Castillo, René Erlín

The Nemytskii operator on bounded p-variation in the mean spaces

Matemáticas: Enseñanza Universitaria, vol. XIX, núm. 1, junio, 2011, pp. 31-41

Escuela Regional de Matemáticas

Cali, Colombia

Available in: http://www.redalyc.org/articulo.oa?id=46818606003
The Nemytskii operator on bounded \( p \)-variation in the mean spaces

René Erlín Castillo
Universidad Nacional de Colombia

Received Aug. 20, 2010 Accepted Nov. 22, 2010

Abstract
We introduce the notion of bounded \( p \)-variation in the sense of \( L_p \)-norm. We obtain a Riesz type result for functions of bounded \( p \)-variation in the mean. We show that if the Nemytskii operator map the bounded \( p \)-variation in the mean spaces into itself and satisfy some Lipschitz condition there exist two functions \( g \) and \( h \) belonging to the bounded \( p \)-variation in the mean space such that
\[
f(t, y) = g(t)y + h(t), \quad t \in [0, 2\pi], \quad y \in \mathbb{R}.
\]

Keywords: Nemytskii’s Operator.

1 Introduction
Two centuries ago, around 1880, C. Jordan (see [2]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones; since then a number of authors such as, Yu Medved’ev (see [5]), N Merentes (see [3] and [4]), D Waterman (see [9]), M Schramm (see [8]) and recently Castillo and Trousselot had been study different spaces with same type of variation (see [1]). The circle group \( T \) is defined as the quotient \( \mathbb{R}/2\pi\mathbb{Z} \), where, as indicated by the notation, \( 2\pi\mathbb{Z} \) is the group of integral multiples of \( 2\pi \). There is a natural identification between functions on \( T \) and \( 2\pi \)-periodic functions on \( \mathbb{R} \), which allows an implicit introduction of notions such as continuity, differentiability, etc for functions on \( T \). The Lebesgue measure on \( T \) also can be defined by means of the preceding identification: a function \( f \) is integrable on \( T \) if the corresponding \( 2\pi \)-periodic function, which we denote again by \( f \), is integrable on \( [0, 2\pi] \), and we set
\[
\int_T f(t)dt = \int_0^{2\pi} f(x)dx.
\]

Let \( f \) be a real-value function in \( L_1 \) on the circle group \( T \). We define the corresponding interval function by \( f(I) = f(b) - f(a) \), where \( I \) denotes the interval \( [a, b] \). Let \( 0 = t_0 < t_1 < \cdots < t_n = 2\pi \) be a partition of \( [0, 2\pi] \) and \( I_{kx} = [x + t_{k-1}, x + t_k] \), if
\[
V_m (f, T) = \sup \left\{ \int_T \sum_{k=1}^n |f(I_{kx})| \, dx \right\} < \infty
\]
where the supremum is taken over all partition of $[0, 2\pi]$, then $f$ is said to be of variation in the mean (or bounded variation in $L_1$-norm).

We denote the class of all functions which are of bounded variation in the mean by $BVM$. This concept was introduce by Móricz and Siddiqi [6], who investigated the convergence in the mean of the partial sums of $S[f]$, the Fourier series of $f$.

If $f$ is of bounded variation ($f \in BV$) with variation $V(f, T)$, then

$$\int_T \sum_{k=1}^n |f(I_{kx})| \, dx \leq 2\pi V(f, T),$$

and so it is clear that $BV \subset BVM$. A straightforward calculation shows that $BVM$ is a Banach space with norm

$$\|f\|_{BVM} = \|f\|_1 + V_m(f, T).$$

In the present paper we introduce the concept of bounded $p$-variation in the mean in the sense of $L_p[0, 2\pi]$ norm (see Definition 2.1) and prove a characterization of the class $BV_pM$ in terms of this concept.

In 1910 in [7], F. Riesz defined the concept of bounded $p$-variation ($1 \leq p < \infty$) and proved that for $1 < p < \infty$ this class coincides with the class of functions $f$, absolutely continuous with derivative $f' \in L_p[a, b]$. Moreover, the $p$-variation of a function $f$ on $[a, b]$ is given by $\|f'\|_{L_p[a, b]}$, that is

$$V_p(f; [a, b]) = \|f'\|_{L_p[a, b]} \quad (1)$$

In this paper we obtain an analogous result for the class $BV_pM$. More precisely we show that if $f \in BV_pM$ is such that $f'$ is continuous on $[0, 2\pi]$, then $f' \in L_p[0, 2\pi]$ and

$$V_p^m(f) = 2\pi \|f'\|_{L_p}^p$$

(See theorem 2.9).

2 Bounded $p$-variation in the mean

Definition 2.1. let $f \in L_p[0, 2\pi]$ with $1 < p < \infty$. Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ if

$$V_p^m(f, T) = \sup \left\{ \sum_{k=1}^n \int_T \frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \, dx \right\} < \infty, \quad (2)$$

where the supremum is taken over all partitions $P$ of $[0, 2\pi]$, then $f$ is said to be of bounded $p$-variation in the mean.
We denote the class of all functions which are of bounded $p$-variation in the mean by $BV_p M$, that is
\[ BV_p M = \{ f \in L_p [0, 2\pi] : V^m_p (f, T) < \infty \} \] (3)

**Remark 2.2.** For $1 < p < \infty$, it is not hard to prove that
\[ \| f \|_{BV_p M} = \| f \|_{L_p} + \{ V^m_p (f, T) \}^{1/p} \] (4)
defines a norm on $BV_p M$.

**Proposition 2.3.** Let $f$ and $g$ be two functions in $BV_p M$, then
i) $f + g \in BV_p M$,
ii) $kf \in BV_p M$, for any $k \in \mathbb{R}$.

In order words, $BV_p M$ is a vector space.
Moreover
\[ V^m_p (f + g, T) \leq 2^{p-1} [ V^m_p (f, T) + V^m_p (g, T) ] , \]
and
\[ V^m_p (kf, T) = |k|^p V^m_p (f, T) . \]

**Theorem 2.4.** For $1 < p < \infty$, $BV_p M \subset BVM$ and
\[ V_m (f, T) \leq (2\pi)^{2 - \frac{2}{p}} [ V^m_p (f, T) ]^{\frac{1}{p}} . \] (5)

**Demostración.** Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ and consider $f \in BV_p M$, then by Hőlder’s inequality we obtain
\[ \sum_{k=1}^{n} \int_{0}^{2\pi} |f (x + t_k) - f (x + t_{k-1})| \, dx \]
\[ \leq \left( \sum_{k=1}^{n} \int_{0}^{2\pi} |t_k - t_{k-1}| \, dx \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f (x + t_k) - f (x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \, dx \right)^{\frac{1}{p}} \]
\[ \leq (2\pi)^{2 - \frac{2}{p}} [ V^m_p (f, T) ]^{\frac{1}{p}} . \] (6)

Thus $f \in BVM$, therefore $BV_p M \subset BVM$. By (6) we obtain (5). This completes the proof of Theorem 2.4.

**Theorem 2.5.** $\text{Lip} [0, 2\pi] \subset BV_p M$, where $\text{Lip} [0, 2\pi]$ denotes the class of all functions which are Lipschitz on $[0, 2\pi]$. \qed
Demostración.\ Let \( f \in \text{Lip} [0, 2\pi] \), then there exists a positive constant \( M > 0 \) such that
\[
|f(x) - f(y)| \leq M |x - y|,
\]
for all \( x, y \in [0, 2\pi] \).
Let \( P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi \) be a partition of \([0, 2\pi]\), thus
\[
|f(x + t_k) - f(x + t_{k-1})| \leq M |t_k - t_{k-1}|,
\]
from (7) we have
\[
\sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \leq 4\pi M^p
\]
by (8) we get \( f \in \text{BV}_{p,M} \). This completes the proof of the Theorem 2.5.

**Remark 2.6.** By Theorem 2.4 and 2.5, we can observe the following embedding:
\[
\text{Lip} [0, 2\pi] \subset \text{BV}_{p,M} \subset \text{BV}_{M}.
\]

**Theorem 2.7.** Let \( f \in \text{Lip} [0, 2\pi] \) and \( g \in \text{BV}_{p,M} \). Then \( f \circ g \in \text{BV}_{p,M} \).

Demostración.\ Let \( P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi \) be a partition of \([0, 2\pi]\) then
\[
\sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f(g(x + t_k)) - f(g(x + t_{k-1}))|^p}{|t_k - t_{k-1}|^{p-1}} dx
\]
\[
\leq M^p \sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|g(x + t_k) - g(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx.
\]
Thus
\[
\sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f(g(x + t_k)) - f(g(x + t_{k-1}))|^p}{|t_k - t_{k-1}|^{p-1}} dx \leq M^p V^m_p (g, T)
\]
for all partitions of \([0, 2\pi]\). By (9) we obtain
\[
V^m_p (f \circ g, T) \leq M^p V^m_p (g, T).
\]
Hence \( f \circ g \in \text{BV}_{p,M} \).

**Theorem 2.8.** \( \text{BV}_{p,M} \), equipped with the norm defined in Remark 2.1, is a Banach space.

Demostración.\ Let \( \{f_n\} \) be a Cauchy sequence in \( \text{BV}_{p,M} \). Then for any \( \epsilon > 0 \) there exists a positive integer no such that
\[
\|f_n - f_m\|_{\text{BV}_{p,M}} < \epsilon \quad \text{whenever} \quad n, m \geq n_0,
\]
for all partitions of \([0, 2\pi]\). By (10) we obtain
\[
V^m_p (f_n \circ g, T) \leq M^p V^m_p (g, T).
\]
Hence \( f_n \circ g \in \text{BV}_{p,M} \).

Hence \( f_n \circ g \in \text{BV}_{p,M} \).

**Theorem 2.9.** \( \text{BV}_{p,M} \), equipped with the norm defined in Remark 2.1, is a Banach space.

Demostración.\ Let \( \{f_n\} \) be a Cauchy sequence in \( \text{BV}_{p,M} \). Then for any \( \epsilon > 0 \) there exists a positive integer no such that
\[
\|f_n - f_m\|_{\text{BV}_{p,M}} < \epsilon \quad \text{whenever} \quad n, m \geq n_0,
\]
for all partitions of \([0, 2\pi]\). By (10) we obtain
\[
V^m_p (f_n \circ g, T) \leq M^p V^m_p (g, T).
\]
Hence \( f_n \circ g \in \text{BV}_{p,M} \).
From (3) and (9) we have
\[ \|f_n - f_m\|_{L^p} \leq \|f_n - f_m\|_{BV^p_M} < \epsilon \]
Whenever \( n, m \geq n_0 \), this implies that \( \{f_n\}_{n \in \mathbb{N}} \) is Cauchy sequence in \( L^p \) since this space is complete, thus \( \lim_{n \to \infty} f_n \) exits, call it \( f \). By Fatou's lemma and (3) we obtain
\[ \|f_n - f_m\|_{BV^p_M} \leq \liminf_{m \to \infty} \|f_n - f_m\|_{L^p} + \liminf_{m \to \infty} \{V^m_p(f_n - f_m, T)\}^{1/p} < \epsilon \]
whenever \( n \geq n_0 \).

Finally we need to prove that \( f \in BV^p_M \). In other to do that we invoke Fatou's lemma again.
\[ \|f\|_{BV^p_M} \leq \liminf_{n \to \infty} \|f_n\|_{L^p} + \liminf_{m \to \infty} \{V^m_p(f_n, T)\}^{1/p} < \infty. \]
Thus \( f \in BV^p_M \).

This completes the proof of Theorem 2.8.

**Theorem 2.9.** Let \( f \in BV^p_M \) such that \( f' \) is continuous on \( [0, 2\pi] \), then \( f' \in L^p[0, 2\pi] \) and
\[ V^m_p(f) = 2\pi\|f'\|_{L^p}^{p} \] (11)

*Demostración.* Let \( P : 0 = t_0 < t_1 < \ldots < t_n = 2\pi \) be a partition of \( [0, 2\pi] \). By the Mean value theorem there exists \( \epsilon_k \in (x + t_{k-1}, x + t_k) \) for any \( x \in [0, 2\pi] \) such that for \( 1 < p < \infty \)
\[ \frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} = |f'(\epsilon_k)|^p(t_k - t_{k-1}) \] (12)
by (12) we obtain
\[ 2\pi \lim_{|p| \to 0} \sum_{k=1}^{n} |f'(\epsilon_k)|^p(t_k - t_{k-1}) \leq \sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \] (13)
from (13) we have
\[ 2\pi \int_{0}^{2\pi} |f'(x)|^p dx \leq V^m_p(f). \] (14)
Thus (14) implies that \( f' \in L^p[0, 2\pi] \) and also we have
\[ 2\pi\|f'\|_{L^p}^{p} \leq V^m_p(f) \] (15)
on the other hand
\[ f(x + t_k) - f(x + t_{k-1}) = \int_{x+t_{k-1}}^{x+t_k} f'(t) dt \] (16)
by Hölder’s inequality we obtain
\[
\left| \int_{x+tk-1}^{x+tk} f'(t) dt \right|^p \leq \left( \int_{x+tk-1}^{x+tk} |f'(t)|^p dt \right) |t_k - tk_{k-1}|^{p-1},
\]
hence by (16) we get
\[
\left| f(x + tk) - f(x + tk_{k-1}) \right|^p \leq \int_{x+tk-1}^{x+tk} |f'(t)|, dt
\]
then
\[
\sum_{k=1}^{n} \int_{0}^{2\pi} \frac{|f(x + tk) - f(x + tk_{k-1})|^p}{|t_k - tk_{k-1}|^{p-1}} \leq 2\pi \int_{0}^{2\pi} |f'(x)|^p dx.
\]
From (17) we finally have
\[
V_{p}^{m}(f) \leq 2\pi \left\| f' \right\|_{L_{p}}^p
\]
Combining (15) and (16) we obtain (11)

3 Substitution Operators
Let \( \Omega \subset \mathbb{R} \) be a bounded open set. A function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is said to satisfy the Carathéodory conditions if

i) For every \( t \in \mathbb{R} \), the function \( f(\cdot, t) : \Omega \to \mathbb{R} \) is Lebesgue measurable

ii) For a.e. \( x \in \Omega \), the function \( f(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is continuous.

Set
\[
\mathcal{M} = \{ \varphi : \Omega \to \mathbb{R} : \varphi \text{ is Lebesgue measurable} \}
\]
for each \( \varphi \in \mathcal{M} \) define the operator
\[
(Nf\varphi)(t) = f(t, \varphi(t))
\]
The operator \( Nf \) is said to be the substitution or Nemytskii operator generated by the function \( f \).

The purpose of this section is to present one condition under which the operator \( Nf \) maps \( BV_pM \) into itself. Also if \( Nf \) satisfy the hypothesis condition from Lemma 3.1 below we will show that these exist two functions \( g \) and \( h \) which belong to the bounded \( p-variation \) in the mean space such that
\[
f(t, y) = g(t)y + h(t), \quad t \in [0, 2\pi], y \in \mathbb{R}.
\]
Lemma 3.1. \( N_f : BV_p M \to BV_p M \) if there exits a constant \( L > 0 \) such that \( |f(s, \varphi(s)) - f(t, \varphi(t))| \leq L|\varphi(s) - \varphi(t)| \) for every \( \varphi \in \mathcal{M} \).

Demostración. Let \( \varphi \in BV_p M \), then
\[
\sup \left\{ \sum_{k=1}^{n} \int_{\Gamma} \frac{|(N_f \varphi)(x + t_k) - (N_f \varphi)(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \, dx \right\} = \sup \left\{ \sum_{k=1}^{n} \int_{\Gamma} \frac{|f(x + t_k, \varphi(x + t_k)) - f(x + t_{k-1}, \varphi(x + t_{k-1}))|^p}{|t_k - t_{k-1}|^{p-1}} \, dx \right\}
\]
\[
\leq L \sup \left\{ \sum_{k=1}^{n} \int_{\Gamma} \frac{|\varphi(x + t_k) - \varphi(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \, dx \right\} < \infty.
\]
Thus \( N_f \in BV_p M \).

Theorem 3.2. Let \( f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R} \) and the Nemytskii operator \( N_f \) generated by \( f \) and defined by
\[
N_f : BV_p M \to BV_p M
\]
\[
u \to N_f \nu,
\]
with \((N_f \nu)(t) = f(t, \nu(t)), t \in [0, 2\pi] \). If there exists a constant \( K > 0 \) such that
\[
||N_f u_1 - N_f u_2||_{BV_p M} \leq K ||u_1 - u_2||_{BV_p M}
\]
(19) for \( u_1, u_2 \in BV_p M \). Then there exist \( g, h \in BV_p M \) such that
\[
f(t, y) = g(t)y + h(t), \quad t \in [0, 2\pi],
\]
y \( \in \mathbb{R} \).

Demostración. Let \( y \in \mathbb{R} \), define
\[
u_0 : [0, 2\pi] \to \mathbb{R}
\]
\[
t \to \nu_0(t) = y, \quad \text{(a constant function)}
\]
and
\[
N_f : BV_p M \to BV_p M
\]
\[
u_0 \to N_f \nu_0
\]
with \((N_f \nu_0)(t) = f(t, \nu_0(t)) = f(t, y) \). Note that \( f(t, y) \in BV_p M, \forall y \in \mathbb{R} \), by hypothesis.
Next, let \( t, t' \in [0, 2\pi] \), \( t < t' \); \( y_1, y_2, y'_1, y'_2 \in \mathbb{R} \). Now, we define \( u_1, u_2 \) by
\[
\begin{align*}
s \to u_i(s) &= \begin{cases} 
y_i, & 0 \leq s < t, 
y'_i - y_i (s - t), & t \leq s \leq t', 
y'_i, & t' < s \leq 2\pi.
\end{cases}
\end{align*}
\]
Note that each \( u_i \) belong to \( Lip[0, 2\pi] \), thus \( u_1 - u_2 \in Lip[0, 2\pi] \). Then
\[
(u_1 - u_2)(s) = \begin{cases} 
y_1 - y_2, & 0 \leq s < t, 
y'_1 - y'_2 (s - t) + y_1 - y_2, & t \leq s \leq t', 
y'_1 - y'_2, & t' < s \leq 2\pi.
\end{cases}
\]
Observe that
\[
(u_1 - u_2)'(s) = \begin{cases} 
0, & 0 \leq s < t, 
y'_1 - y'_2 - (y'_2 - y_2), & t \leq s \leq t', 
0, & t' < s \leq 2\pi.
\end{cases}
\]
and also that \( (u_1 - u_2)' \) is a continuous function on \([0, 2\pi]\). Now, we can apply Theorem 2.5, obtaining
\[
2\pi \|(u_1 - u_2)'\|^p_{L_p} = 2\pi \int_0^{2\pi} \|(u_1 - u_2)'(s)\|^p ds
\]
\[
= 2\pi \int_t^{t'} \left| \frac{y'_1 - y'_2 + y_2 - y_1}{t' - t} \right|^p ds
\]
\[
= 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}} < \infty,
\]
hence
\[
V^m_p (u_1 - u_2) = 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}}.
\]
Therefore
\[
\|u_1 - u_2\|_{BV_p M} = \|u_1 - u_2\|_{L_p} + \left( V^m_p (u_1 - u_2) \right)^{\frac{1}{p}}
\]
\[
= \|u_1 - u_2\|_{L_p} + \left( 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}} \right)^{\frac{1}{p}}.
\]
By hypothesis \( N_f u_1, N_f u_2 \) belong to \( BV_p M \) and thus \( N_f u_1 - N_f u_2 \in BV_p M \) with
\[
N_f u_i : [0, 2\pi] \to \mathbb{R}
\]
\[
s \to (N_f u_i)(s) = f(s, u_i(s)),
\]
where
\[
f(s, u_i(s)) = \begin{cases} 
  f(s, y_i), & 0 \leq s < t, \\
  f(s, \frac{y_i - y_i(s - t) + y_i}{t - s}), & t \leq s \leq t', \\
  f(s, y_i'), & t' < s \leq 2\pi.
\end{cases}
\]

Next, let us consider the partition $\Pi : 0 < t < t' < 2\pi$, then
\[
\left(\frac{|(N_f u_1 - N_f u_2)(t') - (N_f u_1 - N_f u_2)(t)|^p}{|t' - t|^{p-1}}\right)^{\frac{1}{p}}
\leq \left( V_p^m (N_f u_1 - N_f u_2, T) \right)^{\frac{1}{p}}.
\leq \| N_f u_1 - N_f u_2 \|_{L^p} + \left( V_p^m (N_f u_1 - N_f u_2) \right)^{\frac{1}{p}}.
\leq \| N_f u_1 - N_f u_2 \|_{BV_p, M} \leq K \| u_1 - u_2 \|_{BV_p, M}.
\]

Hence, by hypothesis we have
\[
\left(\frac{|f(t', y'_1) - f(t', y'_2) - f(t, y_1) + f(t, y_2)|^p}{|t' - t|^{p-1}}\right)^{\frac{1}{p}}
\leq K \left[ \| u_1 - u_2 \|_{L^p} + \left( 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}} \right)^{\frac{1}{p}} \right].
\]
(20)

Then
\[
\frac{|f(t', y'_1) - f(t', y'_2) - f(t, y_1) + f(t, y_2)|^p}{|t' - t|^{p-1}}
\leq K^p \left[ \| u_1 - u_2 \|_{L^p} + \left( 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}} \right)^{\frac{1}{p}} \right]^p,
\]
since $(A + B)^p \leq 2^p (A^p + B^p)$ for $A, B \geq 0$, we obtain
\[
\frac{|f(t', y'_1) - f(t', y'_2) - f(t, y_1) + f(t, y_2)|^p}{|t' - t|^{p-1}}
\leq 2^p K^p \left[ \| u_1 - u_2 \|_{L^p} + 2\pi \frac{|y'_1 - y'_2 + y_2 - y_1|^p}{|t' - t|^{p-1}} \right],
\]
and
\[
\frac{|f(t', y'_1) - f(t', y'_2) - f(t, y_1) + f(t, y_2)|^p}{|t' - t|^{p-1}}
\leq (2K)^p \left[ \| u_1 - u_2 \|_{L^p} |t' - t|^{p-1} + 2\pi |y'_1 - y'_2 + y_2 - y_1|^p \right].
\]

Since $f(\cdot, y)$ is continuous, if we let $t' \to t$, then we have
\[
|f(t', y'_1) - f(t', y'_2) - f(t, y_1) + f(t, y_2)| \leq 2\pi K |y'_1 - y'_2 + y_2 - y_1|.
\]
(21)
Next, we make the following substitution:

\[
\begin{cases}
    y_1' = w + z, \\
    y_2' = w, \\
    y_1 = z, \\
    y_2 = 0.
\end{cases}
\]  

(22)

Putting (22) into (21) we get

\[
|f(t, w + z) - f(t, w) + f(t, 0) - f(t, z)| \leq 2\pi K |w + z - w - z| = 0,
\]

thus

\[
f(t, w + z) - f(t, w) + f(t, 0) - f(t, z) = 0,
\]

from this latter equation we have

\[
f(t, w + z) - f(t, 0) = (f(t, w) - f(t, 0)) + (f(t, z) - f(t, 0)).
\]

Writing \( P_t(\cdot) = f(t, \cdot) - f(t, 0) \), then

\[
P_t(w + z) = P_t(w) + P_t(z),
\]

which means that \( P_t \) is additive and also \( P_t(\cdot) = f(t, \cdot) - f(t, 0) \) is a continuous function thus \( P_t(\cdot) \) satisfy the functional Cauchy equation and its unique solution is given by

\[
P_t(y) = g(t)y,
\]

with \( g : [0, 2\pi] \to \mathbb{R}, \ y \in \mathbb{R} \). Let

\[
h : [0, 2\pi] \to \mathbb{R}, \quad t \to h(t) = f(t, 0),
\]

then \( h \in BV_pM \) and \( P_t(y) = f(t, y) - f(t, 0) \) can be reduce to

\[
g(t)y = f(t, y) - h(t),
\]

where \( f(t, y) = g(t)y + h(t) \).

Finally, since

\[
f(t, 1) - f(t, 0) = (P_t(1) + f(t, 0)) - f(t, 0) = g(t)
\]

for \( t \in [0, 2\pi] \), we conclude that \( g \in BV_pM \). Now the proof of Theorem 3.2 is complete.

**Remark 3.3.** The converse of Theorem 3.2 does not hold because \( L_p \) is not a Banach algebra.
References


Author’s address
René Erlín Castillo — Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá-Colombia
e-mail: recastillo@unal.edu.co