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Exponentiated extended Weibull-power series class of distributions

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Abstract

In this paper, we introduce a new class of distributions by compounding the exponentiated extended Weibull family and power series family. This distribution contains several lifetime models such as the complementary extended Weibull-power series, generalized exponential-power series, generalized linear failure rate-power series, exponentiated Weibull-power series, generalized modified Weibull-power series, generalized Gompertz-power series and exponentiated extended Weibull distributions as special cases. We obtain several properties of this new class of distributions such as Shannon entropy, mean residual life, hazard rate function, quantiles and moments. The maximum likelihood estimation procedure via a EM-algorithm is presented.

Keywords: EM-algorithm, Exponentiated family, Maximum likelihood estimation, Power series distributions.

1   Introduction

The extended Weibull (EW) family contains various well-known distributions such as exponential, Pareto, Gompertz, Weibull, linear failure rate (Barlow, 1968), modified Weibull (Lai et al., 2003), additive Weibull (Xie and Lai, 1995; Almalki and Yuan, 2013) and Chen (Chen, 2000) distributions. For more details see Nadarajah and Kotz (2005) and Pham and Lai (2007).

Using the given method by Gupta and Kundu (1999), the EW family can be generalized. We call it exponentiated extended Weibull (EEW) distribution. The cumulative distribution function (cdf) of this distribution is

$$G(x; \alpha, \beta, \boldsymbol{\Theta}) = [1 - e^{-\alpha H(x; \boldsymbol{\Theta})}]^{\beta}, \ \alpha > 0, \ \beta > 0, \ x \geq 0, \quad (1)$$

and its probability density function (pdf) is

$$\begin{aligned} g(x; \alpha, \beta, \boldsymbol{\Theta}) &= \alpha \beta h(x; \boldsymbol{\Theta}) e^{-\alpha H(x; \boldsymbol{\Theta})} \\ &\times [1 - e^{-\alpha H(x; \boldsymbol{\Theta})}]^{\beta - 1}, \quad (2) \end{aligned}$$

where $\boldsymbol{\Theta}$ is a vector of parameters, and $H(x; \boldsymbol{\Theta})$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $H(x; \boldsymbol{\Theta}) \to 0$ as $x \to 0^{+}$ and $H(x; \boldsymbol{\Theta}) \to \infty$ as $x \to \infty$. It is denoted by $\text{EEW}(\alpha, \beta, \boldsymbol{\Theta})$.

The EEW distribution is a flexible family and contains many exponentiated distributions such as generalized exponential (Gupta and Kundu, 1999), exponentiated Weibull (Mudholkar and Srivastava, 1993), generalized Rayleigh (Surles and Padgett, 2001; Kundu and Raqab, 2005), generalized modified Weibull (Carrasco et al., 2008), generalized linear failure rate (Sarhan and Kundu, 2009), and generalized Gompertz (El-Gohary et al., 2013) distributions.

In recent years, many distributions to model lifetime data have been introduced. The basic idea of introducing these models is that a lifetime of a system with N (discrete random variable) components and the positive continuous random variable, say X_i (the lifetime of ith component), can be denoted by the non-negative random variable $Y = \min(X_1, \ldots, X_N)$ or $Y = \max(X_1, \ldots, X_N)$, based on whether the components are series or parallel.

In this paper, we compound the EEW family and power series distributions, and introduce a new class of distribution. This class of distributions can be applied to reliability problems and its some properties are investigated in this paper. We call it exponentiated extended Weibull- power series (EEWPS) class of distributions. In similar way, some distributions are proposed in literature: the exponential-power series (EP) distribution by Chahkandi and Ganjali (2009), Weibull-power series (WPS) distributions by Morais and Barreto-Souza (2011), generalized exponential-power series (GEP) distribution by Mahmoudi and Jafari (2012), complementary exponential power series by Flores et al. (2013), extended Weibull-power series (EWPS) distribution by Silva et al. (2013), double bounded Kumaraswamy-power series by Bidram and Nekoukhou (2013), Burr-power series by Silva and Cordeiro (2013), generalized linear failure rate-power series (GLFRP) distribution by Alamatsaz and Shams (2014), Birnbaum-Saunders-power series distribution by Bourguignon et al. (2014), linear failure rate-power series by Mahmoudi and Jafari (2014), and complementary extended Weibull-power series by Cordeiro and Silva (2014). Similar procedures are used by Roman et al. (2012), Lu and Shi (2011), Nadarajah et al. (2015) and Louzada et al. (2014). For compounding continuous distributions with discrete distributions, Nadarajah et al. (2013) introduced the package Compounding in R software (R Development Core Team, 2014).

We provide three motivations for the EEWPS class of distributions, which can be applied in some interesting situations as follows: (i) this new class of distributions due to the stochastic representation $Y = \max(X_1, \ldots, X_N)$, can arises in parallel systems with identical components, where each component has the EEW distribution lifetime. This model appears in many industrial applications and biological organisms which the lifetime of the event is only the maximum ordered lifetime value among all causes. (ii) The EEWPS class of distributions gives a reasonable parametric fit to some modeling phenomenon with non-monotone hazard rates such as the bathtub-shaped, unimodal and increasing-decreasing-increasing hazard rates, which are common in reliability and biological studies. (iii) The time to the last failure can be appropriately modeled by the EEWPS class of distributions.

The remainder of this paper is organized as follows: the pdf and failure rate function of the new class of distributions are given in Section 2. The special cases of the EEWPS distribution are considered in Section 3. Some properties such as quantiles, moments, order statistics, Shannon entropy and mean residual life are given in Section 4. Estimation of parameters by maximum likelihood are discussed in Section 5. Application to a real data set is presented in Section 6.

2 Introducing new family

A discrete random variable, N is a member of power series distributions (truncated at zero) if its probability mass function (pmf) is given by

$$p_n = P(N = n) = \frac{a_n \lambda^n}{C(\lambda)}, \quad n = 1, 2, \ldots, \quad (3)$$

where $a_n \geq 0$, $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$, and $\lambda \in (0, s)$ is chosen in a way such that $C(\lambda)$ is finite and its first, second and third derivatives are defined and shown by $C'(.)$, $C''(.)$ and $C'''(.)$, respectively. The term "power series distribution" is generally credited to Noack (1950). This family of distributions includes many of the most common distributions, including the binomial, Poisson, geometric, negative binomial, logarithmic distributions. For more details about power series distributions, see Johnson et al. (2005), page 75.

Theorem 2.1. *Let N be a random variable denoting the number of failure causes which it is a member of power series distributions with pmf in* (3). *Also, For given N, let $X_1, X_2, \ldots, X_N$ be independent identically distributed random variables from EEW distribution with pdf in* (2). *Then $X_{(N)} = \max_{1 \leq i \leq N}\{X_i\}$ has EEWPS class of distributions is denoted by* EEWPS($\alpha, \beta, \lambda, \Theta$) *and has the following pdf:*

$$\begin{aligned} f(x) &= \alpha\beta\lambda h(x; \Theta)e^{-\alpha H(x;\Theta)}(1 - e^{-\alpha H(x;\Theta)})^{\beta-1} \\ &\quad \times \frac{C'\left(\lambda(1 - e^{-\alpha H(x;\Theta)})^{\beta}\right)}{C(\lambda)}, \quad x > 0. \quad (4) \end{aligned}$$

Proof. The conditional cdf of $X_{(N)} \mid N = n$ has EEW($\alpha, n\beta, \Theta$). Hence,

$$P(X_{(N)} \leq x, N = n) = \frac{a_n\lambda^n}{C(\lambda)}[1 - e^{-\alpha H(x;\Theta)}]^{n\beta}, \quad (5)$$

and the marginal cdf of $X_{(N)}$ is

$$F(x) = \frac{C(\lambda(1 - e^{-\alpha H(x;\Theta)})^{\beta})}{C(\lambda)}, \quad x > 0. \quad (6)$$

The derivative of F with respect to x is (4). Therefore, $X_{(N)}$ has EEWPS distribution. $\qquad \square$

Proposition 1. *The pdf of EEWPS class can be expressed as infinite linear combination of density of order distribution, i.e. it can be written as*

$$f(x) = \sum_{n=1}^{\infty} p_n g_{(n)}(x; \alpha, n\beta, \Theta), \quad (7)$$

where $g_{(n)}(x; \alpha, n\beta, \Theta)$ is the pdf of EEW distribution with parameters α, $n\beta$ and Θ.

Proof. Consider $t = 1 - e^{-\alpha H(x;\Theta)}$. So

$$
\begin{aligned}
f(x) &= \alpha\beta\lambda h(x;\Theta)e^{-\alpha H(x;\Theta)}t^{\beta-1}\frac{C'\left(\lambda t^{\beta}\right)}{C(\lambda)} \\
&= \alpha\beta\lambda h(x;\Theta)e^{-\alpha H(x;\Theta)}t^{\beta-1}\frac{\sum\limits_{n=1}^{\infty} na_n(\lambda t^{\beta})^{n-1}}{C(\lambda)} \\
&= \sum_{n=1}^{\infty}\frac{a_n\lambda^n}{C(\lambda)}n\alpha\beta h(x;\Theta)e^{-\alpha H(x;\Theta)}t^{n\beta-1} \\
&= \sum_{n=1}^{\infty} p_n g_{(n)}(x;\alpha,n\beta,\Theta).
\end{aligned}
$$

$\square$

Proposition 2. *The limiting distribution of* EEWPS(β,λ,Θ) *when* $\lambda \to 0^+$ *is*

$$
\lim_{\lambda\to 0^+} F(x) = [1 - e^{-\alpha H(x;\Theta)}]^{c\beta},
$$

which is a EEW distribution with parameters α, $c\beta$ *and* Θ, *where* $c = \min\{n \in \mathbb{N} : a_n > 0\}$.

Proof. Consider $t = 1 - e^{-\alpha H(x;\Theta)}$. So

$$
\begin{aligned}
\lim_{\lambda\to 0^+} F(x) &= \lim_{\lambda\to 0^+}\frac{C(\lambda t^{\beta})}{C(\lambda)} = \lim_{\lambda\to 0^+}\frac{\sum\limits_{n=1}^{\infty} a_n\lambda^n t^{n\beta}}{\sum\limits_{n=1}^{\infty} a_n\lambda^n} \\
&= \lim_{\lambda\to 0^+}\frac{a_c t^{c\beta} + \sum\limits_{n=c+1}^{\infty} a_n\lambda^{n-c} t^{n\beta}}{a_c + \sum\limits_{n=c+1}^{\infty} a_n\lambda^{n-c}} = t^{c\beta}.
\end{aligned}
$$

$\square$

Proposition 3. *The hazard rate function of the EEWPS class of distributions is given by*

$$
r(x) = \frac{\alpha\lambda\beta h(x;\Theta)(1-t)t^{\beta-1}C'\left(\lambda t^{\beta}\right)}{C(\lambda) - C(\lambda t^{\beta})}, \tag{8}
$$

where $t = 1 - e^{-\alpha H(x;\Theta)}$.

Proof. Using (4), (6) and definition of hazard rate function as $r(x) = f(x)/(1 - F(x))$, the proof is obvious. $\square$

3 Special cases

In this Section, we consider some special cases of the EEWPS distribution.

3.1 Complementary extended Weibull power series

If $\beta = 1$, then the pdf in (4) becomes to

$$
f(x) = \alpha\lambda h(x;\Theta)e^{-\alpha H(x;\Theta)}
$$

$$
\times\frac{C'\left(\lambda(1-e^{-\alpha H(x;\Theta)})\right)}{C(\lambda)}, \quad x > 0, \tag{9}
$$

which is the pdf of complementary extended Weibull power series (CEWPS) class of distributions introduced by Cordeiro and Silva (2014).

3.2 Generalized exponential-power series

If $H(x;\Theta) = x$, then the pdf in (4) becomes to

$$
f(x) = \alpha\beta\lambda e^{-\alpha x}(1-e^{-\alpha x})^{\beta-1}\frac{C'\left(\lambda(1-e^{-\alpha x})^{\beta}\right)}{C(\lambda)}, \quad x > 0. \tag{10}
$$

which is the pdf of generalized exponential-power series (GEPS) class of distributions introduced by Mahmoudi and Jafari (2012). The GEPS class of distributions contains complementary exponentiated exponential-geometric distribution introduced by Louzada et al. (2013), complementary exponential-geometric distribution introduced by Louzada et al. (2011), Poisson-exponential distribution introduced by Cancho et al. (2011) and Louzada-Neto et al. (2011), complementary exponential-power series class of distributions introduced by Flores et al. (2013), generalized exponential distribution introduced by Gupta and Kundu (1999) and generalized exponential-geometric distribution introduced by Bidram et al. (2013) .

3.3 Generalized linear failure rate-power series

If $H(x;\Theta) = \frac{ax}{\alpha} + \frac{bx^2}{2\alpha}$, then the pdf in (4) becomes to

$$
\begin{aligned}
f(x) &= \beta\lambda(a+bx)e^{-ax-\frac{bx^2}{2}}(1-e^{-ax-\frac{bx^2}{2}})^{\beta-1} \\
&\times\frac{C'\left(\lambda(1-e^{-ax-\frac{bx^2}{2}})^{\beta}\right)}{C(\lambda)}, \quad x > 0. \tag{11}
\end{aligned}
$$

which is the pdf of generalized linear failure rate-power series (GLFRPS) class of distributions introduced by Alamatsaz and Shams (2014). It is a modification of generalized linear failure rate distribution introduced by Sarhan and Kundu (2009) and generalized linear failure rate- geometric distribution introduced by Nadarajah et al. (2014). If $b = 0$, it becomes to GEPS class of distributions. Also, If $\beta = 1$, it becomes to linear failure rate-power series introduced by Mahmoudi and Jafari (2014).

3.4 Exponentiated Weibull- power series

If $H(x;\Theta) = x^{\gamma}$, then the pdf in (4) becomes to

$$
f(x) = \alpha\beta\lambda\gamma x^{\gamma-1}e^{-\alpha x^{\gamma}}(1-e^{-\alpha x^{\gamma}})^{\beta-1}
$$

$$\times \frac{C'\left(\lambda(1-e^{-\alpha x^\gamma})^\beta\right)}{C(\lambda)}, \quad x > 0. \quad (12)$$

which is the pdf of exponentiated Weibull-power series (EWPS) class of distributions introduced by Mahmoudi and Shiran (2012). It is a modification of exponentiated Weibull distribution introduced by Mudholkar and Srivastava (1993). It is contain the complementary Weibull geometric distribution introduced by Tojeiro et al. (2014). Also, the Marshall-Olkin extended Weibull distribution introduced by Cordeiro and Lemonte (2013) is a special case of EWPS.

3.5 Generalized modified Weibull-power series

If $H(x;\mathbf{\Theta}) = x^\gamma \exp(\tau x)$, then the pdf in (4) becomes to

$$
\begin{aligned}
f(x) &= \alpha\beta\lambda x^{\gamma-1}(\gamma+\tau x)e^{\tau x - \alpha x^\gamma \exp(\tau x)} \quad (13) \\
&\times \frac{C'\left(\lambda(1-e^{-\alpha x^\gamma \exp(\tau x)})^\beta\right)}{(1-e^{-\alpha x^\gamma \exp(\tau x)})^{1-\beta}C(\lambda)}, \quad x > 0,
\end{aligned}
$$

and we call generalized modified Weibull-power series (GMWPS) class of distributions. It is contained the generalized modified Weibull distribution introduced by Carrasco et al. (2008). If $\tau = 0$, then GMWPS class of distributions becomes to EWPS class of distributions.

3.6 Generalized Gompertz- power series

If $H(x;\mathbf{\Theta}) = \frac{1}{\gamma}(e^{\gamma x}-1)$, then the pdf in (4) becomes to

$$
\begin{aligned}
f(x) &= \alpha\beta\lambda e^{\gamma x}e^{-\frac{\alpha}{\gamma}(e^{\gamma x}-1)}(1-e^{-\frac{\alpha}{\gamma}(e^{\gamma x}-1)})^{\beta-1} \\
&\times \frac{C'\left(\lambda(1-e^{-\frac{\alpha}{\gamma}(e^{\gamma x}-1)})^\beta\right)}{C(\lambda)}, \quad x > 0. \quad (14)
\end{aligned}
$$

and we call generalized Gompertz-power series class of distributions. It is contained the generalized Gompertz distribution introduced by El-Gohary et al. (2013).

4 Statistical properties

In this section, some properties of EEWPS class of distributions such as quantiles, moments, order statistics, Shannon entropy and mean residual life are derived. Using (7), we can obtain

$$F(x) = \sum_{n=1}^{\infty} p_n G_{(n)}(x;\alpha, n\beta, \mathbf{\Theta}) = \sum_{n=1}^{\infty} p_n t^{n\beta}, \quad (15)$$

where $t = 1 - e^{-\alpha H(x;\mathbf{\Theta})}$. Based on the mathematical quantities of the baseline pdf $g_{(n)}(x;\alpha, n\beta, \mathbf{\Theta})$, we can obtain some statistical quantities such as ordinary and incomplete moments, generating function and mean deviations of this family of distributions.

4.1 Quantiles and Moments

Let

$$X = G^{-1}\left(\frac{C^{-1}(C(\lambda)U)}{\lambda}\right),$$

where U has a uniform distribution on $(0,1)$, $G^{-1}(y) = H^{-1}[-\frac{1}{\alpha}\ln(1-y^{\frac{1}{\beta}})]$ and $C^{-1}(.)$ is the inverse function of $C(.)$. Then X has the EEWPS$(\alpha, \beta, \lambda, \mathbf{\Theta})$ distribution. This result helps in simulating data from the EEWPS distribution with generating uniform distribution data.

Theorem 4.1. *Consider $X \sim$ EEWPS$(\alpha, \beta, \lambda, \mathbf{\Theta})$. Then the moment generating function of EEWPS is*

$$M_X(t) = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} p_n \binom{n\beta}{j+1}(-1)^j M_Y(t), \quad (16)$$

where Y has EEW$(\alpha(j+1), 1, \mathbf{\Theta})$.

Proof. The Laplace transform of the EEWPS class can be expressed as

$$L(s) = E(e^{-sX}) = \sum_{n=1}^{\infty} P(N=n)L_n(s),$$

where $L_n(s)$ is the Laplace transform of EEW distribution with parameters α, $n\beta$ and $\mathbf{\Theta}$ given as

$$
\begin{aligned}
L_n(s) &= \int_0^{+\infty} e^{-sx}n\alpha\beta h(x;\mathbf{\Theta})e^{-\alpha H(x;\mathbf{\Theta})} \\
&\times [1-e^{-\alpha H(x;\mathbf{\Theta})}]^{n\beta-1}dx \\
&= n\alpha\beta\int_0^{+\infty} e^{-sx}h(x;\mathbf{\Theta}) \\
&\times \sum_{j=0}^{\infty}\binom{n\beta-1}{j}(-1)^j e^{-(j+1)\alpha H(x;\mathbf{\Theta})}dx \\
&= \sum_{j=0}^{\infty} n\beta\binom{n\beta-1}{j}(-1)^j\int_0^{+\infty}\frac{\alpha(j+1)}{j+1} \\
&\times h(y;\mathbf{\Theta})e^{-(j+1)\alpha H(y;\mathbf{\Theta})-sy}dy \\
&= \sum_{j=0}^{\infty}\binom{n\beta}{j+1}(-1)^j L_1(s),
\end{aligned}
$$

where $L_1(s)$ is the Laplace transform of the EEW$(\alpha(j+1), 1, \mathbf{\Theta})$. Therefore, the moment generating function of EEWPS is

$$
\begin{aligned}
M_X(t) &= \sum_{n=1}^{\infty} p_n L_n(-t) \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} p_n \binom{n\beta}{j+1}(-1)^j L_1(-t) \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} p_n \binom{n\beta}{j+1}(-1)^j M_Y(t).
\end{aligned}
$$

$\square$

Table 1: The four moments of EEWPS model.

α	β	λ	γ	μ_1	μ_2	μ_3	μ_4
0.3	0.3	0.2	2.0	0.936	1.594	3.520	9.164
0.3	0.3	0.2	5.0	0.856	0.884	1.011	1.237
0.3	0.3	0.8	2.0	1.656	3.719	9.722	28.292
0.3	0.3	0.8	5.0	1.150	1.446	1.916	2.631
0.3	2.0	0.2	2.0	2.192	5.446	14.976	44.841
0.3	2.0	0.2	5.0	1.345	1.853	2.606	3.734
0.3	2.0	0.8	2.0	2.835	8.733	28.694	99.531
0.3	2.0	0.8	5.0	1.500	2.285	3.530	5.521
0.8	0.3	0.2	2.0	0.573	0.598	0.808	1.289
0.8	0.3	0.2	5.0	0.704	0.597	0.561	0.565
0.8	0.3	0.8	2.0	1.014	1.394	2.232	3.979
0.8	0.3	0.8	5.0	0.945	0.977	1.064	1.201
0.8	2.0	0.2	2.0	1.342	2.042	3.439	6.306
0.8	2.0	0.2	5.0	1.106	1.252	1.446	1.704
0.8	2.0	0.8	2.0	1.736	3.275	6.589	13.997
0.8	2.0	0.8	5.0	1.233	1.543	1.960	2.519
2.0	0.3	0.2	2.0	0.362	0.239	0.204	0.206
2.0	0.3	0.2	5.0	0.586	0.414	0.324	0.271
2.0	0.3	0.8	2.0	0.641	0.558	0.565	0.637
2.0	0.3	0.8	5.0	0.787	0.677	0.614	0.577
2.0	2.0	0.2	2.0	0.849	0.817	0.870	1.009
2.0	2.0	0.2	5.0	0.921	0.867	0.835	0.819
2.0	2.0	0.8	2.0	1.098	1.310	1.667	2.239
2.0	2.0	0.8	5.0	1.026	1.070	1.131	1.210

Theorem 4.2. *The noncentral moment functions of EEWPS is*

$$
\begin{aligned}
\mu_r &= \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \sum_{j=0}^{\infty} \binom{n\beta}{j+1} (-1)^j \mu_r' \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} p_n \binom{n\beta}{j+1} (-1)^j \mu_r',
\end{aligned}
\tag{17}
$$

where $\mu_r' = E[Y^r]$ and Y has $EEW(\alpha(j+1), 1, \Theta)$.

Proof. We can use $M_X(t)$ to obtain μ_r. But from the direct calculation, proof is obvious. $\qquad\square$

Considering $H(x) = x^\gamma$ and $C(\lambda) = \lambda(1-\lambda)^{-1}$, we calculated the first four moments with different values of parameters for the EEWPS distribution using (17). Also, we computed these values from the direct definition by numerical integration. We found that the results are same. The values are given in Table 1.

4.2 Order statistic

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from EEWPS$(\alpha, \beta, \lambda, \Theta)$, then the pdf of the ith order statistic, say $X_{i:m}$, is given by

$$
\begin{aligned}
f_{i:m}(x) &= \frac{m!}{(i-1)!(m-i)!} f(x) \left[\frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{i-1} \\
&\quad \times \left[1 - \frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{m-i}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m!}{(i-1)!(m-i)!} f(x) \sum_{j=0}^{m-i} \binom{m-i}{j} \\
&\quad \times (-1)^j \left[\frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{j+i-1} \\
&= \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{m-i} p_n g_{(n)}(x; \alpha, n\beta, \Theta) \\
&\quad \times \binom{m-i}{j} (-1)^j \left[\frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{j+i-1} \\
&= \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{m-i} w_j p_n \\
&\quad \times g_{(n)}(x; \alpha, n\beta, \Theta) \left[\frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{j+i-1},
\end{aligned}
$$

where f is the pdf of EEWP class of distributions, $t = 1 - e^{-\alpha H(x;\Theta)}$ and $w_j = \binom{m-i}{j}(-1)^j$. Also, the cdf of $X_{i:m}$ is given by

$$
F_{i:m}(x) = \sum_{k=i}^{m} \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \binom{m}{k} \left[\frac{C(\lambda t^\beta)}{C(\lambda)} \right]^{j+k}.
$$

An analytical expression for rth moment of order statistics $X_{i:m}$ is obtained as

$$
\begin{aligned}
E[X_{i:m}^r] &= \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{m-i} w_j p_n \\
&\quad \times E[Z^r (F(Z))^{j+i-1}],
\end{aligned}
$$

where Z has a EEW distribution with parameters α, $n\beta$ and Θ.

4.3 Shannon entropy and mean residual life

The maximum entropy method is a powerful technique in the field of probability and statistics. It is introduced by Jaynes (1957) and closely related to the Shannon's entropy. Also, it is applied in a wide variety of fields and used for the characterization of pdf's; see, for example, Kapur (1994), Soofi (2000) and Zografos and Balakrishnan (2009). Shore and Johnson (1980) treated the maximum entropy method axiomatically.

Consider a class of pdf's

$$
F = \{ f(x; \alpha, \beta, \lambda, \Theta) : E_f(T_i(X)) = \beta_i, \; i = 0, 1, \ldots, m \},
\tag{18}
$$

where $T_1(X), \ldots, T_m(X)$ are absolutely integrable functions with respect to f, and $T_0(X) = 1$. Also, consider the shannon's entropy of none-negative continuous random variable X with pdf f defined by Shannon (1948)

as

$$H_{sh}(f) = E[-\log f(X)] = -\int_0^{+\infty} f(x) \log(f(x))dx. \tag{19}$$

The maximum entropy distribution is the pdf of the class F, denoted by f^{ME} determined as

$$f^{ME}(x; \lambda, \beta, \boldsymbol{\Theta}) = \arg\max_{f \in F} H_{sh}(f).$$

Now, suitable constraints are derived in order to provide a maximum entropy characterization for the class (18) based on Jaynes (1957). For this purpose, the next result plays an important role.

Proposition 4. *Let X has* EEWPS$(\alpha, \beta, \lambda, \boldsymbol{\Theta})$ *with the pdf given by* (4). *Then,*
i.

$$E\left[\log(C'(\lambda(1 - e^{-\alpha H(X; \boldsymbol{\Theta})})^\beta))\right]$$
$$= \frac{\lambda}{C(\lambda)} E\left[C'(\lambda(1 - e^{-\alpha H(Y; \boldsymbol{\Theta})})^\beta)\right.$$
$$\left. \times \log(C'(\lambda(1 - e^{-\alpha H(Y; \boldsymbol{\Theta})})^\beta))\right],$$

ii.

$$E\left[\log(h(X; \boldsymbol{\Theta}))\right] = \frac{\lambda}{C(\lambda)} E\left[C'(\lambda(1 - e^{-H(Y; \boldsymbol{\Theta})})^\beta)\right.$$
$$\left. \times \log(h(Y; \boldsymbol{\Theta}))\right],$$

iii.

$$E\left[\log(1 - e^{-\alpha H(X; \boldsymbol{\Theta})})\right] = \frac{\lambda}{C(\lambda)}$$
$$\times E\left[C'(\lambda(1 - e^{-\alpha H(Y; \boldsymbol{\Theta})})^\beta) \log(1 - e^{-\alpha H(Y; \boldsymbol{\Theta})})\right],$$

where Y follows the EEW distribution with the pdf in (2).

An explicit expression of Shannon entropy for EEWPS distribution is obtained as

$$H_{sh}(f) = -\log(\alpha\beta\lambda) - \frac{\lambda}{C(\lambda)} E[C'(\lambda(1 - e^{-H(Y; \boldsymbol{\Theta})})^\beta)$$
$$\times \log(C'(\lambda(1 - e^{-H(Y; \boldsymbol{\Theta})})^\beta))]$$
$$+ \log[C(\lambda)] - (\beta - 1)\frac{\lambda}{C(\lambda)}$$
$$\times E[C'(\lambda(1 - e^{-H(Y; \boldsymbol{\Theta})})^\beta) \log(1 - e^{-H(Y; \boldsymbol{\Theta})})]$$
$$- \frac{\lambda}{C(\lambda)} E[C'(\lambda(1 - e^{-H(Y; \boldsymbol{\Theta})})^\beta) \log(h(Y; \boldsymbol{\Theta}))]. \tag{20}$$

Also, the mean residual life function of X is given by

$$m(t) = E[X - t | X > t]$$
$$= \frac{\int_t^{+\infty}(x - t)f(x)dx}{1 - F(t)}$$

$$= \frac{C(\lambda)\sum\limits_{n=1}^{\infty} p_n \int_t^{+\infty} zg_{(n)}(z; \alpha, n\beta, \boldsymbol{\Theta})dz}{C(\lambda) - C(\lambda G(x))} - t$$
$$= \frac{C(\lambda)\sum\limits_{n=1}^{\infty} p_n E[ZI_{(Z>t)}]}{C(\lambda) - C(\lambda G(x))} - t, \tag{21}$$

where Z has a EEW distribution with parameters α, $n\beta$ and $\boldsymbol{\Theta}$.

4.4 Reliability and average lifetime

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X subjected to a random stress Y. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Y$. Hence, $R = P(X > Y)$ is a measure of component reliability. It has many applications especially in engineering concept. Here, we obtain the form for the reliability R when X and Y are independent random variables having the same EEWPS distribution. The quantity R can be expressed as

$$R = \int_0^{\infty} f(x; \alpha, \beta, \lambda, \boldsymbol{\Theta}) F(x; \alpha, \beta, \lambda, \boldsymbol{\Theta})dx$$
$$= \int_0^{\infty} \lambda g(x) \frac{C'(\lambda G(x))C(\lambda G(x))}{C^2(\lambda)}dx \tag{22}$$
$$= \sum_{n=1}^{\infty} p_n \int_0^{\infty} g_{(n)}(x; \alpha, n\beta, \boldsymbol{\Theta}) \frac{C(\lambda G(x))}{C(\lambda)}dx.$$

5 Estimation

In this section, we first study the maximum likelihood estimations (MLE's) of the parameters. Then, we propose an Expectation-Maximization (EM) algorithm to estimate the parameters.

5.1 The MLE's

Let $x_1, \ldots, x_n$ be observed value from the EEWPS distribution with parameters $\boldsymbol{\xi} = (\alpha, \beta, \lambda, \boldsymbol{\Theta})^T$. The log-likelihood function is given by

$$l_n = l_n(\boldsymbol{\xi}; \boldsymbol{x}) = n[\log(\alpha) + \log(\beta) + \log(\lambda)$$
$$- \log(C(\lambda))] + \sum_{i=1}^{n} \log[h(x_i; \boldsymbol{\Theta})]$$
$$- \alpha \sum_{i=1}^{n} H(x_i; \boldsymbol{\Theta}) + (\beta - 1) \sum_{i=1}^{n} \log t_i$$
$$+ \sum_{i=1}^{n} \log(C'(\lambda t_i^\beta)),$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $t_i = 1 - e^{-\alpha H(x_i; \boldsymbol{\Theta})}$. The components of the score function $U(\boldsymbol{\xi}; \boldsymbol{x}) = (\frac{\partial l_n}{\partial \alpha}, \frac{\partial l_n}{\partial \beta}, \frac{\partial l_n}{\partial \lambda},$

$\frac{\partial l_n}{\partial \Theta})^T$ are

$$\frac{\partial l_n}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} H(x_i; \Theta), \qquad (23)$$

$$\frac{\partial l_n}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(t_i) + \sum_{i=1}^{n} \frac{\lambda t_i^\beta \log(t_i) C''(\lambda t_i^\beta)}{C'(\lambda t_i^\beta)}, \qquad (24)$$

$$\frac{\partial l_n}{\partial \lambda} = \frac{n}{\lambda} - \frac{nC'(\lambda)}{C(\lambda)} + \sum_{i=1}^{n} \frac{t_i^\beta C''(\lambda t_i^\beta)}{C'(\lambda t_i^\beta)}, \qquad (25)$$

$$\frac{\partial l_n}{\partial \Theta_k} = \sum_{i=1}^{n} \frac{\partial h(x_i; \Theta)}{\partial \Theta_k} \cdot \frac{1}{h(x_i; \Theta)} - \alpha \sum_{i=1}^{n} \frac{\partial H(x_i; \Theta)}{\partial \Theta_k}$$
$$+ (\beta - 1) \sum_{i=1}^{n} \frac{\frac{\partial t_i}{\partial \Theta_k}}{t_i} + \beta \lambda \sum_{i=1}^{n} \frac{[\frac{\partial t_i}{\partial \Theta_k}] t_i^{\beta-1} C''(\lambda t_i^\beta)}{C'(\lambda t_i^\beta)}, \qquad (26)$$

where Θ_k is the kth element of the vector Θ.

The MLE of ξ, say $\hat{\xi}$, is obtained by solving the nonlinear system $U(\xi; x) = 0$. We cannot get an explicit form for this nonlinear system of equations and they can be calculated by using a numerical method, like the Newton method or the bisection method. Only, for given Θ, from (23) we have

$$\alpha = \frac{n}{\sum_{i=1}^{n} H(x_i; \Theta)}.$$

Therefore, (26) becomes

$$\sum_{i=1}^{n} \frac{\partial h(x_i; \Theta)}{\partial \Theta_k} \cdot \frac{1}{h(x_i; \Theta)}$$
$$- \frac{n}{\sum_{i=1}^{n} H(x_i; \Theta)} \sum_{i=1}^{n} \frac{\partial H(x_i; \Theta)}{\partial \Theta_k} \qquad (27)$$
$$+ (\beta - 1) \sum_{i=1}^{n} \frac{\frac{\partial t_i}{\partial \Theta_k}}{t_i} + \beta \lambda \sum_{i=1}^{n} \frac{[\frac{\partial t_i}{\partial \Theta_k}] t_i^{\beta-1} C''(\lambda t_i^\beta)}{C'(\lambda t_i^\beta)}.$$

Theorem 5.1. *The pdf, $f(x|\Theta)$, of EEWPS distribution satisfies on the regularity condistions, i.e.*

i. *the support of $f(x|\Theta)$ does not depend on Θ,*

ii. *$f(x|\Theta)$ is twice continuously differentiable with respect to Θ,*

iii. *the differentiation and integration are interchangeable in the sense that*

$$\frac{\partial}{\partial \Theta} \int_{-\infty}^{\infty} f(x|\Theta) dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial \Theta} f(x|\Theta) dx,$$
$$\frac{\partial^2}{\partial \Theta \partial \Theta^T} \int_{-\infty}^{\infty} f(x|\Theta) dx = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \Theta \partial \Theta^T} f(x|\Theta) dx.$$

Proof. The proof is obvious and for more details, see Casella and Berger (2001), Section 10. $\qquad \square$

The asymptotic confidence intervals of these parameters will be derived based on Fisher information matrix. It is well-known that under regularity conditions, the asymptotic distribution of $\sqrt{n}(\hat{\xi} - \xi)$ is multivariate normal with mean $\mathbf{0}$ and variance-covariance matrix $J_n^{-1}(\xi)$, where $J_n(\xi) = \lim_{n \to \infty} I_n(\xi)$, and $I_n(\xi)$ is the observed information matrix as

$$I_n(\xi) = - \begin{bmatrix} U_{\alpha\alpha} & U_{\alpha\beta} & U_{\alpha\lambda} & | & U_{\alpha\Theta}^T \\ U_{\alpha\beta} & U_{\beta\beta} & U_{\beta\lambda} & | & U_{\beta\Theta}^T \\ U_{\alpha\beta} & U_{\lambda\beta} & U_{\lambda\lambda} & | & U_{\lambda\Theta}^T \\ - & - & - & - & - \\ U_{\alpha\Theta} & U_{\beta\Theta} & U_{\lambda\Theta} & | & U_{\Theta\Theta} \end{bmatrix},$$

whose elements are obtained by derivative the equations (23)-(26) with respect to parameters.

5.2 EM-algorithm

The traditional methods to obtain the MLE's are numerical methods for solving the equations (23)-(26), and sensitive to the initial values. Therefore, we develop an EM algorithm for obtaining the MLE's of the parameters of EEWPS class of distributions. It is a very powerful tool in handling the incomplete data problem (Dempster et al., 1977). It is an iterative method, and there are two steps in each iteration: expectation step or the E-step and the Maximization step or the M-step. The EM algorithm is especially useful if the complete data set is easy to analyze.

Using (5), we define a hypothetical complete-data distribution with a joint pdf in the form

$$g(x, z; \xi) = \frac{a_z \lambda^z}{C(\lambda)} z \alpha \beta h(x; \Theta)(1 - t) t^{z\beta - 1}, \quad x > 0, \ z \in \mathbb{N},$$

where $t = 1 - e^{-\alpha H(x;\Theta)}$. The E-step of an EM cycle requires the expectation of $(Z|X; \xi^{(r)})$ where $\xi^{(r)} = (\alpha^{(r)}, \beta^{(r)}, \lambda^{(r)}, \Theta^{(r)})$ is the current estimate (in the rth iteration) of ξ.

The expected value of $Z|X = x$ is

$$E(Z|X = x) = 1 + \frac{\lambda t^\alpha C''(\lambda t^\beta)}{C'(\lambda t^\beta)}. \qquad (28)$$

The M-step of EM cycle is completed by using the MLE over Θ, with the missing z's replaced by their conditional expectations given above. Therefore, the log-likelihood for the complete-data $y = (x_1, \ldots, x_n, z_1, ..., z_n)$ is

$$l^*(y; \xi) \propto \sum_{i=1}^{n} z_i \log(\lambda) + n \log(\alpha\beta) + \sum_{i=1}^{n} \log h(x_i; \Theta)$$
$$- \alpha \sum_{i=1}^{n} H(x_i; \Theta) - n \log(C(\lambda))$$
$$+ \sum_{i=1}^{n} (z_i \beta - 1) \log(1 - e^{-\alpha H(x_i;\Theta)}). \qquad (29)$$

On differentiation of (29) with respect to parameters α, β, λ and Θ_k, we obtain the components of the score function as

$$\frac{\partial l_n^*}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} H(x_i; \Theta) + \sum_{i=1}^{n} (z_i \beta - 1)$$
$$\times \frac{H(x_i; \Theta) e^{-\alpha H(x_i; \Theta)}}{1 - e^{-\alpha H(x_i; \Theta)}},$$

$$\frac{\partial l_n^*}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} z_i \log(1 - e^{-\alpha H(x_i; \Theta)}),$$

$$\frac{\partial l_n^*}{\partial \lambda} = \sum_{i=1}^{n} \frac{z_i}{\lambda} - n \frac{C'(\lambda)}{C(\lambda)},$$

$$\frac{\partial l_n^*}{\partial \Theta_k} = \sum_{i=1}^{n} \frac{\partial h(x_i; \Theta)}{\partial \Theta_k} \cdot \frac{1}{h(x_i; \Theta)} - \alpha \sum_{i=1}^{n} \frac{\partial H(x_i; \Theta)}{\partial \Theta_k}$$
$$+ \sum_{i=1}^{n} (z_i \beta - 1) \frac{\frac{\partial H(x_i; \Theta)}{\partial \Theta_k}}{1 - e^{-\alpha H(x_i; \Theta)}}.$$

Therefore, we obtain the iterative procedure of the EM-algorithm as

$$\hat{\beta}^{(j+1)} = \frac{-n}{\sum_{i=1}^{n} \hat{z}_i^{(j)} \log[1 - e^{-\hat{\alpha}^{(j)} H(x_i; \hat{\Theta}^{(j)})}]},$$

$$\hat{\lambda}^{(j+1)} = \frac{C(\hat{\lambda}^{(j+1)})}{nC'(\hat{\lambda}^{(j+1)})} \sum_{i=1}^{n} \hat{z}_i^{(j)},$$

$$\frac{n}{\hat{\alpha}^{(j+1)}} - \sum_{i=1}^{n} H(x_i; \hat{\Theta}^{(j)}) + \sum_{i=1}^{n} (\hat{z}_i^{(j)} \hat{\beta}^{(j)} - 1)$$
$$\times \frac{H(x_i; \hat{\Theta}^{(j)}) e^{-\hat{\alpha}^{(j+1)} H(x_i; \hat{\Theta}^{(j)})}}{1 - e^{-\hat{\alpha}^{(j+1)} H(x_i; \hat{\Theta}^{(j)})}} = 0,$$

$$\sum_{i=1}^{n} \frac{\partial h(x_i; \hat{\Theta}^{(j+1)})}{\partial \Theta_k} \cdot \frac{1}{h(x_i; \hat{\Theta}^{(j+1)})}$$
$$- \hat{\alpha}^{(j)} \sum_{i=1}^{n} \frac{\partial H(x_i; \hat{\Theta}^{(j+1)})}{\partial \Theta_k}$$
$$+ \sum_{i=1}^{n} \frac{\partial H(x_i; \hat{\Theta}^{(j+1)})}{\partial \Theta_k} \cdot \frac{\hat{z}_i^{(j)} \hat{\beta}^{(j)} - 1}{1 - e^{-\hat{\alpha}^{(j)} H(x_i; \hat{\Theta}^{(j+1)})}} = 0,$$

where $\hat{\lambda}^{(j+1)}$, $\hat{\alpha}^{(j+1)}$ and $\hat{\Theta}_k^{(j+1)}$ are found numerically. Here, we have

$$\hat{z}_i^{(j)} = 1 + \frac{\lambda^{*(j)} C''(\lambda^{*(j)})}{C'(\lambda^{*(j)})}, \quad i = 1, 2, ..., n,$$

where $\lambda^{*(j)} = \hat{\lambda}^{(j)} [1 - e^{-\hat{\alpha}^{(j)} H(x_i; \hat{\Theta}_k^{(j)})}]^{\hat{\beta}^{(j)}}$.

We can use the results of Louis (1982) to obtain the standard errors of the estimators from the EM-algorithm. Consider $\ell_c(\Theta; x) = E(I_c(\Theta; y)|x)$, where $I_c(\Theta; y) = -[\frac{\partial U(y; \Theta)}{\partial \Theta}]$ is the $(k+3) \times (k+3)$ observed information matrix. If $\ell_m(\Theta; x) = Var[U(y; \Theta)|x]$, then, we obtain the observed information as

$$J(\hat{\Theta}; x) = \ell_c(\hat{\Theta}; x) - \ell_m(\hat{\Theta}; x). \tag{30}$$

The standard errors of the MLE's based on the EM-algorithm are the square root of the diagonal elements of the $J(\hat{\Theta}; x)$. The computation of these matrices are too long and tedious. Therefore, we did not present the details. Reader can see Mahmoudi and Jafari (2012) how to calculate these values.

6   A real example

In this section, we analyze the real data set given by Murthy et al. (2004) to demonstrate the performance of EEWPS class of distributions in practice. This data set consists of the failure times of 20 mechanical components, and is also studied by Silva et al. (2013):

> 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085
> 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115
> 0.121, 0.125, 0.131, 0.149, 0.160, 0.485

Since the EEWPS distribution can be used for modeling of failure times, we consider this distribution for fitting these data. But, this distribution is a large class of distributions. Here, we consider five sub-models of EEWPS distribution. Some of them are suggested in literature.

(i) The exponentiated Weibull geometric (EWG) distribution, i.e. the EEWPS distribution with $H(x, \Theta) = x^\gamma$ and $C(\lambda) = \lambda(1 - \lambda)^{-1}$.

(ii) The complementary Weibull geometric (CWG) distribution, i.e. the EEWPS distribution with $H(x, \Theta) = x^\gamma$, $C(\lambda) = \lambda(1 - \lambda)^{-1}$ and $\beta = 1$. This distribution is considered by Cordeiro and Silva (2014).

(iii) The generalized exponential geometric (GEG) distribution, i.e. the EEWPS distribution with $H(x, \Theta) = x$ and $C(\lambda) = \lambda(1 - \lambda)^{-1}$. This distribution is considered by Mahmoudi and Jafari (2012).

(iv) The exponentiated Chen logarithmic (ECL) distribution, i.e. the EEWPS distribution with $H(x, \Theta) = \exp(x^\gamma)$ and $C(\lambda) = -\log(1 - \lambda)$.

(v) The complementary Chen logarithmic (CCL) distribution, i.e. the EEWPS distribution with $H(x, \Theta) = \exp(x^\gamma)$, $C(\lambda) = -\log(1 - \lambda)$ and $\beta = 1$. This distribution is considered by Cordeiro and Silva (2014).

The MLE's of the parameters for the distributions are obtained by the EM algorithm given in Section 5. Also, the standard errors of MLE's are computed and given in paracenteses. To test the goodness-of-fit of the distributions, we calculated the maximized

Table 2: Parameter estimates (standard errors), K-S statistic, *p*-value, AIC, AICC, BIC, CM and AD for the data set.

Distribution	EWG	CWG	GEG	ECL	CCL
$\hat{\alpha}$	28.665	25.972	27.752	17.111	22.019
(s.e.)	(4.617)	(11.093)	(6.841)	(1.572)	(10.177)
$\hat{\gamma}$	0.199	1.642	—	0.136	1.586
(s.e.)	(0.052)	(0.407)	—	(0.026)	(0.231)
$\hat{\lambda}$	0.136	0.012	0.001	0. 146	0.261
(s.e.)	(0.918)	(1.122)	(0.658)	(0.032)	(0.332)
$\hat{\beta}$	$5.5\,e^{7}$	—	13.825	$7.4\,e^{7}$	—
(s.e.)	$(1.6\,e^{8})$	—	(8.471)	$2.3\,e^{7}$	—
$\log(L)$	37.978	26.422	32.976	37.794	25.759
K-S	0.124	0.264	0.160	0.121	0.262
p-value	0.917	0.122	0.683	0.931	0.127
AIC	-67.957	-46.845	-59.952	-67.588	-45.518
AICC	-65.29	-45.345	-58.452	-64.922	-44.018
BIC	-63.974	-43.858	-56.965	-63.606	-42.531
CM	0.048	0.436	0.153	0.051	0.463
AD	0.402	2.537	1.136	0.423	2.663

log-likelihood ($\log(L)$), the Kolmogorov-Smirnov (K-S) statistic with its respective p-value, the AIC (Akaike Information Criterion), AICC (AIC with correction), BIC (Bayesian Information Criterion), CM (Cramer-von Mises statistic) and AD (Anderson-Darling statistic) for the five submodels of distribution. The R software (R Development Core Team, 2014) is used for the computations.

The results are given in Table 2, and from K-S, it can be concluded that all five models are appropriate for this data set. But, the EWG and ECL distributions are better than other distributions. In fact, we have a better fit when there is the parameter β (exponentiated parameter) in model. The plots of the densities (together with the data histogram) and cdf's given in Figure 1 confirm this conclusion.
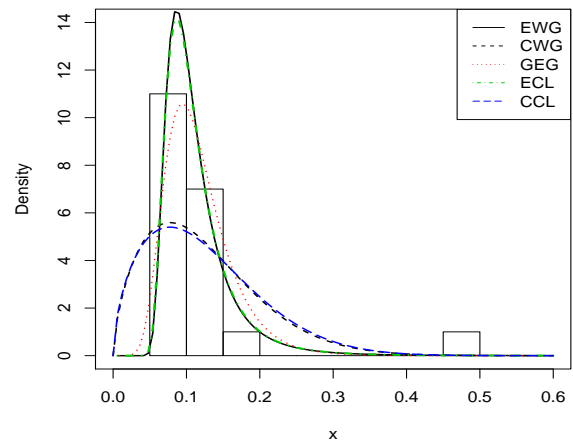
Acknowledgments

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(a) Histogram

(b) Empirical distribution

Figure 1: The histogram of the data set with the estimated pdf's (a), the empirical cdf of the data set, and estimated cdf's (b) for fitted of five submodels.

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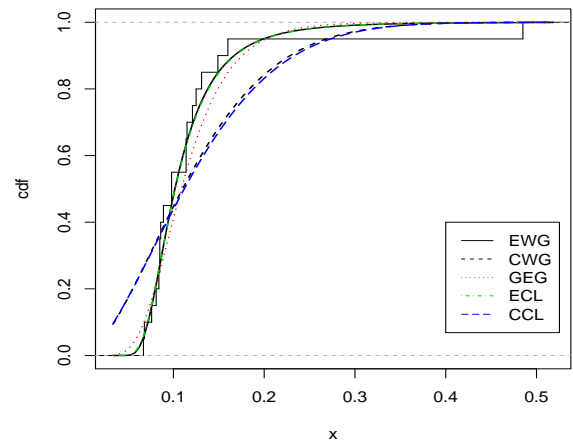
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