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An iterative method for a second order problem with nonlinear two-point boundary conditions

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Abstract
A semi-linear second order ODE under a nonlinear two-point boundary condition is considered. Under appropriate conditions on the nonlinear term of the equation, we define a two-dimensional shooting argument which allows to obtain solutions for some specific situations by the use of Poincaré-Miranda’s theorem. Finally, we apply this result combined with the method of upper and lower solutions and develop an iterative sequence that converges to a solution of the problem.

Keywords: Nonlinear two-point boundary conditions; upper and lower solutions; iterative methods.

MSC(2000): 34B15

1 Introduction
We study the semi-linear second order ODE

\[ u''(t) + g(t, u(t), u'(t)) = 0, \quad 0 < t < T \]  

under a nonlinear two-point boundary condition.

Problem (1) under various boundary conditions has been studied by many authors. In the pioneering work of Picard [18], the existence of a solution for the Dirichlet problem was proved by the well-known method of successive approximations, assuming that $g$ is Lipschitz and $T$ is small. These conditions have been improved by Hamel [9], for the special case of a forced pendulum equation (see also [13], [14]). For general $g = g(\cdot, u)$, the variational approach has been employed already in 1915 by Lichtenstein [12]. However, when $g$ depends on $u'$ the problem has non-variational structure, and different techniques are required. As a historical antecedent of the topological methods, we may mention the shooting method introduced in 1905 by Severini [20]; later on, more abstract topological tools have been applied, such as the Leray-Schauder degree theory. For an overview of the use of topological methods to this kind of problems, we refer the reader to [15].

The above-mentioned two-point boundary conditions, as well as some other standard ones, such as the Neumann or the Sturm-Liouville conditions, are linear;
it is worthy to mention, however, that the general nonlinear case

$$\phi(u(0), u(T), u'(0), u'(T)) = 0,$$  \hspace{1cm} (2)

where $\phi : \mathbb{R}^4 \to \mathbb{R}$ is continuous is very important in applications and, in recent years, a considerable number of works have been developed in this direction.

We shall study the existence of solutions of (1) under a particular case of condition (2): namely, nonlinear boundary conditions of the type

$$u'(0) = f_0(u(0)), \quad u'(T) = f_T(u(T))$$  \hspace{1cm} (3)

where $f_0, f_T : \mathbb{R} \to \mathbb{R}$ are given continuous functions. The special case $f_i(x) = a_i x + b_i$ for $i = 0, T$ corresponds to the Sturm-Liouville conditions, and Neumann conditions when $a_0 = a_T = 0$. Our interest in (3) relies on some models in nonlinear beam theory, usually leading to fourth order problems \cite{7}, but that admit second order analogues (see e.g. \cite{19}). The results in the present paper complement and extend those in \cite{1}.

The paper is organized as follows. In the second section, we impose a growth condition on $g$, which allows to prove the unique solvability of the associated Dirichlet problem. Furthermore, we prove that the trace mapping $Tr : \mathcal{S} \to \mathbb{R}^2$ given by $Tr(u) = (u(0), u(T))$, where

$$\mathcal{S} := \{u \in H^2(0, T) : u''(t) + g(t, u(t), u'(t)) = 0\}$$  \hspace{1cm} (4)

is a homeomorphism for the $H^2$-norm.

Then, we define a two-dimensional shooting argument, which proves to be successful with the aid of the Poincaré-Miranda theorem (see e.g. \cite{11}) in some particular situations, which include the Sturm-Liouville boundary conditions. This generalizes some of the results in \cite{2}, and constitutes the main tool for our iterative method for problem (1)-(3), developed in the third section.

Our method, based on the existence of an ordered couple $(\alpha, \beta)$ of a lower and an upper solution, has been successfully applied to different boundary value problems when $g$ does not depend on $u'$. For general $g$, existence results can still be obtained if one assumes a Nagumo-Bernstein type condition (see \cite{3}, \cite{16}). However, these results are usually proved by fixed point or degree arguments and, in consequence, they are non-constructive.

We shall assume instead a Lipschitz condition on $u'$, which is more restrictive, but allows the construction of a non-increasing (resp. non-decreasing) sequence of upper (lower) solutions that converges to a solution of the problem. Our method is slightly different from the monotone techniques known in the literature for linear boundary conditions, see e.g. \cite{4}, \cite{17} among others (for upper and lower solutions in the reversed order, see \cite{10}).
2 A continuum of solutions of (1)

For simplicity, let us assume that \( g \) is continuous, and write it as

\[
g(t, u(t), u'(t)) = r(t)u'(t) + h(t, u(t), u'(t)),
\]

with \( r \in W^{1,\infty}(0, T) \). We shall assume that \( h \) satisfies a global Lipschitz condition on \( u' \), namely

\[
\left| \frac{h(t, u, A) - h(t, u, B)}{A - B} \right| \leq k < \frac{\pi}{T} \quad \text{for} \quad A \neq B. \tag{5}
\]

Furthermore, in this section we shall assume the following one-side growth condition on \( u' \):

\[
\frac{h(t, u, A) - h(t, v, A)}{u - v} \leq c \tag{6}
\]

for \( u \neq v \), where the constant \( c \in \mathbb{R} \) satisfies

\[
c + \frac{k\pi}{T} < \left( \frac{\pi}{T} \right)^2 + \frac{1}{2} \inf_{0 \leq t \leq T} r'(t). \tag{7}
\]

Under these assumptions, the set \( S \) of solutions of (1) defined by (4) is homeomorphic to \( \mathbb{R}^2 \). More precisely,

**Theorem 1.** Assume that (5) and (6) hold and let \( x, y \in \mathbb{R} \). Then there exists a unique solution \( u_{x,y} \) of (1) satisfying the non-homogeneous Dirichlet condition

\[
u_{x,y}(0) = x, \quad u_{x,y}(T) = y.
\]

Furthermore, the mapping \( Tr : (S, \| \cdot \|_{H^2}) \to \mathbb{R}^2 \) given by \( Tr(u) = (u(0), u(T)) \) is a homeomorphism.

**Proof.** For fixed \( v \in H^1(0, T) \), let \( u := Tv \) be defined as the unique solution of the linear problem

\[
u'' = -[rv' + h(\cdot, v, v')]
\]

\[
u(0) = x, \quad u(T) = y.
\]

It is immediate that \( T : H^1(0, T) \to H^1(0, T) \) is compact. Moreover, if \( S_\sigma : H^2(0, T) \to L^2(0, T) \) is the semilinear operator defined by \( S_\sigma u := u'' + \sigma [rv' + h(\cdot, u, u')] \), with \( \sigma \in [0, 1] \), then using (6) and (7) it is seen that the following a priori bound holds for any \( u, v \in H^2(0, T) \) with \( u - v \in H^1_0(0, T) \):

\[
\|u' - v'\|_{L^2} \leq \mu \|S_\sigma u - S_\sigma v\|_{L^2} \tag{8}
\]

for some constant \( \mu \) independent of \( \sigma \).

Hence, if \( u = \sigma Tu \) for some \( \sigma \in [0, 1] \), then setting \( l_{x,y}(t) = \frac{y-x}{T} t + x \) we obtain:

\[
\|u' - \sigma l'_{x,y}\|_{L^2} \leq \mu \|S_\sigma(\sigma l_{x,y})\|_{L^2} \leq C
\]
for some constant $C$ depending only on $x$ and $y$. Existence of solutions follows from the Leray-Schauder Theorem. Uniqueness is an immediate consequence of (8) for $\sigma = 1$.

Thus, $Tr$ is bijective, and its continuity is clear. On the other hand, if $(x, y) \to (x_0, y_0)$, then applying (8) to $u = u_{x,y} - l_{x,y}$ and $v = u_{x_0,y_0} - l_{x_0,y_0}$ it is easy to see that $u_{x,y} \to u_{x_0,y_0}$ for the $H^1$-norm. As $u_{x,y}$ and $u_{x_0,y_0}$ satisfy (1), we conclude from (5) that also $u''_{x,y} \to u''_{x_0,y_0}$ for the $L^2$-norm and so completes the proof. $\Box$

It is worth noticing that the previous result allows to define a two-dimensional shooting argument as follows: let $\Theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\Theta(x, y) = (u'_{x,y}(0) - f_0(x), u'_{x,y}(T) - f_T(y)).$$

From the previous theorem, we deduce that $\Theta$ is continuous, and it is clear that, if $\Theta(x, y) = (0, 0)$, then $u_{x,y}$ is a solution of the problem.

**Example 1.** Assume that (5) and (6) hold, and that

$$h(t, u, 0) \text{sgn}(u) < 0 \quad \text{for} \quad |u| \geq M, \quad (9)$$

$$f_0(M^+) \geq 0 \geq f_0(M^-), \quad f_T(M^+) \leq 0 \leq f_T(M^-) \quad (10)$$

for some constants $M^- \leq -M < M \leq M^+$. Then (1)-(3) admits at least one solution.

In particular, the result holds for the Sturm-Liouville conditions

$$u'(0) = a_0u(0) + b_0, \quad u'(T) = a_Tu(T) + b_T, \quad a_0 > 0 > a_T. \quad (11)$$

Furthermore, in this case the solution is unique, provided that $c < 0$ in (6).

Indeed, it follows from (9) that $u_{x,y}$ cannot attain in $(0, T)$ neither a maximum value larger than $M$, nor a minimum value smaller than $-M$. Moreover, for $M^- \leq y \leq M^+$ we obtain:

$$u_{M^+,y}(0) = M^+ \geq y = u_{M^+,y}(T), \quad u_{M^-,y}(0) = M^- \leq y = u_{M^-,y}(T).$$

Thus, $u'_{M^+,y}(0) \leq 0 \leq u'_{M^-,y}(0)$, and hence $\Theta_1(M^+, y) \leq 0 \leq \Theta_1(M^-, y)$. In the same way, it follows that $\Theta_2(x, M^+) \geq 0 \geq \Theta_2(x, M^-)$ for $M^- \leq x \leq M^+$. By the Poincaré-Miranda’s generalized intermediate value theorem, we conclude that $\Theta$ has at least one zero $(x, y) \in [M^-, M^+] \times [M^-, M^+]$.

On the other hand, if $u$ and $v$ are solutions of (1)-(11), then

$$(u - v)'' + (r + \psi)(u - v)' + h(\cdot, u, v') - h(\cdot, v, v') = 0$$

where

$$\psi = \frac{h(\cdot, u, u') - h(\cdot, u, v')}{u' - v'} \in L^\infty(0, T).$$
Next, take $p(t) = e^{\int_0^t (r(s)+\psi(s)) \, ds}$, multiply the previous equality by $(u - v)p$ and integrate. We obtain:

$$0 = p(u' - v')(u - v)igg|_0^T - \int_0^T p(u' - v')^2 + \int_0^T p[h(\cdot, u, v') - h(\cdot, v, v')](u - v)$$

$$\leq p(T)a_T(u - v)^2(T) - a_0(u - v)^2(0) - \int_0^T p(u' - v')^2 + c \int_0^T p(u - v)^2.$$ 

Hence, for $c < 0$ it is seen that $u = v$.

3 Iterative sequences of upper and lower solutions

In this section we shall construct solutions of (1) under the two-point boundary condition (3) by an iterative method, based upon the existence of upper and lower solutions.

Let us recall that $(\alpha, \beta)$ is an ordered couple of a lower and an upper solution for (1) if $\alpha \leq \beta$ and

$$\alpha'' + g(\cdot, \alpha, \alpha') \geq 0 \geq \beta'' + g(\cdot, \beta, \beta').$$

Existence results under various boundary conditions in presence of an ordered couple of a lower and an upper solution are known (see e. g. [6]). In our particular case, we shall assume the boundary constraints:

$$\alpha'(0) - f_0(\alpha(0)) \geq 0 \geq \beta'(0) - f_0(\beta(0)),$$

$$\alpha'(T) - f_T(\alpha(T)) \leq 0 \leq \beta'(T) - f_T(\beta(T))$$

and a Nagumo type condition adapted from [5]:

$$|g(t, u, v)| \leq \psi(|v|), \quad \text{for } \alpha(t) \leq u \leq \beta(t), m \leq |v| \leq M \quad (12)$$

where $\psi : [0, +\infty) \to (0, +\infty)$ is continuous and satisfies:

$$\int_m^M \frac{1}{\psi(t)} \, dt > T,$$

and

$$m = \min \left\{ \frac{\alpha(0) - \beta(T)}{T}, \frac{\alpha(T) - \beta(0)}{T}, \max_{\alpha(0) \leq s \leq \beta(0)} |f_0(s)|, \max_{\alpha(T) \leq s \leq \beta(T)} |f_T(s)| \right\}$$

$$M > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, m\}.$$ 

Then, the following existence result can be obtained as in [1]:

**Theorem 2.** Assume there exists an ordered couple $(\alpha, \beta)$ of a lower and an upper solution as before, and that (12) holds. Then the boundary value problem (1)-(3) admits at least one solution $u$, with $\alpha \leq u \leq \beta$. 

Sketch of the proof: The proof follows the outline of the standard results on the subject. Let $P(t,u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$ and $Q(v) = sgn(v)\min\{|v|, M\}$, and apply Schauder’s Theorem in order to obtain a solution of the problem

$$u''(t) - \lambda u(t) = -g(t, P(t, u(t)), Q(u'(t))) - \lambda P(t, u(t)), \quad u'(0) = f_0(P(0, u(0)), \quad u'(T) = f_T(P(T, u(T))$$

for some fixed $\lambda > 0$. It is easy to see that $\alpha \leq u \leq \beta$, and hence $P(t, u(t)) = u(t)$ for every $t$. Furthermore, if we suppose that for example $u'(t_1) = M$, then there exists $t_0$ such that $u'(t_0) = m$ and $m < u'(t) < M$ for $t$ between $t_0$ and $t_1$. Hence

$$T < \int_M^1 \frac{1}{\psi(s)} ds = \int_{t_0}^{t_1} \frac{u''(t)}{\psi(u'(t))} dt \leq |t_1 - t_0|,$$

a contradiction. The same conclusion holds if we suppose $u'(t_1) = -M$; thus, $|u'(t)| < M$ and the proof is complete.

\[ \square \]

Example 2. The previous result applies when (9) and (10) hold: indeed, in this case it is clear $(M^-, M^+)$ is an ordered couple of a lower and an upper solution. Thus, conditions (3) and (6) in example 1 can be dropped.

Also, we may consider the forced pendulum equation with friction

$$u'' + ru' + \sin u = \theta,$$

and assume that the forcing term $\theta$ is a measurable function satisfying:

$$-1 \leq \theta(t) \leq 1 \quad \forall t \in [0, T].$$

Then $\alpha \equiv \frac{\pi}{2}$ and $\beta \equiv \frac{3}{2}\pi$ are respectively a lower and an upper solution. Hence, (1)-(3) has a solution for any continuous $f_0$ and $f_T$ such that

$$f_0\left(\frac{\pi}{2}\right) \leq 0 \leq f_0\left(\frac{3\pi}{2}\right)$$

and

$$f_T\left(\frac{\pi}{2}\right) \geq 0 \geq f_T\left(\frac{3\pi}{2}\right).$$

Our last result is concerned with the construction of solutions by iteration, provided that $h$ and $f$ satisfy some stronger assumptions.

Let us firstly establish the following auxiliary lemmas:

**Lemma 1.** Assume that (5) holds and let $\lambda$ be a positive constant satisfying $\lambda \geq k_T^2 - \left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\text{infr}'$. Then for any $z, \theta \in C([0, T])$ the equation

$$u'' + ru' + h(\cdot, z, u') - \lambda u = \theta$$

is uniquely solvable under the Sturm-Liouville conditions (11). Furthermore, the mapping $K : C([0, T])^2 \rightarrow C([0, T])$ given by $K(z, \theta) = u$ is compact.
Proof. Existence and uniqueness follow as a particular case of example 1, with $\bar{g}(\cdot,u,u') = ru' + \bar{h}(\cdot,u,u')$, where

$$
\bar{h}(\cdot,u,u') = h(\cdot,z,u') - \lambda u - \theta.
$$

Indeed, it is clear that $\bar{h}$ satisfies (5) and (6) with $c = -\lambda$. Moreover,

$$
\bar{h}(t,u,0)\text{sgn}(u) = (h(t,z(t),0) - \theta(t))\text{sgn}(u) - \lambda|u| < 0
$$

when $|u| > \|h(\cdot,z,0) - \theta\|_{\infty}$. Thus, (9) is also satisfied.

Let $(z,\theta)$ tend to $(z_0,\theta_0)$, and set $u = \mathcal{K}(z,\theta)$, $u_0 = \mathcal{K}(z_0,\theta_0)$. Then

$$(u - u_0)'' + (r + \psi)(u - u_0)' - \lambda(u - u_0) = h(\cdot,z,u_0') - h(\cdot,z_0,u_0') + \theta - \theta_0$$

where $\psi = \frac{h(\cdot,z,u') - h(\cdot,z_0,u_0')}{u - u_0}$. Hence, continuity of $\mathcal{K}$ is a consequence of the following estimate, which is valid for any $w$ satisfying (11) with $b_0 = b_T = 0$ and some constant $c$ depending only on $k$:

$$
\|w\|_{H^1} \leq c\|w'' + (r + \psi)w' - \lambda w\|_{L^2}.
$$

Indeed, this is easily deduced by applying the Cauchy-Schwartz inequality to the integral $\int_0^T pLw.w$, where $Lw = w'' + (r + \psi)w' - \lambda w$ and $p(t) = e^{\int_0^t (r(s) + \psi(s)) \, ds}$, and the fact that $0 < m \leq p \leq M$ for some $m$ and $M$ depending only on $k$.

Finally, compactness of $\mathcal{K}$ follows from the imbedding $H^1(0,T) \hookrightarrow C([0,T])$. \qed

Remark 1. In the previous proof, an analogous estimate can be also obtained for the $H^2$-norm of $w$. This implies the compactness of $\mathcal{K}$, but now regarded as an operator from $C([0,T])^2$ to $C^1([0,T])$. More generally, one might consider also $a_i$ and $b_i$ as variables for $i = 0,T$: in this case, $\mathcal{K}$ could be seen as a compact operator from $\mathbb{R}^4 \times C([0,T])^2$ to $C^1([0,T])$.

Lemma 2. Let $\phi \in L^\infty(0,T)$ and assume that $w'' + \phi w' - \lambda w \geq 0$ (in the weak sense) for some $\lambda \geq 0$, and

$$
w'(0) - a_0 w(0) \geq 0 \geq w'(T) - a_T w(T)
$$

with $a_0 > 0 > a_T$. Then $w \leq 0$.

Proof. If $w(0),w(T) \leq 0$, the result is the well-known maximum principle for Dirichlet conditions.

If for example $w(0) > 0$, then restricting $w$ up to its first zero if necessary, it suffices to consider only the case $w \geq 0$. Taking $p(t) = e^{\int_0^t \phi(s) \, ds}$, it is observed that $(pw')' \geq \lambda pw \geq 0$. Thus, $pw'$ is nondecreasing on $[0,T]$, and hence

$$
0 \geq p(T)w(T) \geq p(T)w'(T) \geq p(0)w'(0) \geq p(0)a_0 w(0) > 0,
$$

a contradiction. The proof is similar when $w(T) > 0$. \qed

In order to define our iterative scheme, we shall assume that $f_0$ and $f_T$ satisfy a one-side Lipschitz condition:
There exists a positive constant $R$ such that
\[ f_0(y) - f_0(x) \leq R(y - x) \]
if $\alpha(0) \leq x < y \leq \beta(0)$, and
\[ f_T(y) - f_T(x) \geq -R(y - x) \]
if $\alpha(T) \leq x < y \leq \beta(T)$.

In virtue of Lemma 1, if (5) holds then for $\lambda = \min \{R, k \pi - \left( \frac{\pi}{T} \right)^2 - \frac{1}{2} \inf r' \}$, we may define the compact operator $T : C([0, T]) \to C([0, T])$ given by $Tv = u$, where $u$ is the unique solution of the problem
\[ u'' + ru' + h(\cdot, v, u') - \lambda u = -\lambda v \]
satisfying the following Sturm-Liouville condition:
\[ u'(0) - Ru(0) = f_0(v(0)) - Rv(0), \quad u'(T) + Ru(T) = f_T(v(T)) + Rv(T). \]

From Remark 1, we observe, moreover, that the set $T(\{v : \alpha \leq v \leq \beta\})$ is bounded for the $C^1$-norm. In particular, this implies the existence of a constant $M = M(R)$ such that if $u = Tv$ for some $v$ lying between $\alpha$ and $\beta$, then $\|u'\|_\infty \leq M$. This suggests to consider the following Lipschitz condition on $h$:
\[ |h(t, u, A) - h(t, v, A)| \leq R|u - v| \]
for $u, v$ such that $\alpha(t) \leq u < v \leq \beta(t)$ and $|A| \leq M(R)$.

**Remark 2.** Conditions (F) and (H) are trivially satisfied if $f_0$, $f_T$ and $h$ are $C^1$ functions, and $\frac{\partial h}{\partial u}$ is bounded with respect to $u'$.

**Theorem 3.** Assume there exists an ordered couple $(\alpha, \beta)$ of a lower and an upper solution as before. Further, assume that (5), (H) and (F) hold. Set $\lambda$ as before, and define the sequences $\{u_n\}$ and $\{\overline{u}_n\}$ recursively by
\[ u_0 = \alpha, \quad \overline{u}_0 = \beta \]
and
\[ u_{n+1} = Tu_n, \quad \overline{u}_{n+1} = T\overline{u}_n \]
Then $(u_n, \overline{u}_n)$ is an ordered couple of a lower and an upper solution. Furthermore, $(\{\overline{u}_n\})$ (resp. $(\{u_n\})$ is non-increasing (non-decreasing) and converges to a solution of the problem.
Proof. Let us firstly prove that $\alpha \leq u_1 \leq \beta$. From the definition,

$$u''_1 + r u'_1 + h(\cdot, \beta, u'_1) - \lambda u_1 = -\lambda \beta \geq -\lambda \beta + \beta'' + r \beta' + h(\cdot, \beta, \beta').$$

Hence, setting

$$\psi = \frac{h(\cdot, \beta, u'_1) - h(\cdot, \beta, \beta')}{u'_1 - \beta'} \in L^\infty(0, T)$$

we deduce that

$$(\overline{u} - \beta)'' + (r + \psi)(\overline{u} - \beta)' - \lambda (\overline{u} - \beta) \geq 0.$$ 

On the other hand,

$$\overline{u}'_1(0) - R\overline{u}_1(0) = f_0(\beta(0)) - R\beta(0)$$

and

$$\overline{u}'_1(T) + R\overline{u}_1(T) = f_T(\beta(T)) + R\beta(T).$$

Thus,

$$(\overline{u} - \beta)'(0) - R(\overline{u} - \beta)(0) = 0 = (\overline{u} - \beta)'(T) - R(\overline{u} - \beta)(T),$$

and from Lemma 2 we obtain that $\overline{u}_1 \leq \beta$.

In the same way,

$$\overline{u}'' + r \overline{u}' + h(\cdot, \beta, \overline{u}') - \lambda \overline{u} \leq -\lambda \beta + \alpha'' + r \alpha' + h(\cdot, \alpha, \alpha')$$

and hence

$$(\overline{u} - \alpha)'' + (r + \psi)(\overline{u} - \alpha)' - \lambda (\overline{u} - \alpha) \geq 0$$

where

$$\psi = \frac{h(\cdot, \alpha, u'_1) - h(\cdot, \alpha, \alpha')}{u'_1 - \alpha'} \in L^\infty(0, T).$$

Also

$$\overline{u}'_1(0) - R\overline{u}_1(0) = f_0(\alpha(0)) - R\alpha(0) \leq f_0(\alpha(0)) - R\alpha(0)$$

and

$$\overline{u}'_1(T) + R\overline{u}_1(T) = f_T(\alpha(T)) + R\alpha(T) \geq f_T(\alpha(T)) + R\alpha(T),$$

and we conclude that $\overline{u}_1 \geq \alpha$.

On the other hand,

$$\overline{u}'' + r \overline{u}' + h(\cdot, \overline{u}, \overline{u}') = (\lambda - R)(\overline{u} - \beta) + h(\cdot, \overline{u}, \overline{u}') + R\overline{u} - |h(\cdot, \beta, \overline{u}') + R\beta| \leq 0,$$

and we deduce that $\overline{u}_1$ is an upper solution of the problem. Inductively, it follows that $\overline{u}_n$ is an upper solution for every $n$, with $\alpha \leq \overline{u}_{n+1} \leq \overline{u}_n$, which by Dini’s
theorem implies that $\bar{u}_n$ converges uniformly to a function $\bar{u}$. From the definition of $\{\bar{u}_n\}$,

$$\bar{u}_{n+1}'' + r\bar{u}_{n+1}' + h(\cdot, \bar{u}_n, \bar{u}_n') \to 0$$

uniformly. Moreover, from Lemma 1 and Remark 1 we know that $\{\bar{u}_n\}$ is bounded in $H^2(0,T)$, and it follows easily that

$$\bar{u}'' + r\bar{u}' + h(\cdot, \bar{u}, \bar{u}') = 0.$$ 

Thus, $\bar{u}$ is a solution of the problem. The proof for $u_n$ is analogous. Moreover, if we assume as inductive hypothesis that $u_n \leq \bar{u}_n$, then

$$\bar{u}_{n+1}'' + r\bar{u}_{n+1}' + h(\cdot, \bar{u}_n, \bar{u}_n') - \lambda \bar{u}_{n+1} = -\lambda \bar{u}_n$$

$$\leq -\lambda \bar{u}_n = u_{n+1}'' + r\bar{u}'_{n+1} + h(\cdot, \bar{u}_n, \bar{u}'_{n+1}) - \lambda \bar{u}_{n+1}.$$ 

In the same way as before, we may define

$$\psi = \frac{h(\cdot, \bar{u}_n, \bar{u}'_{n+1}) - h(\cdot, \bar{u}_n, \bar{u}'_{n+1})}{\bar{u}'_{n+1} - \bar{u}'_{n+1}} \in L^\infty(0,T),$$ 

and hence for $w = \bar{u}_{n+1} - \bar{u}_{n+1}$ we deduce:

$$w'' + (r + \psi)w' - \lambda w \leq h(\cdot, \bar{u}_n, \bar{u}'_{n+1}) - h(\cdot, \bar{u}_n, \bar{u}'_{n+1}) \leq -R(\bar{u}_n - \bar{u}_n) \leq -Rw.$$ 

From Lemma 2, we conclude that $w \geq 0$, i.e. $u_{n+1} \leq \bar{u}_{n+1}$. \qed

**Remark 3.** It is interesting to observe that, even if (5) is somewhat too restrictive, some condition regarding the growth of $h$ with respect to $u'$ is required. We may recall, for instance, the following example by Habets and Pouso [8] for the mean curvature operator:

$$
\left( \frac{u'}{\sqrt{1 + uu'}} \right)' = u + a,
$$

where the function $a \in L^\infty(0,T)$ is defined by

$$a(t) = 2[\chi_{[0,\frac{T}{2}]}(t) - \chi_{(\frac{T}{2},T]}(t)] = \begin{cases} 
2 & 0 \leq t \leq \frac{T}{2} \\
-2 & \frac{T}{2} < t \leq T
\end{cases}$$

Under conditions (11) with $b_0 = b_T = 0$, it is seen that $\alpha = -3$ and $\beta = 3$ is an ordered couple of a lower and an upper solution, but the equation has no solutions when $T > 2\sqrt{2}$. However, here

$$h(\cdot, u, u') = (u + a) \left( \sqrt{1 + uu'} \right)^{3/2},$$

and (9) is satisfied. This explains the need of the Nagumo condition, or at least a similar one, in Theorem 2.
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