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Hopf's theorem for constant mean curvature spheres in $\mathbb{E}^3$ and the Abresch-Rosenberg extension in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

We review Hopf's theorem on immersed spheres with constant mean curvature in the Euclidean space and then the recent work of Abresch and Rosenberg, where they obtain a similar theorem in product spaces.

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1 Introduction

A compact simply-connected surface immersed in the Euclidean space $\mathbb{E}^3$ may be shaped in infinitely many ways, but the only one with *constant mean curvature*, abbreviated by cmc, is an embedded round sphere. This beautiful result was proved by Heinz Hopf ([7]) in 1955 and since then geometers have been moved to generalize it. The analogous problem in any dimension was proved to be true by Alexandrov [3], for embedded hypersurfaces in $\mathbb{E}^n$, so in particular, a compact embedded cmc surface in $\mathbb{E}^3$ is a round sphere, independently of its topological type. The existence of a compact non-spherical immersed (with self-intersections) cmc surface in $\mathbb{E}^3$ defied many generations of geometers until the exhibition in [14] of such a surface of genus one, the famous Wente's torus in $\mathbb{E}^3$. Further developments may be found in the introduction of [1].

In the proof of his theorem, Hopf devised a smooth complex-valued quadratic differential $\mathcal{A}$, defined on the Riemann surface associated to an immersion in $\mathbb{E}^3$, suited to satisfy two properties:

(i) the Hopf diferential $\mathcal{A}$ is holomorphic if, and only if, the immersed surface has constant mean curvature;

(ii) the zeroes of $\mathcal{A}$ coincide with the umbilical points of the surface.

Certainly, Hopf had in mind two significant facts:

(iii) a Riemann surface of genus zero has a unique complex structure, which forces any holomorphic quadratic differential to vanish everywhere.

(iv) a compact and totally umbilical surface in $\mathbb{E}^3$ is a round sphere.

We will show later in this paper how Hopf used (i)-(iv) to prove his theorem.

In 2004, Uwe Abresch and Harold Rosenberg [1] extended Hopf's work to cmc surfaces in 3-dimensional product spaces $\mathbb{M}_c \times \mathbb{R}$, where $\mathbb{M}_c$ denotes the universal 2-dimensional space of constant curvature c, that is, the Euclidean plane $\mathbb{E}^2$ of curvature $c = 0$, a round sphere with radius $1/\sqrt{c}$ in $\mathbb{E}^3$ of curvature $c > 0$, or the hyperbolic plane of curvature $c < 0$. Their study of cmc surfaces invariant by rotations around a vertical axis $\{p_0\} \times \mathbb{R}$ in those ambient spaces was the key to discover a quadratic differential $\mathcal{Q}$, known as the *Abresch-Rosenberg differential*, with the properties of being holomorphic for cmc surfaces and of vanishing everywhere on compact simply-connected rotational cmc surfaces. The main theorem in [1] actually classifies the cmc immersed surfaces of any topological type in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ with $\mathcal{Q} \equiv 0$. One item of that classification asserts that the compact ones are rotational, leading to the Abresch-Rosenberg extension of Hopf's theorem - stated in Theorem 9 of this paper. It is worth to mention that the study of cmc spheres in $\mathbb{S}^2 \times \mathbb{R}$, including their classification, started with Pedrosa's thesis (see [10], [11] and[12]), while cmc spheres in $\mathbb{H}^2 \times \mathbb{R}$ were treated in [8].

Our purpose is to discuss Hopf's work and its recent generalization in [1]. In the preliminaries, we review Hopf's differential and prove his theorem, filling in some details. Thereafter, we discuss the extension of the *Hopf differential* obtained by Abresch and Rosenberg and sketch a proof of the extension result previously mentioned.

2 Preliminaries

Any surface with a riemannian metric admits an atlas of isothermal coordinates (see [5] for a proof), so in particular, an immersed surface in $\mathbb{M}_c \times \mathbb{R}$ with the induced metric is locally parametrized as $F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{M}_c \times \mathbb{R}$ satisfying $F_x.F_x = F_y.F_y = E > 0$ and $F_x.F_y = 0$, where the dot notation is used for the product metric. Assume that both the surface and $\mathbb{M}_c \times \mathbb{R}$ have fixed orientations and let $\{F_x, F_y\}$ be positive. The unit normal field N is chosen to obtain that $\{F_x, F_y, N\}$ is a positive frame in $\mathbb{M}_c \times \mathbb{R}$. A system of *isothermal coordinates* compatible with the surface orientation induces a complex structure. Identifying a pair of positively-oriented isothermal coordinates (x, y) with the local complex coordinate $z = x + iy$, one obtains holomorphic changes of complex coordinates, since the corresponding change of isothermal coordinates satisfies the Cauchy-Riemann equations. The complex derivatives $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and its conjugate $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ act on functions in the standard way. Direct computations give us that a global complex-valued function is holomorphic if, and only if, $\frac{\partial f}{\partial \bar{z}} = 0$ holds locally. Let us write $2F_z = F_x - iF_y$, $2F_{\bar{z}} = F_x + iF_y$ and extend the metric in a formal algebraic way to complex-valued vector fields. As usual, ∇ and $\overline{\nabla}$ denote the riemannian connections in the surface and in the ambient space, respectively.

The connections are algebraically extended to complex-valued vector fields.

The mean curvature of the immersed surface is $H = (tr\tilde{A})/2$, where $\tilde{A}$ denotes the *shape operator* $\tilde{A}(V) = -\overline{\nabla}_V N$ corresponding to the second fundamental form $A(V,W) = \tilde{A}(V).W = (\overline{\nabla}_V W).N$.

It is convenient to shorten the notation in local coordinates. For instance, writting N_x instead of $\overline{\nabla}_{F_x} N$, F_{xx} instead of $\overline{\nabla}_{F_x} F_x$, and so on. Likewise, F_{zz} will mean $\overline{\nabla}_{F_z} F_z$, etc. Locally, one has that $2H = \frac{l+n}{2E}$, where $l = F_{xx}.N$, $m = F_{xy}.N$ and $F_{yy}.N$ are the local coefficients of the second fundamental form A.

3 The theorem of Hopf and its extension

Definition 3.1. *The Hopf differential of an immersed oriented surface in $\mathbb{M}_c \times \mathbb{R}$ is a complex-valued quadratic form $\mathcal{A}$, locally defined in complex coordinates by*

$$\mathcal{A} = \alpha(z)dz^2 \quad \textit{with coefficient} \quad \alpha = \frac{l-n}{2} - im, \tag{1}$$

where l, m and n are the second fundamental form coefficients.

Remark 3.2. *To verify that $\mathcal{A}$ is well-defined in the sense that it does not depend on the coordinates, one uses that a holomorphic change of coordinates $z \to w$ transforms $dz^2 = (dx + idy) \otimes (dx + idy)$ into $[z'(w)]^2 dw^2$. Moreover, the zeroes of $\mathcal{A}$ coincide with the umbilical points of the immersion. Indeed, $\alpha = 0$ means that $l = n$ and $m = 0$, that is, $\tilde{A}$ has both eigenvalues equal to l/E at the point, which is umbilical.*

Proposition 3.3 (Hopf)**.** *The Hopf differential of an oriented and connected immersed surface in $\mathbb{E}^3$ is holomorphic if, and only if, the mean curvature is constant.*

To prove it, Hopf derived the Codazzi equations in complex coordinates (see [7] or the appendix in [9]): $\alpha_{\overline{z}} = EH_z$. Since $E > 0$, the conclusion follows immediately.

Theorem 3.4 (Hopf)**.** *An immersed cmc simply-connected compact surface in $\mathbb{E}^3$ is an embedded round sphere.*

Proof. (following [7] and [13]) Topology implies that the surface is diffeomorphic to a sphere, since it is simply-connected and compact. The uniformization theorem from Complex Analysis (see p.142 of [2] or p. 194 of [6]) asserts that a sphere has a unique complex structure. Therefore, we may assume that the surface of this theorem is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Note that $\hat{\mathbb{C}}$ is covered by two coordinate neighborhoods $\phi_1(\mathbb{C}) = \hat{\mathbb{C}} - \{\infty\}$ and $\phi_2(\mathbb{C}) = \hat{\mathbb{C}} - \{0\}$, respectively parametrized by

$$\phi_1(z) = z,\ z \in \mathbb{C} \quad \text{and} \quad \phi_2(0) = \infty,\ \phi_2(w) = 1/w,\ w \in \mathbb{C} - \{0\}.$$

The hypothesis of cmc guarantees that the Hopf differential is holomorphic, hence it must vanish everywhere. Indeed, consider $\mathcal{A} = \alpha_1(z)dz^2$ and $\mathcal{A} = \alpha_2(w)dw^2$ to be the local expressions of $\mathcal{A}$ in coordinates. On one hand, α_1 and α_2 are entire functions. On the other hand, $\alpha_1(z(w))[z'(w)]^2 = \alpha_2(w)$ must hold in the neighborhoods intersection, for $\mathcal{A}$ is globally defined (see Remark 2). Since the change of coordinates in $\hat{\mathbb{C}} - \{0, \infty\}$ is given by $z = 1/w,\ w \in \mathbb{C} - \{0\}$, and $z'(w) = -1/w^2$, one arrives at

$$(\alpha_1(1/w))\,(1/w^4) = \alpha_2(w),\ w \in \mathbb{C} - \{0\};$$

as $w \to \infty$, one has that $\alpha_1(1/w)$ converges to $\alpha_1(0)$, for α_1 is continuous on the entire complex plane. Thus, the left hand side of the above equality converges to 0 and so does α_2; the latter turns out to be constantly equal to zero, by Liouville theorem applied to an entire bounded function.

Finally, $\mathcal{A} \equiv 0$ means that the surface is totally umbilical. Being compact and connected, it follows that the surface is mapped onto a round sphere $\mathbb{S}^2(a)$ of radius $a > 0$, with $H \equiv 1/a$. The immersion is actually injective, for it is a covering map and $\mathbb{S}^2(a)$ is simply-connected. Conclusion: the surface is embedded onto $\mathbb{S}^2(a)$. □

Remark 3.5. *In [7], Hopf also presents a second proof of his theorem, where he uses the Poincaré theorem on the index of a vector field, instead of the uniformization theorem.*

Definition 3.6. *The Abresch-Rosenberg differential of an immersed oriented surface in $\mathbb{M}_c \times \mathbb{R}$, c any real constant, is the complex-valued quadratic differential form*

$$\mathcal{Q} = 2H\mathcal{A} - c\mathcal{T}, \tag{2}$$

where $\mathcal{A}$ is the Hopf differential, H is mean curvature of the surface and $\mathcal{T}$ is the quadratic differential locally defined by

$$\mathcal{T} = \tau(z)dz^2 \quad \textit{with coefficient} \quad \tau = \left(\frac{r^2 - s^2}{2} - irs\right), \tag{3}$$

where $r = F_x.\mathbf{T}_0$, $s = F_y.\mathbf{T}_0$ and $\mathbf{T}_0 = (\vec{0}, 1)$ is the unit vertical vector field pointing upwards on $\mathbb{M}_c \times \mathbb{R}$.

Remark 3.7. *Note that $\mathbf{T}_0(p, \xi) = (\vec{0}_p, 1) \in T_p\mathbb{M}_c \times \mathbb{R}$ may be interpreted as the gradient of the height function ξ on $\mathbb{M}_c \times \mathbb{R}$, thus $d\xi(V) = V.\mathbf{T}_0$ and in particular, $d\xi(2F_z) = r - is$. In other words, $\mathcal{T} = [(r-is)^2/2]dz^2$ is determined by the quadratic form $d\xi \otimes d\xi$, in the same way the second fundamental form determines the Hopf differential. We finally observe that when $c = 0$ and H is a non-zero constant, then $\mathcal{Q} = 2H\mathcal{A}$, so it extends the Hopf differential up to a constant.*

Proposition 3.8. *([1]) The Abresch-Rosenberg differential of an immersed cmc surface in $\mathbb{M}_c \times \mathbb{R}$ is holomorphic if the mean curvature H is constant.*

This proposition is proved in [1]. A different proof is found in [9], where it is shown that the Codazzi equation in complex coordinates has the form

$$2H(EH_z - \alpha_{\overline{z}}) = -c\tau_{\overline{z}}. \tag{4}$$

When H is constant, the last equation is reduced to $(2H\alpha - c\tau)_{\overline{z}} = 0$, implying that $\mathcal{Q}$ is holomorphic.

Theorem 3.9. *([1]) An immersed cmc simply-connected compact surface in $\mathbb{M}_c \times \mathbb{R}$ is rotationally invariant and embedded.*

Sketch of a proof. The same argument used to prove Hopf's theorem implies that $\mathcal{Q} \equiv 0$, that is,

$$2H\mathcal{A} = \mathcal{T}. \tag{5}$$

The case $H = 0$ is possible only for $c > 0$, since the surface is compact and totally geodesic, thus an embedded slice $\mathbb{M}_c \times \{\xi_0\}$ of constant height.

We follow [9] in the general case $H \neq 0$. Consider the smooth global vector field $\mathbf{T}$ given by the tangential component of $\mathbf{T}_0 = (\vec{0}, 1)$ and rotate it along the surface by an angle of $\pi/2$, obtaining the field $\mathbf{JT}$, clearly orthogonal to $\mathbf{T}_0$. Use that $d\xi(\mathbf{T}) = ||\mathbf{T}||^2$ and $d\xi(\mathbf{JT}) = 0$ in (5) to obtain that the principal curvatures are given by $H + \frac{c||\mathbf{T}||^2}{4H}$ and $H - \frac{c||\mathbf{T}||^2}{4H}$; at non-umbilical points, the corresponding principal directions are $\mathbf{T}$ and $\mathbf{JT}$, respectively.

After this, one proves (technical details in [9]) that:

(i) the orbits of $\mathbf{JT}$ are geodesic circles of constant height, whose centers are expressed by a nice smooth function. As a curiosity, the same analysis shows that the orbits of $\mathbf{JT}$ for non-compact (not our case) immersed cmc surfaces with $\mathcal{Q} \equiv 0$ could also be horocycles.

(ii) the centers of the geodesic circles described in (i) project onto the same point $\{p_0\} \in \mathbb{M}_c$, so the surface is invariant by rotations around the vertical axis $\{p_0\} \times \mathbb{R}$.

To prove that the surface is embedded, the method used in [1] is the analysis of the ordinary differential system describing the rotational surfaces in $\mathbb{M}_c \times \mathbb{R}$. The authors integrate that system and derive analytical equations for the meridians spanning the surfaces, whose figures are clearly embedded. Using coordinates (r, ξ), where r measures the geodesic distance to the vertical axis of rotation, they describe a rotational compact surface in $\mathbb{M}_c \times \mathbb{R}$ with cmc $H \neq 0$ and $\mathcal{Q} \equiv 0$ by the equations:

$$H^2(\xi^2 + r^2) = 1, \tag{$c = 0$}$$

$$4H^2\ \sinh^2\left(\sqrt{\frac{c(4H^2+c)}{16H^2}}\ \xi\right)+(4H^2+c)\sin^2\left(\sqrt{c/4}\ r\right)=c, \qquad (c>0)$$

and, with $4H^2>-c>0$ *required to hold,*

$$4H^2\sin^2\left(\sqrt{\frac{-c(4H^2+c)}{16H^2}}\ \xi\right)+(4H^2+c)\ \sinh^2\left(\sqrt{-c/4}\ r\right)=-c. \quad (c<0)$$

Remark 3.10. *In [1], the authors actually integrate the ordinary differential system (of Riccati equations) corresponding to a cmc surface with null Abresch-Rosenberg differential, whose solutions are essential to their proof of the classification theorem. Other surfaces, distinct from those in the previous theorem, occur when* $c<0$ *and* $4H^2\leq -c$*, corresponding to rotational surfaces with the conformal type of a disk or of a catenoid, besides the special surfaces foliated by horocycles, all conformally equivalent to planes.*

Dedicated: In memory of José Escobar (Chepe)

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