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GENERAL AND APPLIED PHYSICS

Mathematical Aspects of Quantum Systems with a Pseudo-Hermitian Hamiltonian

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Abstract A non-self-adjoint bosonic Hamiltonian H possessing real eigenvalues is investigated. It is shown that the operator can be diagonalized by making use of pseudo-bosonic operators. The biorthogonal sets of eigenvectors for the Hamiltonian and its adjoint are explicitly constructed. The positive definite operator which connects both sets of eigenvectors is also given. The dynamics of the model is briefly analyzed.

Keywords Pseudo-Hermitian Hamiltonians · Non-Hermitian Hamiltonians with real eigenvalues · Pseudo-bosons

1 Introduction

In classical quantum mechanics, one of the most fundamental axioms is that the Hamiltonian H of the physical system, which acts on a Hilbert space, is self-adjoint, $H=H^*$. Also, all observables are described by self-adjoint operators. The self-adjointness ensures that the eigenvalues of the operators are real, a relevant issue in the basis of the theory. The appearance of quantum systems described by non-self-adjoint operators motivated the investigation of this kind of systems in finite and infinite-dimensional Hilbert spaces.

In the last decade, the interest of researchers on non-self-adjoint operators having real eigenvalues has developed in different aspects (we refer the readers to [1–16] and references therein).

In this paper, a non-self-adjoint bosonic Hamiltonian H acting on an infinite dimensional Hilbert space $\mathcal{H}$ and possessing real eigenvalues, is investigated. In section 2, it is shown that the operator can be diagonalized by making use of pseudo-bosonic operators. The biorthogonal sets of eigenvectors for the Hamiltonian and its adjoint are explicitly constructed. A bosonic operator S is determined such that $\exp(-S)H\exp(S)$ is Hermitian, being $\exp(-S^*)\exp(S)$ the positive definite operator which connects the set of eigenvectors of H^* with those of H.

Some considerations concerning the physical interpretation of states with positive and negative J-norm, useful for a deeper understanding of this kind of physical systems, are presented.

In section 3, a numerical example illustrates the procedure presented in section 2. Finally, in section 4, some discussions are carried out.

2 Pseudo-Hermitian Hamiltonian with Real Eigenvalues

Let $\mathcal{H}$ be a Hilbert space endowed with inner product $\langle\cdot,\cdot\rangle$ and related norm $\|\cdot\|$. Let $\mathcal{D}$ be a dense domain of $\mathcal{H}$. The operators $a_i, a_i^* : \mathcal{D} \rightarrow \mathcal{D},\ i = 1, \dots, N$ are bosonic

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operators, i.e., they satisfy the Weil-Heisenberg commutation relations (CRs),

$$\left[a_i, a_j^*\right] = \delta_{ij}, \quad \left[a_i^*, a_j^*\right] = 0, \; \left[a_i, \; a_j\right] = 0,$$
$$i, j = 1, \; \ldots, \; N,$$

where δ_{ij} denotes the Kronecker symbol (δ_{ij} equals 1 for $i=j$ and 0 otherwise). Conventionally, a_i and a_i^* are called annihilation and creation operators, respectively, and they are unbounded. Let us consider the non-self-adjoint Hamiltonian

$$H = \sum_{ij} \left(A_{ij} a_i^* a_j + B_{ij} a_i^* a_j\right),$$

where $A=(A_{ij})$, $B=(B_{ij})$ are real matrices of order $N\times N$, such that

$$A_{ij} = A_i \delta_{ij}, \;\; B_{ij} = B_i \delta_{j,i+1} - B_{i-1} \delta_{j,i-1},$$

that is, $A=\mathrm{diag}\ (A_1, A_2, \ldots, A_N)$ and

$$B = \begin{bmatrix} 0 & B_1 & 0 & 0 & \ldots & 0 & 0 \\ -B_1 & 0 & B_2 & 0 & \ldots & 0 & 0 \\ 0 & -B_2 & 0 & B_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & B_{N-1} \\ 0 & 0 & 0 & 0 & \ldots & -B_{N-1} & 0 \end{bmatrix}$$

For physical convenience, we assume that $A_1 > \ldots > A_N \geq 0$. Notice that $A+B$ is called a pseudo-Jacobi matrix, since $J\cdot(A+B)$ is a Jacobi matrix for $J=\mathrm{diag}(1,-1,1,\ldots,-(-1)^N)$, that is, a real tridiagonal symmetric matrix. We observe that a complete system of eigenvectors for H is provided by the vectors

$$\Phi_{n1,\ldots,n_N} = a_1^{*n_1} \ldots a_N^{*\,n_N} \Phi_0,$$

where $\Phi_0 \in \mathcal{D}$ is the vacuum state of the operators a_i, i.e., a vector such that

$$a_i \Phi_0 = 0, \;\; i = 1, \ldots, N,$$

and $n_1, \ldots, n_N$ are non-negative integers. These vectors are orthogonal

$$\left\langle \Phi_{p_1,\ldots,p_N}, \Phi_{n_1,\ldots,n_N} \right\rangle = n_1! \cdots n_N! \delta_{n_1 p_1} \cdots \delta_{n_N p_N} \langle \Phi_0, \Phi_0 \rangle.$$

Clearly, Φ_0 is an eigenvector of H corresponding to the eigenvalue 0. The spectral analysis of H requires the determination of the remaining eigenvalues. The physical requirement that the spectrum of H is real imposes important restrictions on the matrices A and B.We may observe that even a self-adjoint Hamiltonian quadratic in bosonic operators may not have a real spectrum, as is the case of the self-adjoint operator

$$a_1^{*2} + a_1^2,$$

which does not have real eigenvalues.In order to determine the eigenvalues of H, we investigate the so called equation of motion method (EMM) condition

$$\left[H, \sum_i \left(X_i a_i^* - Y_i a_i\right)\right] = \lambda \sum_i \left(X_i a_i^* - Y_i a_i\right), \tag{1}$$

with λ a real parameter and $[X,Y] = XY - YX$ denoting the commutator of X and Y. From (1), we get the block matrix equation

$$\begin{bmatrix} A+B & 0 \\ 0 & -A+B \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} X \\ Y \end{bmatrix}, \tag{2}$$

where $X=(X_i)$, $Y=(Y_i)$ are column matrices with N real entries. Since the diagonal block matrix

$$M = \begin{bmatrix} A+B & 0 \\ 0 & -A+B \end{bmatrix}$$

is real and pseudo-Hermitian, because $(J\oplus J)M^T(J\oplus J)=M$, its eigenvalues λ are either real or pairs of complex conjugate numbers. The absence of complex eigenvalues required by the physical interpretation of the model restricts the admissible matrices A and B. The spectrum of H is real only if the eigenvalues of M are real, which is supposed to be the case. Since the eigenvalues of M are real, they coincide with those of M^T, and so, these matrices are isospectral. From (2), it follows that

$$\begin{bmatrix} A+B & 0 \\ 0 & -A+B \end{bmatrix} \begin{bmatrix} Y' \\ X' \end{bmatrix} = -\lambda \begin{bmatrix} Y' \\ X' \end{bmatrix},$$

where $[X'^{\,T}, Y'^{\,T}]^T$ is an eigenvector of M^T corresponding to the same eigenvalue λ, so if λ is an eigenvalue of (2), so is $-\lambda$. We easily conclude that either $X\neq 0, Y=0$ or $Y\neq 0, X=0$, so that we conveniently replace (2) by the eigenvalue problem for $A+B$ and for its transpose

$$(A+B)X = \lambda X, \tag{3}$$

$$(A-B)X' = \lambda X'. \tag{4}$$

Let us consider a set of eigenvectors of (3) and (4), namely, $X^{(r)}=(X_1^{(r)},\ldots,X_N^{(r)})^T$ and $X'^{(r)}=(X_1'^{(r)},\ldots,X_N'^{(r)})^T$. These are complete systems in $\mathbb{R}^N$, because the eigenvalues of $A+B$ as well as those of $A-B$, are mutually distinct. Also, biorthogonality occurs, as

$$\left\langle \mathrm{X}^{(\mathrm{r})}, \mathrm{X}'^{(\mathrm{s})} \right\rangle = X'^{(s)T} X^{(r)} = \delta_{rs}.$$

Notice that $X'^{(\mathrm{r})} = \epsilon_\mathrm{r} J X^{(r)}$, for $J=\mathrm{diag}(\epsilon_1, \ldots \epsilon_\mathrm{N})=\mathrm{diag}(1,-1, 1, \ldots, (-1)^{N+1})$. This is equivalent to considering the J-normalization,

$$\left\langle JX^{(r)}, \; X^{(s)} \right\rangle = X^{(s)T} J X^{(r)} = \varepsilon_r \delta_{rs}.$$

The matrix

$$U = \left[X^{(1)}, \ldots, X^{(N)}\right] \in R^{N\times N},$$

whose columns are the eigenvectors $X^{(i)}$, belongs to the real pseudo-orthogonal group, defined by the property $JU^T JU = I$. The matrix S given by $U = \exp S$ belongs to the algebra, closed under commutation relation, defined by the property $JS^T J = -S$, which is isomorphic to the algebra of the operators

$$S = \sum_{i=1}^{N} s_{ij} a_i^* a_j, \quad S = (s_{ij}).$$

Consider the pseudo-bosonic operators also acting on $\mathcal{H}$,

$$c_r^\ddagger = \sum_{i=1}^{N} X_i^{(r)} a_i^* = e^S a_r^* e^{-S},$$

$$c_r = \sum_{i=1}^{N} \varepsilon_r \varepsilon_i X_i^{(r)} a_i = e^S a_r e^{-S}$$

being $c_r^\ddagger$ a creation operator, i.e., it increases the number of bosons in the dynamical state r by one unit, and c_r a destruction operator, i.e., it decreases the number of bosons in the dynamical state r by one unit.Expressed in terms of $c_r^\ddagger$, c_r, the Hamiltonian becomes

$$H = \sum_{r=1}^{N} \lambda_r c_r^\ddagger cr = e^S \sum_{i=1}^{N} \lambda_i a_i^* a_i e^{-S}.$$

The vacuum Ψ_0 of c_r coincides with Φ_0 and is the groundstate eigenvector of H, if $\lambda_i \geq 0$, $i = 1, \ldots, N$. The corresponding excited state eigenvectors are given by

$$\Psi_{n_1,\ldots,n_N} = c_1^{\ddagger\, n_1} \cdots c_N^{\ddagger\, n_N} \Phi_0 = e^S \Phi_{n_1,\ldots,n_N}, \tag{5}$$

the associated eigenvalues being

$$E_{n_1,\ldots,n_N} = n_1 \lambda_{n_1} + \ldots + n_N \lambda_{n_N},$$

where and $n_1, \ldots, n_N$ are non-negative integers. Similarly, the vacuum Ψ^0 of c^{r*} coincides with Φ_0 and is the groundstate eigenvector of H*. The corresponding excited state eigenvectors are given by

$$\Psi'_{n_1,\ldots,n_N} = c_1^{*n_1} \cdot \cdot \cdot c_N^{*\, nN} \Phi_0 = e^{-S^*} \Phi_{n_1,\ldots,n_N}. \tag{6}$$

The systems of eigenvectors $\Psi_{n_1,\ldots,n_N}$ and $\Psi'_{n1,\ldots,n_N}$ are biorthogonal:

$$\left\langle \Psi'_{p_1,\ldots,p_N}, \Psi_{n_1,\ldots,n_N} \right\rangle = n_1! \cdots n_N! \delta_{n_1 p_1} \cdots \delta_{n_N PN} \langle \Phi_0, \Phi_0 \rangle$$

From (5) and (6) it follows that

$$\Psi'_{p_1,\ldots,p_N} = e^{-S^*} e^{-S} \Psi_{n_1,\cdots,n_N}.$$

We remark that the operator e^{-S^*} e^{-S} which connects both sets of eigenvectors is positive definite. Thus,

$$\left\langle \Psi_{p1,\ldots,p_N}, e^{-S^*} e^{-S} \Psi_{n_1,\ldots,n_N} \right\rangle = n_1! \cdots n_N! \delta_{n_1 p_1} \cdots \delta_{n_N p_N} \langle \Phi_0, \Phi_0 \rangle, \tag{7}$$

i.e., the vectors $\Psi_{p_1,\ldots,p_N}$ are orthogonal with respect to the positive norm operator

$$e^{-S^*} e^{-S}.$$

Recall that, in the finite dimensional case, the system of eigenvectors of a matrix M is biorthogonal to the system of eigenvectors of M^* if the eigenvalues are real. Consider, for instance, the case of the matrices $A+B$ and $(A+B)^T$. The sets of eigenvectors of both matrices are connected by a positive definite operator. Indeed, it is clear that

$$X'^{(r)} = JUU^T JX^{(r)}, \; r = 1, \ldots, N,$$

and JUU^TJ is obviously positive definite.

As a consequence of (7), an arbitrary $\Psi \in H$ may be expanded as

$$\Psi = \sum_{n_1,\ldots,n_N} C_{n_1,\ldots n_N} \Psi_{P_1,\ldots P_n},$$

its $e^{-S} * e^{-S}$ norm being

$$\left\langle \Psi, e^{-S^*} e^{-S} \Psi \right\rangle = \sum_{n1,\ldots,nN} \left| C_{n_1,\ldots,n_N} \right|^2 n_1! \cdots n_N! \langle \Phi_0, \Phi_0 \rangle,$$

an expression which is easily amenable to a probabilistic interpretation.We consider C^N endowed with the already mentioned inner product $\langle X, Y \rangle_J = Y^* JX$, for any $X, Y \in C^N$, and respective J-norm $\|X\|^J = X^* JX$. The matrix $A+B$ gives the possible dynamical states of each bosonic particle. Some considerations are in order concerning the physical interpretation of states with positive J -norm and states with negative J -norm. Several interpretations connected with transition probabilities may be possible. For instance, by analogy with the well-known situation of π-mesons described by the Klein-Gordon equation, we may associate positive J-norm with positive electric charge and negative J-norm with negative electric charge. The J-norm of some state may be associated with the electric charge of that state. Let us consider the state described, at the instant $t=0$, by the vector

$$X = c_1 X^{(1)} + \ldots + c_N X^{(N)} \in C^N.$$

According to quantal dynamics and the superposition principle, at the instant t, the state is described by the vector

$$X(t) = c_1 X^{(1)} e^{i\lambda_1 t} + \ldots + c_N X^{(N)} e^{i\lambda_N t}.$$

Assuming N even, the J-norm of this vector, which gives the total electric charge of the state, is expressed as

$$\langle X, X\rangle_J = X^* J X = |c_1|^2 - |c_2|^2 + |c_3|^2 - |c_4|^2 - \ldots - |c_N|^2.$$

The probability of positive electric charge states is

$$P_+ = \frac{|c_1|^2 + |c_3|^2 + \ldots + |c_{N-1}|^2}{|c_1|^2 + |c_2|^2 + |c_3|^2 + \ldots + |c_N|^2}$$

and the probability of negative electric charge states is

$$P_- = \frac{|c_2|^2 + |c_4|^2 + \ldots + |c_N|^2}{|c_1|^2 + |c_2|^2 + |c_3|^2 + \ldots + |c_N|^2}.$$

3 A Numerical Example

Let us consider the pseudo-Hermitian Hamiltonian $H = 3a_1^* a_1 + a_1^* a_2 - a_2^* a_1$.In terms of pseudo-bosonic operators, it is expressed as

$$H = \frac{3+\sqrt{5}}{2} c_1^\ddagger c_1 + \frac{3-\sqrt{5}}{2} c_2^\ddagger c_2,$$

being

$$c_1^\ddagger = \sqrt{\frac{3+\sqrt{5}}{2\sqrt{2}}} a_1^* - \sqrt{\frac{3-\sqrt{5}}{2\sqrt{2}}} a_2^*,$$

$$c_2^\ddagger = \sqrt{\frac{3+\sqrt{5}}{2\sqrt{2}}} a_2^* - \sqrt{\frac{3-\sqrt{5}}{2\sqrt{2}}} a_1^*,$$

and

$$c_1 = \sqrt{\frac{3+\sqrt{5}}{2\sqrt{2}}} a_1 + \sqrt{\frac{3-\sqrt{5}}{2\sqrt{2}}} a_2,$$

$$c_2 = \sqrt{\frac{3+\sqrt{5}}{2\sqrt{2}}} a_2 - \sqrt{\frac{3-\sqrt{5}}{2\sqrt{2}}} a_1.$$

Moreover

$$c_i^\ddagger = e^S a_i^* e^{-S}, \ \ \mathrm{c}_i = e^S a_i e^{-S}, \ \ i = 1, 2,$$

with

$$S = \arctan \frac{3-\sqrt{5}}{2} \left(a_1^* a_2 - a_2^* a_1\right).$$

The eigenvectors of H are given by

$$\Psi_{n_1, n_2} = c_1^{\ddagger\, n1} c_2^{\ddagger\, n2} \Phi_0,$$

The eigenvectors of H^* are given by

$$\Psi'_{n_1, n_2} = c_1^{*n_1} c_2^{*n_2} \Phi_0 = e^{-S^*} a_1^{*n_1} a_2^{*n_2} \Phi_0.$$

Here, Φ_0 is the vacuum of the operators a_i, $i = 1, 2$. These sets of eigenvectors are biorthogonal. Finally, we note that H is similar to the Hermitian operator,

$$H_0 = \frac{3+\sqrt{5}}{2} a_1^* a_1 + \frac{3-\sqrt{5}}{2} a_2^* a_2.$$

Indeed, $H = e^S H_0 e^{-S}$.

4 Discussion

In section 2, a certain non self-adjoint Hamiltonian H with real eigenvalues, expressed as a quadratic combination of bosonic operators, is diagonalized by means of dynamical pseudo-bosons, which are determined by the EMM, with the help of a real and pseudo-Hermitian matrix M of size $2N$. A complete system of eigenvectors of the investigated Hamiltonian, expressed in terms of pseudo-bosonic creation and annihilation operators of has been obtained. This system of eigenvectors is biorthogonal to the system of eigenvectors of the adjoint Hamiltonian. Both systems of eigenvectors are connected through a positive definite operator explicitly constructed in terms of bosonic creation and annihilation operators. A bosonic operator S is determined such that $\exp(-S)H\exp(S)$ is Hermitian, being $\exp(-S^*)\exp(-S)$ the positive definite operator which connects the set of eigenvectors of H^* with those of H.

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