by numerical simulations.

2 The Model and Its Reduction

By considering inhomogeneity along the propagation distance, the respective governing equation for the paraxial optical wave propagating in a pure quintic medium can be written as

$$i\frac{\partial u}{\partial z}-\frac{\beta_2(z)}{2}\frac{\partial^2 u}{\partial \tau^2}-\delta(z)|u|^4u=\frac{ig(z)}{2}u, \tag{1}$$

where $u(z, \tau)$ is the complex envelope of the electrical field, with z and τ being the normalized propagation distance and the retarded time, respectively. The distributed parameters $\beta_2(z)$ and $g(z)$ represent the second-order dispersion and amplification, respectively, while $\delta(z)$ is the quintic nonlinear coefficient of the media.

Besides nonlinear optics, Eq. (1) also appears in Bose-Einstein condensates, where z and τ, respectively, represent the time and spatial coordinate, $u(z, \tau)$ is the wave function, $\beta_2(z)$ stands for the diffraction coefficient, $\delta(z)$ describes the three-body interaction, and $g(z)$ is the gain or loss coefficient, which is phenomenologically incorporated to account for the interaction of atomic or thermal clouds. In this case, Eq. (1) can be derived by setting the s-wave scattering length $a_s(t)$ to zero via the Feshbach resonance technique [25]. The demonstration by Inouye et al. [26] shows that matter waves can be amplified and maintain their phase in Bose-Einstein condensates. Therefore, the results obtained in this paper that optical waves can be amplified in the quintic nonlinear media could also be applied to Bose-Einstein condensates. The QNLSE also appears in general NLSE-type systems near the transition from supercritical to subcritical bifurcations [27, 28], pattern formation [29], and dissipative solitons [30].

Note that the construction of exact solutions in the form of chirped self-similar Townes solitons for Eq. (1) has been demonstrated in ref. [31]. However, there is no work on Eq. (1) for obtaining self-similar asymptotic solutions, which have important applications in many fields of nonlinear science, especially in nonlinear optical-fiber amplifier. For this reason, in this work, we will construct the self-similar asymptotic optical pulses for Eq. (1), which are the first to our knowledge.

Using the ansatz $u(z, \tau)=\rho(z, \tau)\exp[i\Phi(z, \tau)]$ with $\rho(z, \tau)$ and $\Phi(z, \tau)$ being real functions, the self-similar linearly chirped solution of Eq. (1) has the form

$$\rho(z,\tau)=\exp\left(\frac{1}{2}G\right)A(z)F(T), \tag{2}$$

$$\Phi(z,\tau)=\phi(z)+C(z)(\tau-\tau_c)^2, \tag{3}$$

where $G\equiv\int_0^z g(z')\,dz'$ and $T=(\tau-\tau_c)/w(z)$ with $w(z)$ being the width of the solution.

Equations (1)–(3) yield the following differential equations for the functions $A(z)$, $C(z)$, $w(z)$ and $\phi(z)$ as

$$w_z+2\beta_2C_w=0, \tag{4}$$

$$A_z-\beta_2CA=0, \tag{5}$$

$$\frac{\beta_2}{2\delta w^2A_4}\exp(G)\frac{\partial^2F}{\partial T^2}+\frac{\phi_z+\left(C_z-2\beta_2C^2\right)w^2T^2}{\delta A^4\exp(2G)}F+F^5=0. \tag{6}$$

From Eqs. (4) and (5), we obtain

$$C=-\frac{w_z}{2\beta_2w},\quad A=\frac{1}{\sqrt{w}}. \tag{7}$$

Substituting Eq. (7) into Eq. (6), one gets

$$\frac{\beta_2}{2\delta}\exp(-2G)\frac{\partial^2 F}{\partial T^2}+\frac{\phi_z w^2}{\delta\exp(2G)}F+\frac{(w_z\beta_{2z}-\beta_2 w_{zz})w^3}{2\delta\beta_2^2\exp(2G)}T^2F+F^5=0. \tag{8}$$

To obtain the asymptotic solution of Eq. (1), we consider the situation when the relative strength of the diffraction term is much less than that of the quintic nonlinearity (the initial power is large enough). In this case, the first term on the left-hand side of Eq. (8) can be neglected. Without the loss of generality, we let $\phi_z w^2\exp(-2G)/\delta\equiv\mu$, and Eq. (8) can be rewritten as

$$F^4+\mu-KT^2=0, \tag{9}$$

at $|T|\leq\sqrt{\mu/K}$, and $F(T)=0$ otherwise, where the following relations are satisfied:

$$\frac{\beta_2}{\delta\exp(2G)}\ll 1,\quad K=\frac{w^3}{2\delta\beta_2^2\exp(2G)}(\beta_2 w_{zz}-\beta_{2z}w_z), \tag{10}$$

with μ, $K<0$. Note that the reduction of Eq. (8) to Eq. (9) is essentially the same as that produced by the Thomas-Fermi approximation for the ground-state solution of the one-dimensional Gross-Pitaevskii equation with the harmonic potential [25].

3 Self-Similar Asymptotic Optical Waves

In this section, we will investigate the properties of the asymptotic optical waves to Eq. (1).

We now solve Eq. (9) which admits the following parabolic solution for $F(T)$:

$$F^2=\sqrt{-\mu+KT^2}, \tag{11}$$

with μ being determined by the input power.

$$P_{\text{in}}=\int_{-\infty}^{\infty}|u(0,\tau)|^2d\tau=\int_{-\sqrt{u/K}}^{\sqrt{u/K}}F^2\mathrm{d}T=-\frac{\mu\pi}{2\sqrt{-K}}. \tag{12}$$

In the following, we consider the simplest case where β_2, δ, and g are constants. In this case, by solving Eq. (10), we have

$$w=\sqrt[4]{\frac{8K\delta\beta_2}{g^2}}\exp\left(\frac{g}{2}z\right). \tag{13}$$

From Eq. (13), we find that $\delta\beta_2<0$. This implies that the asymptotic compact waves can be generated in quintic nonlinear media with the following two cases: (i) normal dispersion and self-focusing quintic nonlinearity ($\beta_2>0$ and $\delta<0$) and (ii) anomalous dispersion and self-defocusing quintic nonlinearity ($\beta_2<0$ and $\delta>0$). These are confirmed by the direct numerical simulations of Eq. (1). Experimentally, the self-similar solutions in both normal and anomalous dispersion regimes are possible in semiconductor double-doped optical fibers, depending on the doping materials, the operating frequency, and the optical pulse intensities [32–34]. Next we mainly focus on the propagation of asymptotic compact waves in the anomalous dispersion and self-defocusing quintic nonlinear media.

One obtains the asymptotic self-similar asymptotic solution after the substitution of Eqs. (2), (3) (7), and (9) into Eq. (1). It is found that the effective width of the compact solution increases exponentially as $\exp(gz/2)$ while the amplitude of the compact solution increases exponentially as $\exp(gz/4)$. At the same time, the condition $\frac{\beta_2}{\delta\exp(2G)}\ll 1$ can be easily satisfied after a short propagation distance because $\delta\exp(2G)$ varies as $\sim\exp(2gz)$. Therefore, one may control the shape of output asymptotic optical waves in the quintic nonlinear media by appropriately choosing the gain g, which is possible in experiments [12–14].

The analytical predictions are confirmed by direct numerical simulations of Eq. (1), as shown in Fig. 1, where the input pulse is $u(0,\tau)=\exp(-\tau^2/2)/\pi^{1/4}$. It is observed that the general forms of the intensity profile, width, amplitude, and phase are in good agreement with the analytical results. Note that the chirp of the asymptotic compact solution is $g/4$, which is also confirmed by numerical simulations using a fast phase unwrapping algorithm [35], see Fig. 1c.

In addition, we consider other numerical simulations that involve pulses with the same input power but with different

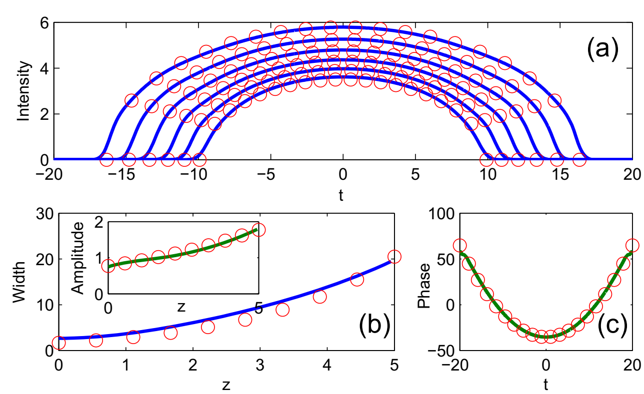

Fig. 1 **a** The evolution of the self-similar asymptotic optical pulse in quintic nonlinear media, starting with the Gaussian input pulse $u(0,\tau)=\exp(-\tau^2/2)/\pi^{1/4}$. From the *top* to *bottom*, the propagation distance is $z=5$, 4.8, 4.6, 4.4, 4.2, and 4, respectively. **b** The pulse's width and amplitude, which are the functions of z. **c** The phase (phase offset is ignored) of the pulse at propagation distance $z=5$. Here, and in the following figures, the *solid lines* and *circles* represent results of the direct numerical simulations of Eq. (1) and the analytical predictions, respectively. The parameters are $\delta=g=-K=-\beta_2=1$ and $\mu=-2/\pi$

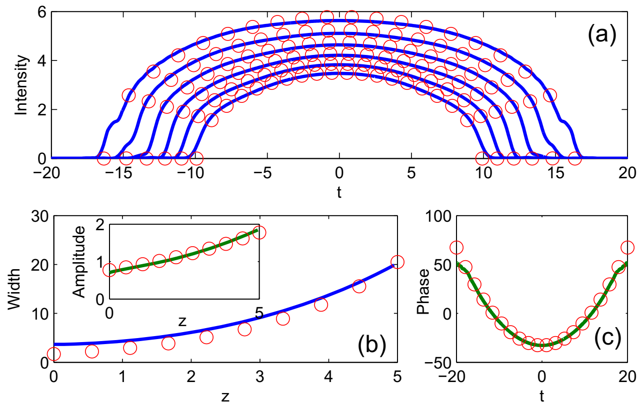

Fig. 2 The same as in Fig. 1 except that the input pulse is $u(0,\tau)=\mathrm{sech}(\tau)/2^{1/2}$, i.e., the hyperbolic secant input pulse

input profiles. Two types of initial input pulses are used, i.e., a hyperbolic secant pulse, $u(0,\tau)=\mathrm{sech}(\tau)/2^{1/2}$, and a super-Gaussian pulse, $u(0,\tau)=\exp(-\tau^6/2)/\{2\pi/[3\Gamma(5/6)]\}^{1/2}$, where $\Gamma(s)$ is a gamma function. The results of numerical simulations and analytical predictions for the general form of the pulse agree well with each other as well, see Figs. 2 and 3. However, we find that this agreement is not as good as that in Fig. 1 after the same units of propagation distance. Therefore, we may infer that the amplifier output corresponding to the Gaussian input profile is closer to the analytical predictions than the output obtained with hyperbolic secant and super-Gaussian profile input.

The asymptotic self-similar compact solution for Eq. (1) can also be generated in the normal dispersion and self-focusing quintic nonlinear media. An example of such calculations is shown in Fig. 4, where the input pulse is $u(0,\tau)=\exp(-\tau^2/2)/\pi^{1/4}$. Similarly, the general forms of the intensity profile, width, amplitude, and phase are in good agreement with the analytical results.

Furthermore, it is possible to obtain an analytical expression for the spectrum of the asymptotic parabolic pulse, defined by $\tilde{u}(z,w\omega)=\int_{\infty}^{\infty}u(z,\tau)\exp(i\omega\tau)\mathrm{d}\tau/\sqrt{2\pi}$ and the stationary phase method [36], which yields

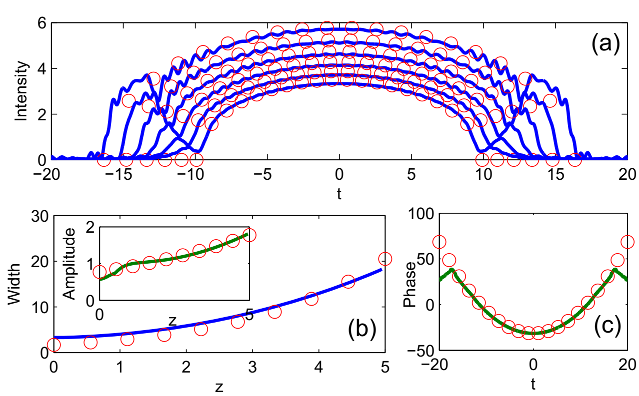

Fig. 3 The same as in Fig. 1 except that the input pulse is $u(0,\tau)=\exp(-\tau^6/2)/\{2\pi/[3\Gamma(5/6)]\}^{1/2}$, i.e., the super-Gaussian input pulse

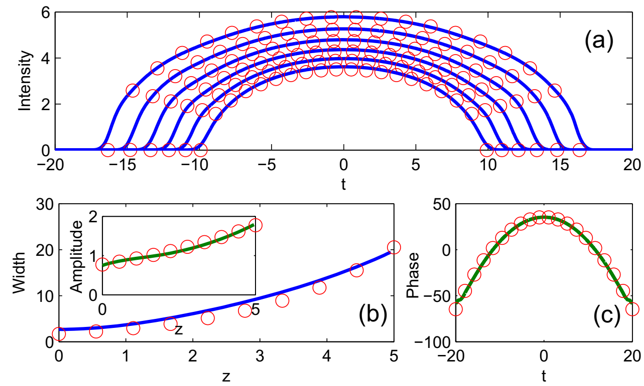

Fig. 4 The same as in Fig. 1 except that $\beta_2=-\delta=1$

$$\left|\tilde{u}(z,\omega)\right|^2=\frac{2\exp(G)}{gw}\sqrt{-\mu+K\frac{\omega^2}{\omega_s^2}}, \tag{14}$$

at $|\omega|\le\omega_s\sqrt{\mu/K}$ and $|\tilde{u}(z,\omega)|^2=0$ otherwise, where $\omega_s=gw/(2|\beta_2|)$ and ω is the spatial frequency. Similar to the spatial distribution, the spectrum of the asymptotic pulse is also a parabolic function [12–14], which was confirmed by our numerical simulations (see Fig. 5).

Compared to the work in ref. [37], there are some significant differences in our paper. First, our model is described by the QNLSE that appears in quintic nonlinear media, while the model in ref. [37] appears in cubic-quintic nonlinear media. Second, in our paper, the effective width of the compact solution increases exponentially as $\exp(gz/2)$ while the amplitude of the compact solution increases exponentially as $\exp(gz/4)$. However, the effective width and amplitude of the compact solution in ref. [37] both increase exponentially as $\exp(gz/3)$. Thus, it is quicker to obtain the compact solution in our situation after the same units of propagation distance. Third, we have considered other numerical simulations that involve pulses with the same input power but with different input profiles and have compared their numerical profiles with the corresponding analytical profiles, while the numerical simulations in ref. [37] only included the Gaussian input pulse.

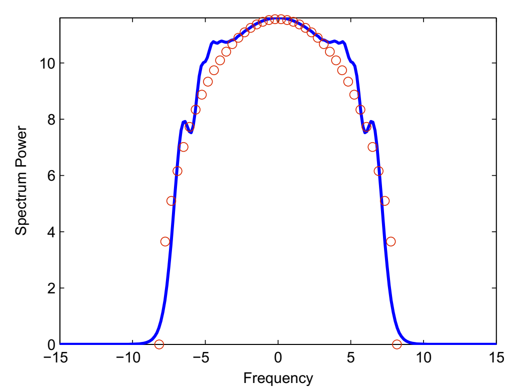

Fig. 5 The output spectrum power as a function of spatial frequency corresponding to Fig. 1 at $z=5$

Further, the spectrum properties of the compact solution were discussed, which has not been studied in ref. [37]. Finally, we found that it is possible to generate the compact solution in anomalous dispersion and self-defocusing quintic nonlinear region.

4 Conclusions

In conclusion, we have shown that the self-similar asymptotic optical waves can be supported in the quintic nonlinear media with distributed coefficients. Our results revealed that the asymptotic self-similar waves can exist in (i) normal dispersion and self-focusing quintic nonlinear media and (ii) anomalous dispersion and self-defocusing quintic nonlinear media. The possibility of controlling the shape of output asymptotic optical waves was demonstrated. The analytical results were confirmed by numerical simulations.

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