Braz J Phys (2016) 46:10–19
DOI 10.1007/s13538-015-0384-1

ATOMIC PHYSICS

Dynamics of a Generalized Two-mode Husimi Function in a Kerr Medium

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Received: 17 July 2015 / Published online: 23 November 2015

Abstract We describe the dynamics of a generalized Husimi function for a two-mode harmonic oscillator representation. The time evolution equation is derived through a normally ordered technique which is shown to be convenient for describing dynamical systems with $SU(2)$ symmetry. We obtain an exact solution in phase space for a quantum nonlinear system described by a Kerr medium initially in a two-mode coherent state. We study in detail the evolution of this system and explain some particular effects exhibited during the process.

Keywords Husimi function · Kerr medium · Phase space · Nonlinear · $SU(2)$ symmetry

1 Introduction

The phase space formulation of spin-like systems was initiated by Stratonovich and Beresin [1, 2] and Agarwal [3], where different types of quasiprobability distribution functions on the sphere $(\theta, \varphi \in \mathscr{S}_2)$ were introduced. These functions are quite useful to visualize nonclassical properties of a collection of two-level atoms [4, 5], and they can be naturally applied for describing atomic systems with $SU(2)$ dynamical symmetries group. Quasiprobability distribution functions are also widely used not only for graphical representation of quantum states in the corresponding phase space but also for analysis of the quantum system's evolution. Since the seminal paper of Wigner [6], where he introduced his famous function, numerous applications in many areas of physics and electronics have taken place [7]. Heisenberg–Weyl quasidistribution functions [8–12] are also of special importance for describing the dynamics of a quantum system in a flat (q, p) space. For a general theory of quasiprobability distribution functions see [13].

The dynamics of quasiprobability distributions in phase space is essentially governed by quantum Liouville-like equations, which in the semiclassical limit reduce to the classical Liouville equation [14–16]. A semiclassical expansion of the evolution equations for the generalized $SU(2)$ Wigner function was considered in Klimov and Romero, and Preciado et al. [17, 18]. The semiclassical expansion parameter depends on the symmetry of the system and is usually inversely related to the number of excitations stored in the system. The main idea consists of an approximation to the exact evolution equation for the quasidistribution function by the Liouville-like equation using a power expansion of the semiclassical parameter. However, this approximation is not valid when interference effects become important. Moreover, such truncated evolution equation does not describe well the dynamics of nonlinear quantum systems [18].

In the present paper, we use a known alternative approach for describing the exact quantum evolution equations based on a normally ordered technique [19] without using the usual machinery of $SU(2)$ quasiprobability distribution functions. From this formulation, we derive equations of

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motion for several $SU(2)$ symmetry Hamiltonian cases which are compared with those found in the literature; this is done for completeness and clarity in our analysis. A familiar Kerr-medium Hamiltonian has been widely solved, but here in contrast we have extended the analysis to the case of two independent harmonic oscillators, in which case we solved the time evolution equation for the corresponding Husimi function analytically.

In Section 2, we summarize the normally ordered method used to derive correspondence rules. These rules are employed to construct evolution equations for Husimi function for several $SU(2)$ symmetry Hamiltonians, whose results are compared with those in the literature. In Section 3, we describe the dynamics of a generalized two-mode Husimi function under a Kerr medium, where a general solution to the equation of motion in phase space has been found and several situations are discussed such as evolution under a double coherent initial state and interference effects. Here, in order to strengthen the consistency of our results, we also study the dynamics of the system under unitary evolution using a convenient representation of two-mode coherent states. This last situation is compared with the evolution of the exact solution found in the previous subsection and its implications are discussed. Concluding remarks are shown in Section 4.

2 The Model and Correspondence Rules

As is well known, the phase space formulation of quantum mechanics has been a remarkable framework in quantum theory, where the concept of quasidistributions plays a fundamental role. As we have mentioned, there exist several types of quasidistribution functions. Here, we only have studied the Husimi function (also known as Q-function) because of its relatively simple definition and, unlike the Wigner function, it is always non-negative, therefore closely resembles a true distribution function.

Also, Husimi function has been useful for several situations; in [20], the Husimi function shows particular advantages compared to the Wigner function. Husimi function has helped to compare exact and semiclassical aspects for a one-dimensional integrable Hamiltonian system exhibiting two resonances [21]. In [22], Husimi function plays an important role for the phase space analysis of quantum phase transitions in the two-dimensional $U(3)$ vibron model for N-size molecules. In our work, we mainly follow Husimi function definition and properties given in [23].

The action of an operator on a density operator $\hat{\rho}$ can be mapped by the action of the corresponding differential operator on the Husimi function by using the so-called correspondence rules [24–27]. If we use them, one can derive the equation of motion for the density matrix $\hat{\rho}$ from $i\dot{\hat{\rho}} = \left[\hat{H}, \hat{\rho}\right]$, into a c-number differential equation for the Husimi function. Here, $\hat{H}$ is the system's Hamiltonian.

In order to get the correspondence rules algebraically, we apply the method used by Wang [28], which briefly consists of expressing the density matrix and physically observable operators in normal ordering. Then, take the expectation value over the corresponding product using the coherent states basis (for details see [19]).

This procedure shows an alternative way to convert the quantum evolution equation into a c-number differential equation for the Husimi function in an algebraic fashion. The method described above exhibits an advantage over traditional forms to derive correspondence rules where standard parametrization becomes important. The use of a quantization kernel to construct quasidistribution functions and then correspondence relations was mainly motivated by Brif and Mann [29]. That approach was used in [17, 18] and [30, 31]. For example, in [31], two orthogonal oscillators were described by the direct product of the corresponding kernel operators but in order to construct the quantization kernel, it is necessary to integrate by some parameters so as to map the dynamics onto the sphere, which leads us to a kernel form not suitable for the case $s = 1$ (P-function). Moreover, integration by parameters induces some ambiguity in the way correspondence relations are defined, thus, the use of an algebraic method independent of parametrization is more attractive and convenient. The efficiency of the method considered here is justified by reviewing limit cases of equations that already exist in literature. The generalized two-mode Husimi function is analyzed by a direct generalization of this technique.

We start by considering two orthogonal harmonic oscillators and make use of the Schwinger representation for the generators of $SU(2)$

$$\hat{S}_+ = \hat{a}_1^\dagger \hat{a}_2, \quad \hat{S}_- = \hat{a}_2^\dagger \hat{a}_1, \tag{1}$$

$$\hat{S}_z = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right), \tag{2}$$

the operator giving the total number of excitations is

$$\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2. \tag{3}$$

In order to represent the operator $\hat{\rho}\hat{S}_+ = \hat{\rho}\hat{a}_1^\dagger \hat{a}_2$ in phase space, we normally order this product, that is

$$\hat{\rho}\hat{a}_1^\dagger \hat{a}_2 = \hat{a}_1^\dagger \hat{\rho}\hat{a}_2 - \left[\hat{a}_1^\dagger, \hat{\rho}\right]\hat{a}_2, \tag{4}$$

and

$$\hat{\rho}\hat{a}_1^\dagger \hat{a}_2 = \hat{a}_1^\dagger \hat{\rho}\hat{a}_2 + \frac{\partial \hat{\rho}}{\partial \hat{a}_1}\hat{a}_2. \tag{5}$$

Next, taking matrix elements in coherent states basis

$$\left\langle \alpha_1, \alpha_2 | \hat{\rho} \hat{a}_1^{\dagger} \hat{a}_2 | \alpha_1, \alpha_2 \right\rangle = \alpha_2 \left\langle \alpha_1, \alpha_2 \left| \hat{a}_1^{\dagger} \hat{\rho} + \frac{\partial \hat{\rho}}{\partial \hat{a}_1} \right| \alpha_1, \alpha_2 \right\rangle = \left(\alpha_1^* \alpha_2 + \alpha_2 \frac{\partial}{\partial \alpha_1} \right) Q, \tag{6}$$

then the correspondence rule takes place

$$\hat{\rho} \hat{S}_+ \rightarrow \left(\alpha_1^* \alpha_2 + \alpha_2 \frac{\partial}{\partial \alpha_1} \right) Q. \tag{7}$$

In the same way, we can obtain the correspondence rules for operators (1–3) (see Appendix A).

Correspondence rules in terms of polar angles were calculated in [32] and [33] under a different technique. Once again, it is worth noticing that the normally ordered method used here provides a direct way of obtaining correspondence rules without using the more exhaustive machinery of $SU(2)$ quasidistribution functions.

2.1 Linear Hamiltonians

In order to show consistency in the construction of the correspondence rules, we give some examples. First, consider the Hamiltonian for the population inversion operator $\hat{S}_z$

$$\hat{H} = \omega \hat{S}_z. \tag{8}$$

In the rest of the paper, we will consider $\hbar = 1$. Using the correspondence relations given in (42, 46), we obtain the equation of motion for the Husimi function

$$\dot{Q}(r, \theta, \varphi, \psi) = -\omega \frac{\partial}{\partial \varphi} Q(r, \theta, \varphi, \psi), \tag{9}$$

where the dot indicates a time derivative. This equation was obtained in [34] based on the quasiclassical approximation; thus, at first order, both approaches are consistent. As another example, we consider the Hamiltonian $\hat{H} = \omega \hat{S}_x$ and by using its respective correspondence relations (40, 41, 44, 45), we obtain

$$\dot{Q}(r, \theta, \varphi, \psi) = \omega \left(-\frac{\cos\varphi}{\sin\theta} \frac{\partial}{\partial \psi} + \cot\theta \cos\varphi \frac{\partial}{\partial \varphi} + \sin\varphi \frac{\partial}{\partial \theta} \right) Q(r, \theta, \varphi, \psi). \tag{10}$$

In this case, the evolution (10) shows a similar structure to that reported in [17] for the Wigner function. It is worth mentioning that for linear Hamiltonians both approaches give the same evolution equations; however, for nonlinear systems, the formulation considered in previous literature is based on the quasiclassical approximation and therefore cannot describe the dynamics of nonlinear quantum systems. The technique used here provides the advantage of obtaining the evolution equation of such systems in a natural way.

2.2 Kerr Medium

It is well known that a field coherent state evolving under the influence of a Hamiltonian linear in the quadrature $\hat{a}$ and $\hat{a}^{\dagger}$ and number operator $\hat{a}^{\dagger}\hat{a}$ evolves to another field coherent state [19]. However, when the interaction is nonlinear the evolution to a Schrödinger cat state from a coherent state can be generated [35, 36]. A nonlinearity of particular interest for generating a Schrödinger cat state is of the Kerr type [37]. The Kerr medium is due to a Hamiltonian that is quadratic in the field number operator. It was shown in [35] that a macroscopic superposition of coherent states with different phases can be generated in the course of the evolution of a single-field mode in a Kerr medium with the Hamiltonian

$$\hat{H}_{Kerr} = \omega \hat{n} + \chi \hat{n}^2. \tag{11}$$

Here, the operators $\hat{a}$, $\hat{a}^{\dagger}$, and $\hat{n} = \hat{a}^{\dagger}\hat{a}$ describe a single-field mode of frequency ω, and χ is the Kerr constant.

The Kerr Hamiltonian (11) deserves special importance because it characterizes some general properties of quantum dynamics. For example, it can be used to generate both quadrature [38] and amplitude squeezing [39] for short times, when the initial wave packet spreads in phase and evolves around the origin in the phase plane [40]. When the phase spread exceeds 2π, the front of the wave packet interferes with its tail. This self interference is a quantum feature that has no classical counterpart [41]. For atomic systems, the effective Kerr Hamiltonian [5] is given by

$$\hat{H}_{eff} = \chi \left[\hat{S}_+ \hat{S}_- + 2\bar{n} \hat{S}_z \right] = \chi \left[\frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right) - \hat{S}_z^2 + (2\bar{n} + 1) \hat{S}_z \right], \tag{12}$$

where $\hat{S}_{\pm}$, $\hat{S}_z$ are spin operators, the total number of excitations is given by operator $\hat{N}$, and $\bar{n}$ is the average number of thermal photons in the cavity. Notice the nonlinearity in (12) due to quadratic term in the population inversion operator $\hat{S}_z$.

Using relations (42, 46) in the Liouville equation, we derive the equation of motion for the Husimi function corresponding to the specific Kerr-type Hamiltonian

$$\hat{H} = \chi \hat{S}_z^2, \tag{13}$$

where χ gives the strength of the interaction. The evolution equation is

$$\dot{Q} = -\chi \left(r^2 \cos\theta \frac{\partial}{\partial \varphi} + \frac{1}{2} r \cos\theta \frac{\partial^2}{\partial r \partial \varphi} - \sin\theta \frac{\partial^2}{\partial \theta \partial \varphi} \right) Q. \tag{14}$$

This expression involves different subspaces labeled by the parameter r. Note that for a fixed subspace, (14) is similar to that reported in [30] where the middle term on the right hand side of (14) describes "diffusion" between different subspaces.

3 Dynamics of a Generalized Two-mode Husimi Function

The Husimi function is widely used for its relatively simple definition, so it is of interest to see how it behaves in a nonlinear medium such as a finite Kerr medium. The Husimi function is defined as $Q(\alpha) = \langle\alpha\left|\hat{\rho}\right|\alpha\rangle$, where $|\alpha\rangle$ denotes a one-mode coherent state [9–12, 23]. For the two-mode Husimi function, we use two-mode coherent states associated with a two-mode realization of $SU(2)$ [42]. The two-mode $SU(2)$ coherent states have been discussed in [43, 44] and are connected with the well known Schwinger realization of the angular momentum algebra in terms of two sets of boson operators. These states are a superposition of a finite number of states of the form $|N, k\rangle$ where N is the total number of excitations in both modes.

The Kerr Hamiltonian is quadratic in the population inversion operator, therefore, the Liouville-von Neumann evolution equation is given by

$$i\frac{\partial\hat{\rho}}{\partial\tau} = \left[\hat{S}_z^2, \hat{\rho}\right], \tag{15}$$

where $\tau = \chi t$ is dimensionless time. If we make use of the correspondence rules (33, 37), averaging over the coherent states and substituting into (15), we obtain the equation of motion

$$\begin{aligned}\frac{\partial Q}{\partial\tau} &= \frac{i}{4}\left[\alpha_1\left(1+2|\alpha_1|^2\right)\frac{\partial}{\partial\alpha_1} + \alpha_1^2\frac{\partial^2}{\partial\alpha_1^2}\right]Q\\ &+\frac{i}{4}\left[\alpha_2\left(1+2|\alpha_2|^2\right)\frac{\partial}{\partial\alpha_2} + \alpha_2^2\frac{\partial^2}{\partial\alpha_2^2}\right]Q\\ &+\frac{i}{2}\left(\alpha_1^*|\alpha_2|^2\frac{\partial}{\partial\alpha_1^*} + |\alpha_1|^2\alpha_2^*\frac{\partial}{\partial\alpha_2^*} + \alpha_1^*\alpha_2^*\frac{\partial^2}{\partial\alpha_1^*\partial\alpha_2^*}\right)Q\\ &+c.c.,\end{aligned} \tag{16}$$

where $Q = Q(\alpha_1, \alpha_2)$ and $c.c.$ means complex conjugate. Note also that (16) corresponds to that reported in [41] for one mode of the field, i.e., with $\alpha_2 = 0$. The exact solution to (16) is given by (see Appendix B)

$$\begin{aligned}Q(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*; \tau) &= \exp\left(-|\alpha_1|^2 - |\alpha_{10}|^2\right)\\ &\times\exp\left(-|\alpha_2|^2 - |\alpha_{20}|^2\right)|S|^2,\end{aligned} \tag{17}$$

where α_{10} (α_{20}) is the complex amplitude for an initial coherent state $|\alpha_{10}\rangle$ $(|\alpha_{20}\rangle)$, respectively, and

$$S = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\left(\alpha_{10}^*\alpha_1\right)^{n_1}}{n_1!}\frac{\left(\alpha_{20}^*\alpha_2\right)^{n_2}}{n_2!}\exp\left[i\frac{\tau}{4}\left(n_1-n_2\right)^2\right]. \tag{18}$$

Solution (18) shows a direct generalization of the results reported in [41], also note the crossed terms in the exponential, which is a clear signature of quantum interference between both oscillators.

In order to study the behavior of the solution discussed above, we use the parametrization given in (38, 39) for two different initial states: a product of coherent states and a Schrödinger cat state, which are the main subject of this section. Note that such parametrization allows us to plot system's behavior on the Bloch sphere in a clearer way.

3.1 Initial State as a Product of Coherent States

We begin by studying the evolution of an initial state given by the product of two coherent states. If we choose $n_1 = N - k$ and $n_2 = k$ in (18), we obtain

$$\begin{aligned}S^* &= \sum_{k=0}^{N}\frac{1}{(N-k)!k!}\left(\alpha_{10}\alpha_1^*\right)^{N-k}\left(\alpha_{20}\alpha_2^*\right)^k\\ &\times\exp\left[-i\frac{\tau}{4}(N-2k)^2\right].\end{aligned}$$

With the initial state oriented along the x-axis, it follows from the substitution of $r = r_0$, $\theta_0 = \frac{\pi}{2}$, $\varphi_0 = 0$ and $\psi_0 = 0$ into (38, 39) that

$$\alpha_{10} = \frac{r_0}{\sqrt{2}}, \qquad \alpha_{20} = \frac{r_0}{\sqrt{2}},$$

and thus the solution reads

$$\begin{aligned}S^* &= \sum_{k=0}^{N}\frac{1}{(N-k)!k!}\left(\frac{r_0\, r}{\sqrt{2}}\right)^N e^{i\frac{\psi}{2}N}e^{i\frac{\varphi}{2}(N-2k)}\\ &\times\left(\cos\frac{\theta}{2}\right)^{N-k}\left(\sin\frac{\theta}{2}\right)^k e^{-i\frac{\tau}{4}(N-2k)^2},\end{aligned} \tag{19}$$

where according to (17)

$$Q(r, \theta, \varphi, \psi; \tau) = e^{-\left(r^2-r_0^2\right)}|S|^2,$$

is the full solution that describes the dynamics of a generalized two-mode Husimi function in terms of Euler angles.

In Fig. 1a, we show a spherical plot of the Husimi function initially in a coherent state. We note in Fig. 1b that at $\tau = \pi/2$, the state evolves into a superposition of two macroscopic states (Schrödinger cat state). Figure 2 is a 3D plot of the Husimi function at $\tau = \pi/2$. We show the results for a coherent amplitude of $r = 1$ in both modes at $\psi = 0$ and for $N = 5$.

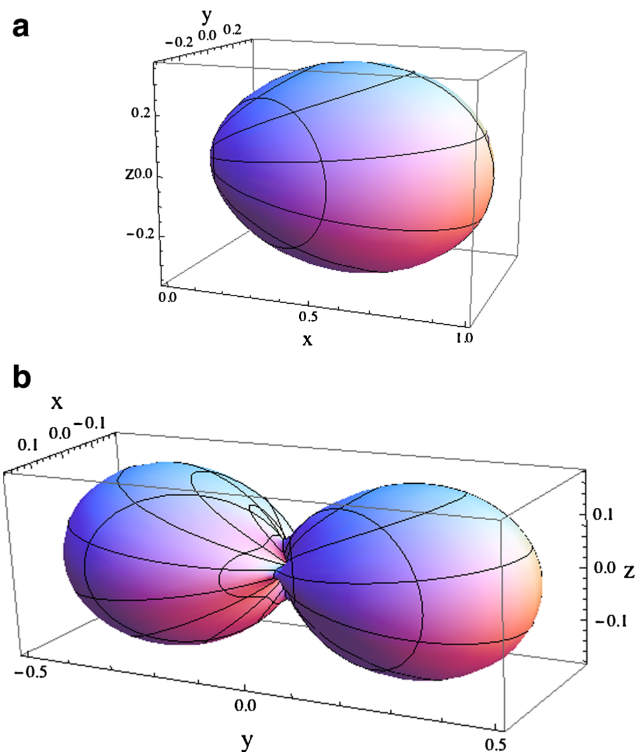

Fig. 1 Spherical polar plots of the Husimi function when the initial state is given by a product of two coherent states for **a** $\tau = 0$ and **b** $\tau = \pi/2$. Both amplitudes are equal ($r = 1$) and $N = 5$. Here, $x \to Q(\theta, \varphi) \sin\theta \cos\varphi$, $y \to Q(\theta, \varphi) \sin\theta \sin\varphi$, $z \to Q(\theta, \varphi) \cos\theta$

3.2 Initial State as a Schrödinger Cat State and Interference Effects

Here, we analyze the behavior of a generalized two-mode Husimi function when the initial state is a Schrödinger cat state. The standard semiclassical treatment of quantum evolution neglects interference effects due to higher order derivatives in the equation of motion (16), however, here we study those terms due to the importance on significant quantum mechanical effects like quantum superposition.

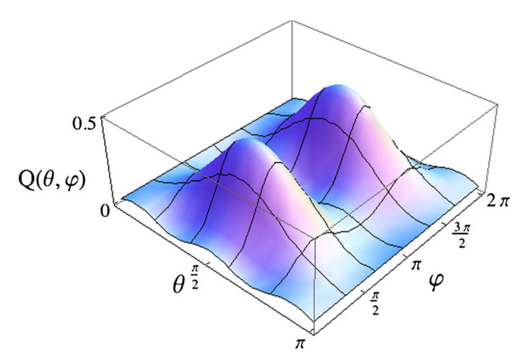

Fig. 2 A 3D plot of the generalized Husimi function under a Kerr medium. The state is given by the product of two coherent states with amplitude $r = 1$, $N = 5$ and the (dimensionless) propagation time $\tau = \pi/2$

The two-mode optical field can be measured by using balanced homodyne detection [45]. In addition, we discuss the issue of generating Schrödinger cat states of a system of spins. Schrödinger cat states are realizable in the experiments on two-level atoms in a high-Q cavity highly detuned from the atomic transition frequency [46].

We take the initial state as a superposition of two diametrically opposed coherent states oriented along the x-axis

$$|\Psi_0\rangle_{cat} = \frac{1}{\sqrt{2}}\left(\left|\Psi_0^{(1)}\right\rangle + e^{i\Phi}\left|\Psi_0^{(2)}\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|\alpha_{10}^{(1)}, \alpha_{20}^{(1)}\right\rangle + e^{i\Phi}\left|-\alpha_{10}^{(2)}, -\alpha_{20}^{(2)}\right\rangle\right),$$

where the Cayley-Klein parameters (38, 39)

$$\alpha_{10}^{(1)}\left(r_0, \frac{\pi}{2}, 0, 0\right) = \frac{r_0}{\sqrt{2}}, \quad \alpha_{20}^{(1)}\left(r_0, \frac{\pi}{2}, 0, 0\right) = \frac{r_0}{\sqrt{2}}, \tag{20}$$

$$\alpha_{10}^{(2)}\left(r_0, \frac{\pi}{2}, \pi, 0\right) = \frac{-i\, r_0}{\sqrt{2}}, \quad \alpha_{20}^{(2)}\left(r_0, \frac{\pi}{2}, \pi, 0\right) = \frac{i\, r_0}{\sqrt{2}}, \tag{21}$$

give the orientation of the two-mode coherent state and Φ is the relative phase of the superposition.

Since the Husimi function is given by

$$Q = \frac{1}{\pi}\left|\langle\alpha_1, \alpha_2|\Psi_0\rangle_{cat}\right|^2, \tag{22}$$

we may express $Q = \frac{1}{\pi}(Q_0 + Q_1)$ as in [5], where

$$Q_0 = \frac{1}{2}\left(\left|\langle\alpha_1, \alpha_2\left|\Psi_0^{(1)}\right\rangle\right|^2 + \left|\langle\alpha_1, \alpha_2\left|\Psi_0^{(2)}\right\rangle\right|^2\right),$$

and

$$Q_1 = \frac{1}{2}\left(\langle\alpha_1, \alpha_2\left|\Psi_0^{(1)}\right\rangle\left\langle\Psi_0^{(2)}\right|\alpha_1, \alpha_2\rangle\, e^{-i\Phi} + c.c.\right).$$

Note that Q_0 corresponds to a situation when there are no interferences and Q_1 gives the interference contribution. Thus, to visualize how the quantum character of the state reflects itself in the properties of the generalized Husimi function, we shall also plot Q/Q_0 on the unit sphere.

We define using (20) and (21)

$$S_{cat1} \equiv \langle\alpha_1, \alpha_2 \,|\, \alpha_{10}^{(1)}, \alpha_{20}^{(1)}\rangle = e^{-(r^2+r_0^2)/2}\sum_{k=0}^{N}\frac{e^{i\frac{\psi}{2}N}e^{i\frac{\varphi}{2}(N-2k)}}{(N-k)!\,k!}\left(\frac{r_0\, r}{\sqrt{2}}\right)^N \times\left(\cos\frac{\theta}{2}\right)^{N-k}\left(\sin\frac{\theta}{2}\right)^k,$$

$$S_{cat2} \equiv \langle\alpha_1, \alpha_2| - \alpha_{10}^{(2)}, -\alpha_{20}^{(2)}\rangle e^{i\Phi} = e^{-(r^2+r_0^2)/2}\sum_{k=0}^{N}(i)^N(-1)^k e^{i\Phi}\frac{e^{i\frac{\psi}{2}N}e^{i\frac{\varphi}{2}(N-2k)}}{(N-k)!\,k!}\left(\frac{r_0\, r}{\sqrt{2}}\right)^N \times\left(\cos\frac{\theta}{2}\right)^{N-k}\left(\sin\frac{\theta}{2}\right)^k.$$

Therefore,

$$Q_0 = \frac{1}{2}\left(|S_{cat1}|^2 + |S_{cat2}|^2\right),$$

and

$$Q_1 = \frac{1}{2}\left(S_{cat1}S^*_{cat2} + S^*_{cat1}S_{cat2}\right).$$

From the definition of the Husimi function (22)

$$\langle\alpha_1,\alpha_2|\Psi_0\rangle_{cat} = \frac{e^{-(r^2+r_0^2)/2}}{\sqrt{2}}\sum_{k=0}^{N}\left[1+(i)^N(-1)^k e^{i\Phi}\right] \times e^{i\frac{\psi}{2}N} \frac{e^{i\frac{\varphi}{2}(N-2k)}}{(N-k)!\,k!}\left(\frac{r_0\, r}{\sqrt{2}}\right)^N\left(\cos\frac{\theta}{2}\right)^{N-k}\left(\sin\frac{\theta}{2}\right)^k,$$

we note three limiting cases of Φ. For $N = 5$ and when $\Phi = 0$ or π, all states contribute to the quasiprobability distribution. When $\Phi = \pi/2$ only states of odd k contribute.

In Fig. 3, we show the spherical plot of Q_0 for an initial state given by a Schrödinger cat state with a coherent amplitude of $r = 1$, $N = 5$, $\psi = 0$, and relative phase $\Phi = 0$. In this case, we must note that no interference effects appear. In Fig. 4, we plot $Q/Q_0 = 1 + Q_1/Q_0$, and see that interference effects come into play, which are in agreement with the previous description of the evolution from a different framework. These interference effects have been studied in different situations [47, 48]. Interference generated in pure-field-state superposition leads to nonclassical oscillations in photon-number distributions and squeezing in quadratures of the field [49]. The point we want to stress here is that when studying the dynamics of systems with $SU(2)$ symmetry, and if interference effects are relevant, then it is suitable to address this problem on the phase space

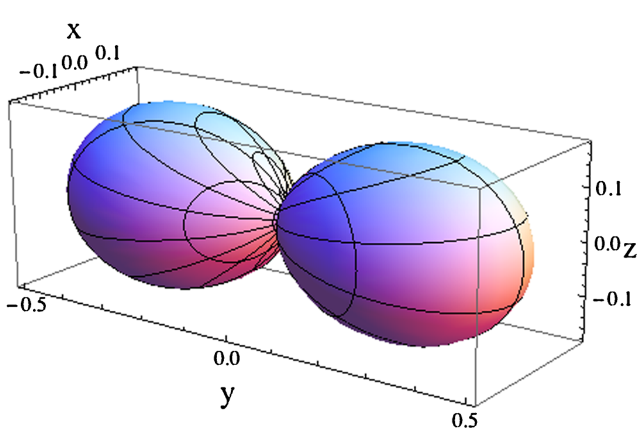

Fig. 3 Spherical polar plot of Q_0 when the initial state is given by a Schrödinger cat state at $\tau = 0$. The coherent amplitude is $r = 1$, $N = 5$, and $\Phi = 0$. Note that no interference structures ares present

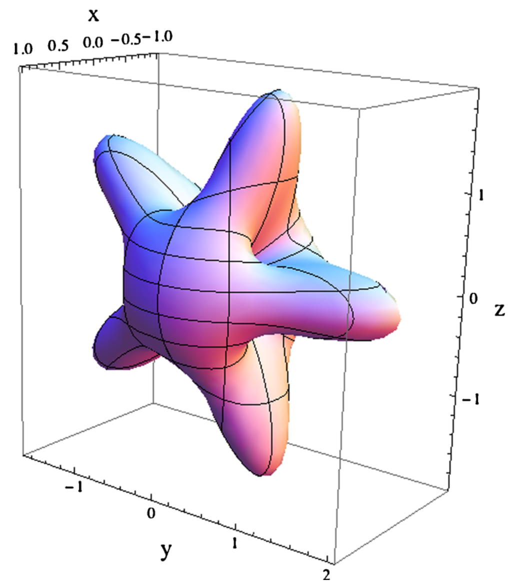

Fig. 4 Spherical polar plot of Q/Q_0 when the initial state is given by a Schrödinger cat state at $\tau = 0$. The coherent amplitude is $r = 1$, $N = 5$, and $\Phi = 0$. Note interference structures which are a clear signature of quantum effects

of the product of two harmonic oscillators and then use the parameterization given by the Cayley–Klein parameters.

3.3 Unitary Evolution

In order to give more consistency to our results, we explore a simple case developed in [5], where according to Schrö–dinger's picture, a unitary evolution generated by the Kerr-type Hamiltonian (13) transforms an atomic coherent state into a superposition of distinct atomic coherent states. In this subsection, we have followed the analysis presented in [5], but instead of the atomic coherent state approach we have used a two-mode coherent state realization. This unitary evolution permits us to have a known result to compare with that one shown in Section 3.2.

Consider a system prepared in a two-mode coherent state of the form [42]

$$|\psi(0)\rangle = |\theta_0, \varphi_0\rangle = \sum_{k=0}^{N}\sqrt{\frac{N!}{k!(N-k)!}}e^{ik\varphi_0} \times \left(\cos\frac{\theta_0}{2}\right)^{N-k}\left(\sin\frac{\theta_0}{2}\right)^k |N,k\rangle. \tag{23}$$

An evolution generated by a linear combination of atomic operators transforms a two-mode coherent state into another one, this is viewed as motion on Bloch's sphere. The two-mode coherent state (23), under the action of the non-linear unitary evolution generated by (13), transforms to

$$|\psi(t)\rangle \equiv \exp[-i\hat{H}t]|\theta_0, \varphi_0\rangle = \sum_{k=0}^{N}\sqrt{\frac{N!}{k!(N-k)!}}e^{ik\varphi_0} \times \left(\cos\frac{\theta_0}{2}\right)^{N-k}\left(\sin\frac{\theta_0}{2}\right)^{k} e^{-i\tau\left(\frac{N^2}{4}-Nk+k^2\right)}|N,k\rangle. \quad (24)$$

From results reported in [37], it can be shown that expression (24) evolves precisely into a Schrödinger cat state for times $\tau_0 = \pi/2$, that is

$$|\psi(\tau_0)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1(\tau_0)\rangle + |\psi_2(\tau_0)\rangle\right), \quad (25)$$

where

$$|\psi_1(\tau_0)\rangle = e^{-i\frac{\pi}{2}\frac{N^2}{4}}e^{-i\frac{\pi}{4}}\left|\theta_0, \varphi_0 + \frac{\pi}{2}N\right\rangle, \quad (26)$$

and

$$|\psi_2(\tau_0)\rangle = e^{-i\frac{\pi}{2}\frac{N^2}{4}}e^{i\frac{\pi}{4}}\left|\theta_0, \varphi_0 + \frac{\pi}{2}N - \pi\right\rangle. \quad (27)$$

We now reload the question of quantum interference but under the perspective of unitary evolution of the atomic cat state (25). For this purpose, we write the Husimi function as

$$Q(\theta, \varphi) = \frac{1}{\pi}\langle\theta, \varphi|\psi(\tau_0)\rangle\langle\psi(\tau_0)|\theta, \varphi\rangle. \quad (28)$$

It is worth noting that in (25), $\langle\psi_1|\psi_2\rangle = 0$, thus the Husimi function can be expressed similarly as in previous subsection

$$Q = \frac{1}{\pi}(V_0 + V_1), \quad (29)$$

where the expressions V_0 and V_1 are given by

$$V_0 \equiv \frac{1}{2}\left(|\langle\theta, \varphi|\psi_1\rangle|^2 + |\langle\theta, \varphi|\psi_2\rangle|^2\right), \quad (30)$$

$$V_1 \equiv \frac{1}{2}\left(\langle\theta, \varphi|\psi_1\rangle\langle\psi_2|\theta, \varphi\rangle + c.c.\right). \quad (31)$$

Note that V_1 will correspond to a situation when there are interferences and V_0 when there are not. Thus, a plot of V/V_0 will exhibit only interference structures. We show this behavior in Fig. 5 for $N = 5$ and at the propagation (dimensionless) time $\tau = \pi/2$. We should note that both descriptions display the same interference behavior. Observe well defined maxima and minima on the surface of the sphere which can be interpreted as interference. For the two-mode case in particular, strong correlations between the modes are responsible for various non-classical effects. Note that for these states it is the superposition that creates the correlations between the modes. Since Schrödinger cat states for harmonic oscillator systems are known to exhibit interesting quantum interferences, thus correlations described above are important as they are a clear manifestation of quantum effects.

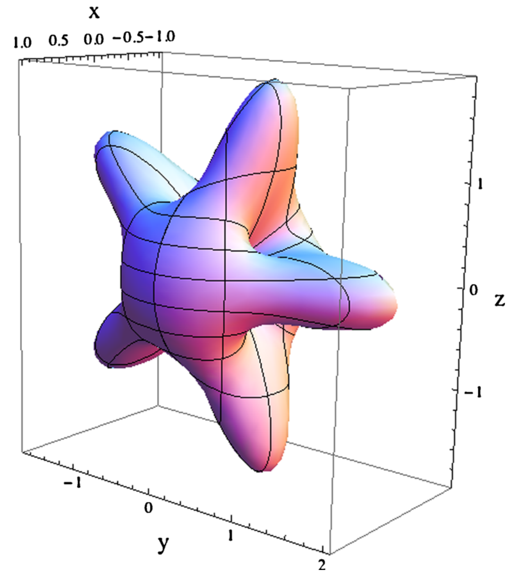

Fig. 5 Plot of V/V_0 showing interference structures for a Schrödinger cat state. Here the $N = 5$ and $\tau = \pi/2$

4 Conclusions

In summary, we have obtained an exact solution in phase space for a quantum non-linear system described by a Kerr medium initially in a double coherent state. The time evolution equation of a generalized two-mode Husimi function has been derived through a normally ordered technique which has been shown to be convenient for describing $SU(2)$ dynamical systems. This characteristic has been evidenced by comparing with some well-known equations in the literature. We have briefly described some effects observed from the study of the exact solution such as Schrödinger cat states which arises from a nonlinear interaction in the atomic population inversion operator. Also from the analysis of the exact solution quantum interference effects are described. They deserve special attention for instance in generating a pure-field-state superposition which has no classical analogue.

Acknowledgments One of the authors (B.E.O.M) would like to thank Consejo Nacional de Ciencia y Tecnologia (CONACyT), Mexico for financial support.

Appendix A: Correspondence Rules

According to Section 2 we list the correspondence rules for $SU(2)$ operators:

$$\hat{\rho}\hat{S}_- \to \left(\alpha_1\alpha_2^* + \alpha_1\frac{\partial}{\partial\alpha_2}\right)Q, \quad (32)$$

$$\hat{\rho}\hat{S}_z \rightarrow \frac{1}{2}\left(|\alpha_1|^2 - |\alpha_2|^2 + \alpha_1\frac{\partial}{\partial\alpha_1} - \alpha_2\frac{\partial}{\partial\alpha_2}\right)Q, \tag{33}$$

$$\hat{\rho}\hat{N} \rightarrow \left(|\alpha_1|^2 + |\alpha_2|^2 + \alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2}\right)Q, \tag{34}$$

and their corresponding adjoints

$$\hat{S}_+\hat{\rho} \rightarrow \left(\alpha_1^*\alpha_2 + \alpha_1^*\frac{\partial}{\partial\alpha_2^*}\right)Q, \tag{35}$$

$$\hat{S}_-\hat{\rho} \rightarrow \left(\alpha_1\alpha_2^* + \alpha_2^*\frac{\partial}{\partial\alpha_1^*}\right)Q, \tag{36}$$

$$\hat{S}_z\hat{\rho} \rightarrow \frac{1}{2}\left(|\alpha_1|^2 - |\alpha_2|^2 + \alpha_1^*\frac{\partial}{\partial\alpha_1^*} - \alpha_2^*\frac{\partial}{\partial\alpha_2^*}\right)Q, \tag{37}$$

$$\hat{N}\hat{\rho} \rightarrow \left(|\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^*\frac{\partial}{\partial\alpha_1^*} + \alpha_2^*\frac{\partial}{\partial\alpha_2^*}\right)Q,$$

where $Q = Q(\alpha_1, \alpha_2)$. Cayley–Klein parameters are used in [17] and [18] where a generalization of $SU(2)$ Wigner function was studied. Such parameters provide a way to uniquely characterize the orientation of a solid body, therefore are physically more intuitive. In terms of angles (θ, φ, ψ), the Cayley–Klein parameters are

$$\alpha_1 = re^{-i(\psi+\varphi)/2}\cos(\theta/2), \quad \alpha_1^* = re^{i(\psi+\varphi)/2}\cos(\theta/2), \tag{38}$$
$$\alpha_2 = re^{-i(\psi-\varphi)/2}\sin(\theta/2), \quad \alpha_2^* = re^{i(\psi-\varphi)/2}\sin(\theta/2), \tag{39}$$

thus using the parameters above, the correspondence rules are

$$\hat{\rho}\hat{S}_+ \rightarrow \left[\frac{1}{2}r^2e^{i\varphi}\sin\theta + \frac{1}{4}re^{i\varphi}\sin\theta\frac{\partial}{\partial r} - e^{i\varphi}\left(\frac{1-\cos\theta}{2}\right)\frac{\partial}{\partial\theta} + \frac{i}{2}e^{i\varphi}\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\varphi} + \frac{i}{2}e^{i\varphi}\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\psi}\right]Q, \tag{40}$$

$$\hat{\rho}\hat{S}_- \rightarrow \left[\frac{1}{2}r^2e^{-i\varphi}\sin\theta + \frac{1}{4}re^{-i\varphi}\sin\theta\frac{\partial}{\partial r} + e^{-i\varphi}\left(\frac{1+\cos\theta}{2}\right)\frac{\partial}{\partial\theta} - \frac{i}{2}e^{-i\varphi}\left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\varphi} + \frac{i}{2}e^{-i\varphi}\left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\psi}\right]Q, \tag{41}$$

$$\hat{\rho}\hat{S}_z \rightarrow \left[\frac{1}{2}r^2\cos\theta + \frac{1}{4}r\cos\theta\frac{\partial}{\partial r} - \frac{1}{2}\sin\theta\frac{\partial}{\partial\theta} + \frac{i}{2}\frac{\partial}{\partial\varphi}\right]Q, \tag{42}$$

$$\hat{\rho}\hat{N} \rightarrow \left[r^2 + \frac{1}{2}r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\psi}\right]Q, \tag{43}$$

and the corresponding adjoint relations are

$$\hat{S}_+\hat{\rho} \rightarrow \left[\frac{1}{2}r^2e^{i\varphi}\sin\theta + \frac{1}{4}re^{i\varphi}\sin\theta\frac{\partial}{\partial r} + e^{i\varphi}\left(\frac{1+\cos\theta}{2}\right)\frac{\partial}{\partial\theta} + \frac{i}{2}e^{i\varphi}\left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\varphi} - \frac{i}{2}e^{i\varphi}\left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\psi}\right]Q, \tag{44}$$

$$\hat{S}_-\hat{\rho} \rightarrow \left[\frac{1}{2}r^2e^{-i\varphi}\sin\theta + \frac{1}{4}re^{-i\varphi}\sin\theta\frac{\partial}{\partial r} - e^{-i\varphi}\left(\frac{1-\cos\theta}{2}\right)\frac{\partial}{\partial\theta} - \frac{i}{2}e^{-i\varphi}\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\varphi} - \frac{i}{2}e^{-i\varphi}\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\psi}\right]Q, \tag{45}$$

$$\hat{S}_z\hat{\rho} \rightarrow \left[\frac{1}{2}r^2\cos\theta + \frac{1}{4}r\cos\theta\frac{\partial}{\partial r} - \frac{1}{2}\sin\theta\frac{\partial}{\partial\theta} - \frac{i}{2}\frac{\partial}{\partial\varphi}\right]Q, \tag{46}$$

$$\hat{N}\hat{\rho} \rightarrow \left[r^2 + \frac{1}{2}r\frac{\partial}{\partial r} - i\frac{\partial}{\partial\psi}\right]Q, \tag{47}$$

where $Q = Q(r, \theta, \varphi, \psi)$. Note that parameters $r^2 \rightarrow 2\mathscr{S}$ are related, where $\mathscr{S}$ is the size of angular momentum irreducible representation.

Appendix B: Solution on Phase Space

Here, we solve an extension to the method developed in [41], where a solution to the equation of motion (16) in phase space is found. We assume the initial state to be given by $\hat{\rho}(0) = |\alpha_{10}, \alpha_{20}\rangle\langle\alpha_{10}, \alpha_{20}|$, i.e., a product of two pure coherent states of complex amplitudes α_{10} and α_{20}. The corresponding Husimi function is a product of two Gaussians centered on α_{10} and α_{20} respectively

$$Q(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*; 0) = \exp\left(-|\alpha_1 - \alpha_{10}|^2\right) \times \exp\left(-|\alpha_2 - \alpha_{20}|^2\right). \tag{48}$$

Solution can be conveniently expressed as a product of two Gaussians and functions involving complex spaces $P_1(\alpha_1, \alpha_2; \tau)$ and their conjugates $P_2(\alpha_1^*, \alpha_2^*; \tau)$, i.e.,

$$Q(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*; \tau) = e^{-\left(|\alpha_1|^2+|\alpha_{10}|^2\right)}e^{-\left(|\alpha_2|^2+|\alpha_{20}|^2\right)} \times P_1(\alpha_1, \alpha_2; \tau)P_2(\alpha_1^*, \alpha_2^*; \tau), \tag{49}$$

if we substitute the solution proposed above into the equation of motion (16), after some algebra and making the substitutions $\beta_1 = \ln\alpha_1$ and $\beta_2 = \ln\alpha_2$, we derived a simplified equation of the form

$$\frac{\partial P_1}{\partial\tau} = \frac{i}{4}\left(\frac{\partial^2 P_1}{\partial\beta_1^2} + \frac{\partial^2 P_1}{\partial\beta_2^2}\right) - \frac{i}{2}\frac{\partial^2 P_1}{\partial\beta_1\partial\beta_2}. \quad (50)$$

A similar expression can be obtained for function P_2. We observe that (50) is a linear second-order differential equation with constant coefficients, this fact allow us to apply a Fourier transform defined by

$$\bar{P}_1(z_1, z_2; \tau) = \int_\zeta d\beta_1 d\beta_2 e^{-i\beta_1 z_1} e^{-i\beta_2 z_2} P_1(\beta_1, \beta_2; \tau), \quad (51)$$

where ζ is a suitably chosen contour. If we apply transformation (51) to (50) then, we obtain

$$\frac{\partial \bar{P}_1}{\partial\tau} = -\frac{i}{4}(z_1 - z_2)^2 \bar{P}_1, \quad (52)$$

whose solution is $\bar{P}_1(z_1, z_2; \tau) = C_0 e^{\eta\tau}$, where $\bar{P}_1(z_1, z_2; 0) = C_0$ and $\eta = -\frac{i}{4}(z_1 - z_2)^2$. The initial condition (48) implies

$$P_1(\alpha_1, \alpha_2; 0) = e^{\alpha_{10}^*\alpha_1} e^{\alpha_{20}^*\alpha_2},$$

therefore

$$\begin{aligned}\bar{P}_1(z_1, z_2; 0) &= \int_\zeta d\beta_1 d\beta_2 e^{-i\beta_1 z_1} e^{-i\beta_2 z_2} P_1(\beta_1, \beta_2; 0) \\ &= \int_\zeta d\beta_1 d\beta_2 e^{-i\beta_1 z_1} e^{-i\beta_2 z_2} \exp\left(\alpha_{10}^* e^{\beta_1} + \alpha_{20}^* e^{\beta_2}\right).\end{aligned}$$

Expanding this last expression and inserting $\bar{P}_1(z_1, z_2; \tau) = C_0 e^{\eta\tau}$ together with the initial condition (48), we have

$$\begin{aligned}\bar{P}_1(z_1, z_2; \tau) = &\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\left(\alpha_{10}^*\right)^{n_1}}{n_1!}\frac{\left(\alpha_{20}^*\right)^{n_2}}{n_2!} \\ &\times\delta(z_1 + in_1)\delta(z_2 + in_2)e^{\eta\tau}.\end{aligned}$$

Finally, applying the inverse Fourier transform, we obtain

$$\begin{aligned}P_1(\alpha_1, \alpha_2; \tau) = &\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\left(\alpha_{10}^*\alpha_1\right)^{n_1}}{n_1!}\frac{\left(\alpha_{20}^*\alpha_2\right)^{n_2}}{n_2!} \exp\left[i\frac{\tau}{4}(n_1 - n_2)^2\right] \\ &\equiv S.\end{aligned}$$

Therefore, the solution to (16) is given by

$$Q(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*; \tau) = e^{\left(-|\alpha_1|^2 - |\alpha_{10}|^2\right)} e^{\left(-|\alpha_2|^2 - |\alpha_{20}|^2\right)} |S|^2.$$

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