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Iterative strategies associated with the normal flow technique on the nonlinear analysis of structural arches

Estratégias de iteração associadas à técnica do fluxo normal na análise não linear de arcos estruturais

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Abstract

A large part of the numerical procedures for obtaining the equilibrium path or load-displacement curve of structural problems with static nonlinear behavior is based on the Newton-Raphson iterative scheme to which are coupled the path-following methods. In this context, this study uses one technique, referred to as normal flow, in the process of obtaining the approximate nonlinear static response of structural systems. Basically, this technique is an adaptation made with in the Newton-Raphson iterative scheme in an attempt to speed up the nonlinear solution process and/or remove convergence problems. To overcome the critical points and to trace the whole nonlinear equilibrium path, three different strategies are used in association with the normal flow technique: the cylindrical arc-length, the minimum residual displacement norm and the generalized displacement. With this procedure, the performance of these strategies when associated with the normal flow technique is valued. Two arches with highly nonlinear load-displacement curves are used in the study. The results obtained demonstrated that the association of the generalized displacement strategy with the normal flow technique contributes to the improvement of the nonlinear solution methodology.

Keywords: *Static analysis, Geometric nonlinearity, Equilibrium paths, Incremental-iterative scheme.*

Resumo

Grande parte dos procedimentos numéricos para obtenção de caminhos de equilíbrio dos problemas estruturais com comportamento estático não linear baseia-se no método de Newton-Raphson, ao qual são acoplados métodos de continuação. Nesse contexto, esse trabalho usa uma técnica, referida como fluxo normal, no processo de obtenção da resposta aproximada não linear estática de sistemas estruturais. Tal técnica trata-se, basicamente, de uma modificação estabelecida no esquema iterativo de Newton-Raphson, na tentativa de acelerar o processo de solução e/ou contornar problemas de convergência. Para ultrapassar os pontos limites e traçar a trajetória de equilíbrio completa das estruturas, adotam-se três diferentes estratégias de iteração: comprimento de arco, norma mínima dos deslocamentos residuais e deslocamento generalizado. Com esse procedimento, é avaliado o desempenho dessas estratégias, quando associadas à técnica do fluxo normal. Dois arcos com caminhos de equilíbrio não lineares são usados no estudo. Os resultados encontrados permitem concluir que a associação da estratégia do deslocamento generalizado com a técnica do fluxo normal contribui para a eficiência da metodologia de solução não linear.

Palavras chave: *Análise estática, não linearidade geométrica, trajetória de equilíbrio, esquema incremental-iterativo.*

1. Introduction

Structural stability analyses using the Finite Element Method (FEM) usually involve solving a nonlinear equations system. Purely incremental methods or schemes that combine incremental and iterative procedures are used to obtain this nonlinear equation system solution. When only based on the Newton-Raphson method (Bathe, 1996), many schemes are not capable of passing through the critical points (bifurcation or limit points) that can appear in the equilibrium path. This occurs due to poor conditioning of the tangent stiffness matrix that becomes singular at these points.

An efficient methodology for nonlinear system solving should be able to trace the complete equilibrium path, and identify and pass through all of the existent singular or critical points of the structural system under analysis. According to Crisfield (1991), although many times the structure results before reaching its critical points is sufficient for the design purposes, determining the response in a post-critical interval is essential within the large displacements domain.

This work's objective is to associate the normal flow technique with path-following iterative strategies. Such an association should help overcome convergence problems proper to numerical nonlinear structural analysis and/or improve the computational performance of these strategies. This technique makes it possible to modify the Newton-Raphson iterative process, where the iterations are conducted along the normal direction to the Davidenko flows (Allgower and Georg, 1980). The normal flow technique was implemented into the Computational System for Advanced Structural Analysis program (CS-ASA; Silva, 2009). CS-ASA is based on the finite element method and performs nonlinear static and dynamic analyses of structures. Two slender arches with strongly nonlinear behavior are analyzed herein to show the performance of three iterative strategies associated with the normal flow technique. The strategies adopted and available in CS-ASA are: the arch-length control idealized by Riks (1972), Ramm (1981), and Crisfield (1981), and used by many other researchers (Sousa and Pimenta, 2010; Lee *et al.*, 2011; Moghaddasie and Stanciulescu, 2013); the minimum residual displacement norm proposed by Chan (1988); and the generalized displacement control proposed by Yang and Kuo (1994). The next section presents details of the methodology used in the solution of nonlinear structural problems, characterized by an incremental-iterative scheme.

It is worth mentioning some researches involving normal flow technique in literature. Watson et al. (1987) and Watson et al. (1997) introduced the normal flow algorithm in the HOMPACK and HOMPACK90 software, respectively. Ragon *et al.* (2002) presents a study involving variants of the arc-length method and the normal flow algorithm. These authors stated that the algorithm could be more efficient than the arc-length method in cases where the equilibrium path is strongly nonlinear. Besides this, they wrote that the algorithm maintains large increments even when the nonlinearity is accentuated. In addition, the use of iterations steps in the normal direction to the Davidenko flows insures that convergence during the iterative process occurs faster. Saffari *et al.* (2008) and Tabatabaei *et al.* (2009) also adopted this technique.

2. Methodology for solving nonlinear structural problems

The equation that governs the static equilibrium of a structural system with geometrically nonlinear behavior can be written by Equation (1):

$$\mathbf{F}_i(\mathbf{U}) = \lambda \mathbf{F}_r \tag{1}$$

where $\mathbf{F}_r$ is a reference vector characterizing the external load direction and λ is the load parameter; $\mathbf{F}_i$ is the internal force vector, which is a function of the displacement, U, at the structure nodal points.

The structural problem solution of Equation (1) is obtained by using an incremental and iterative scheme. As such, for an incremental sequence of the load parameter $\Delta\lambda$, the respective nodal displacement increments $\Delta\mathbf{U}$ are calculated. As $\mathbf{F}_i$ is a displacement nonlinear function, iterations for correcting $\Delta\mathbf{U}$ are necessary to obtain the solution.

Equation (1) can be rewritten as:

$$\mathbf{g} = \lambda \mathbf{F}_r - \mathbf{F}_i(\mathbf{U}) \tag{2}$$

where **g** represents the gradient vector or the unbalance between the external and internal forces.

The Newton-Raphson method has been one of the most utilized in the solution of Equation (2). The objective is to determine the roots of this nonlinear relationship, which refers to the configurations of the static equilibrium of the structure. In this method, it is admitted that given an initial estimate for the root, the problem is to determine a sequence of corrections until a solution is obtained with the desired precision. For this, Equation (2) is approximated using the Taylor series (Press *et al.*, 1986). Therefore, for a load increment at the instant $t + \Delta t$, and at each iteration k, from the approximated solution of the displacement field $\mathbf{U}^{(k-1)} = {}^t\mathbf{U} + \Delta\mathbf{U}^{(k-1)}$, its correction $\delta\mathbf{U}^k$ is calculated so that:

$$\mathbf{g} = (\mathbf{U}^{(k-1)}+\delta\mathbf{U}^k) = \mathbf{0} \tag{3}$$

Note that the terms k and (k-1) are used herein to respectively refer to the current and previous iterations. Expanding the Taylor series to Equation (3), and considering the two first terms of the series, the expression for the correction of the nodal displacements becomes:

$$\delta\mathbf{U}^k = \mathbf{K}^{-1}\,\mathbf{g}\,(\mathbf{U}^{(k-1)}) \tag{4}$$

Thus, the new estimate for the solution, given by:

$$\mathbf{U}^k=\mathbf{U}^{(k-1)}+\delta\mathbf{U}^k \tag{5}$$

is considered to be the solution of the problem when a determined convergence criteria is satisfied.

The modified Newton-Raphson method is an alternative to the standard technique, in which the inclination of the tangent is maintained constant in all of the iterations. For the structural analysis, the stiffness matrix remains unaltered.

In both approaches to the Newton-Raphson method, the load parameter is maintained constant during the iterative cycle. In case the entire equilibrium path is to be accompanied, it is necessary to permit its variation from iteration to iteration. Followed then was the general technique proposed by Batoz and Dhatt (1979), where the alteration of the load parameter is permitted and the change in the nodal displacement is established by the following equation for equilibrium:

$$\mathbf{K}^{(k-1)}\delta\mathbf{U}^k = \mathbf{g}\big(\mathbf{U}^{(k-1)}, \lambda^k\big),\ k \geq 1 \tag{6}$$

where the vector $\mathbf{g}$ becomes a function of the displacement nodes $\mathbf{U}^{(k-1)}$ calculated in the last iteration, and also of the current load parameter λ^k, which is now an unknown element and written as:

$$\lambda^k = \lambda^{(k-1)} + \delta\lambda^k \tag{7}$$

where $\delta\lambda^k$ is a load parameter correction obtained using some iteration strategy (Silva, 2009).

Substituting (7) in (6) and using Equation (2), gives:

$$\mathbf{K}^{(k-1)}\delta\mathbf{U}^k = \left[(\lambda^{(k-1)}+\delta\lambda^k)\,\mathbf{F}_r-\mathbf{F}_i^{(k-1)}\right],\ \text{or},\ \mathbf{K}^{(k-1)}\,\delta\mathbf{U}^k = \mathbf{g}^{(k-1)}+\delta\lambda^k\mathbf{F}_r \tag{8}$$

which is the new equation used in the iterative cycle.

When using Equation. (8), the vector of the iterative nodal displacement nodes can then be decomposed into two parts and is written as:

$$\delta\mathbf{U}^k = \delta\mathbf{U}^k_g + \delta\,\lambda^k\delta\mathbf{U}^k_r \tag{9}$$

with $\delta\mathbf{U}^k_g=\mathbf{K}^{-1(k-1)}\mathbf{g}^{(k-1)}$ and $\delta\mathbf{U}^k_r = \mathbf{K}^{-1(k-1)}\mathbf{F}_r$. The usage of this equation is referred to herein as the conventional process for the nonlinear solution methodology.

In the normal flow technique, the equilibrium between the internal and external forces is obtained by performing iterative corrections along of the normal direction to the Davidenko curves (Allgower and Georg, 1980; Maximiano, 2012). With this technique, the expression used to obtain the nodal displacement correction is given by:

$$\delta\mathbf{U}^k = \left(\delta\mathbf{U}_g^k + \delta\lambda^k\delta\mathbf{U}_r^k\right) - \frac{\left(\delta\mathbf{U}_g^k + \delta\lambda^k\delta\mathbf{U}_r^k\right)^T\delta\mathbf{U}_r^k}{\left(\delta\mathbf{U}_r^k\right)^T\delta\mathbf{U}_r^k}\delta\mathbf{U}_r^k \tag{10}$$

which is, according to Watson *et al.* (1997), the unique solution for the minimum Euclidian norm of Equation (6). Using Equation (10), the vectors $\delta\mathbf{U}$ and $\delta\mathbf{U}_r$ in the current iteration are always perpendicular because the second term of the difference vector is a projection of the first in the direction of vector $\delta\mathbf{U}_r^k$.

Once the corrections $\delta\lambda^k$ and $\delta\mathbf{U}^k$ are obtained, the incremental variables $\Delta\mathbf{U}$ and $\Delta\lambda$, together with the totals $\mathbf{U}$ and λ, are updated. The methodology described in this section is detailed in Table 1.

3. Results - numerical examples

In this section, the objective is to verify the computational efficiency of the following iterative strategies associated with the normal flow technique: cylindrical arc-length control (AL), minimum residual displacement norm (MD) and generalized displacement control (GD). Such verification is made using the static analysis of the two arches that have nonlinear geometrical behavior. These arches are illustrated in Fig. 1 and will be described in subsections below.

From all the nonlinear finite elements formulations implemented in CS-ASA, the one proposed by Pacoste and Eriksson (1997) is used in this analysis. Details of this formulation, which adopts the total Lagrangian reference and is based on the theory of Timoshenko, can also be found in Maximiano (2012) and Silva (2009).

The iterative process determining the displacement correction through Equation (9) — conventional process — was also used for comparison. In the two analyzed problems, were compared: the total number of load increments (N_{tot}) and iterations (I_{tot}), the average number of iterations per load increment (I_{avg}), the processing time in seconds (CPU), and the number of restarting (*Rest*). Notice that a restart occurs when the maximum number of desired iterations is reached (*nmax*) and no convergence is obtained for a given load increment. In this case, the last configuration for the known equilibrium is returned to and the incremental-iterative process is restarted, reducing the value of $\Delta\lambda^0$ by half. In the two analyses, the maximum number of iterations nmax was 21.

Table 1
Incremental-iterative algorithm adopted

1. Load increment control: $i = 1, 2, 3, \ldots, N_{tot}$

2. Incremental Tangent Solution ($\Delta\lambda^0$, $\Delta\mathbf{U}^0$)

2a. Calculate the tangent stiffness matrix: $\mathbf{K}$

2b. Solve: $\delta\mathbf{U}_r = \mathbf{K}^{-1}\mathbf{F}_r$

2c. Define the initial load increment $\Delta\lambda^0$:

$$\begin{cases} \textit{Increment based on arc-length} \\ \Delta\lambda^0 = \pm\Delta l \Big/ \sqrt{\delta\mathbf{U}_r^T\delta\mathbf{U}_r + \mathbf{F}_r^T\mathbf{F}_r} \\ \textit{Increment based on Generalized Stiffness Parameter (GSP)} \\ \Delta\lambda^0 = \pm\Delta\lambda_1^0\sqrt{\left(\left({}^1\delta\mathbf{U}_r^T\right){}^1\delta\mathbf{U}_r\right)\Big/\left(\left({}^t\delta\mathbf{U}_r^T\right)\delta\mathbf{U}_r\right)} \end{cases}$$

2d. Determine: $\Delta\mathbf{U}^0 = \Delta\lambda^0\delta\mathbf{U}_r$

2e. Update the variable in configuration $t + \Delta t$: ${}^{(t+\Delta t)}\lambda = {}^t\lambda + \Delta\lambda^0$ and ${}^{(t+\Delta t)}\mathbf{U} = {}^t\mathbf{U} + \Delta\mathbf{U}^0$

3. Newton-Raphson Iterative Process: $k = 1, 2, 3, \ldots, nmax$

3a. Calculate the internal force vector: ${}^{(t+\Delta t)}\mathbf{F}_i^{(k-1)} = {}^t\mathbf{F}_i + \mathbf{K}\Delta\mathbf{U}^{(k-1)}$

3b. Calculate the unbalanced force vector: $\mathbf{g}^{(k-1)} = {}^{(t+\Delta t)}\lambda^{(k-1)}\mathbf{F}_r - {}^{(t+\Delta t)}\mathbf{F}_i^{(k-1)}$

3c. If standard Newton-Raphson, update the stiffness matrix $\mathbf{K}$

3d. Calculate the load parameter correction, $\delta\lambda^k$:

$$\begin{cases} \textit{Based on cylindrical arc-length (AL)} \\ A\left(\delta\lambda^k\right)^2 + B\delta\lambda^k + C = 0 \text{ (Silva, 2009)} \\ \textit{Based on minimum residual displ. norm (MD)} \\ \delta\lambda^k = -\left(\left(\delta\mathbf{U}_r^k\right)^T\delta\mathbf{U}_g^k\right)\Big/\left(\left(\delta\mathbf{U}_r^k\right)^T\delta\mathbf{U}_r^k\right) \\ \textit{Based on generalized displacement (GD)} \\ \delta\lambda^k = -\left({}^t\delta\mathbf{U}_r^T\delta\mathbf{U}_g^k\right)\Big/\left({}^t\delta\mathbf{U}_r^T\delta\mathbf{U}_r^k\right) \end{cases}$$

3e. Calculate the nodal displacement correction vector:

$\delta\mathbf{U}^k = \delta\mathbf{U}_g^k + \delta\lambda^k\delta\mathbf{U}_r^k$, if **Conventional process**, or,

$$\delta\mathbf{U}^k = \left(\delta\mathbf{U}_g^k + \delta\lambda^k\delta\mathbf{U}_r^k\right) - \frac{\left(\delta\mathbf{U}_g^k + \delta\lambda^k\delta\mathbf{U}_r^k\right)^T\delta\mathbf{U}_r^k}{\delta\mathbf{U}_r^{kT}\delta\mathbf{U}_r^k}\delta\mathbf{U}_r^k,$$ if **Normal flow technique**,

with $\delta\mathbf{U}_g^k = \mathbf{K}^{-1(k-1)}\mathbf{g}^{(k-1)}$ and $\delta\mathbf{U}_r^k = \mathbf{K}^{-1(k-1)}\mathbf{F}_r$

3f. Update the load parameter, λ, and the nodal displacement vector, $\mathbf{U}$:

a) *Incremental*: $\Delta\lambda^k = \Delta\lambda^{(k-1)} + \delta\lambda^k$ and $\Delta\mathbf{U}^k = \Delta\mathbf{U}^{(k-1)} + \delta\mathbf{U}^k$

b) *Total*: ${}^{(t+\Delta t)}\lambda^k = {}^t\lambda + \Delta\lambda^k$ and ${}^{(t+\Delta t)}\mathbf{U}^k = {}^t\mathbf{U} + \Delta\mathbf{U}^k$

3g. Verify the convergence (based on displacement): $\left\|\delta\mathbf{U}^k\right\|\Big/\left\|\Delta\mathbf{U}^k\right\| \leq \zeta$

Yes: *Stop the iterative process and go to step 4*

No: *If k < nmax, return to step 3*

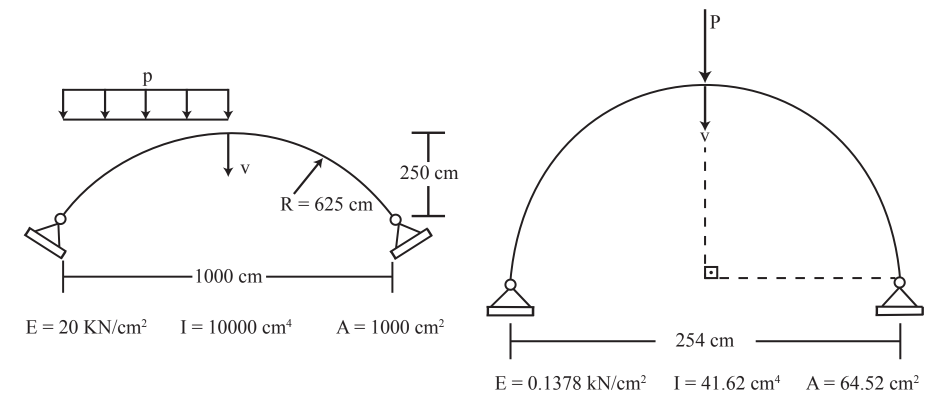
If k = nmax, reduce $\Delta\lambda^0$ *by half and restart the process, step 2*

4. IF $i < N_{tot}$, **Return to step 1 and make a new load increment**

IF $i = N_{tot}$, **Stop the incremental process**

3.1 Partially-loaded circular arch

The first example is the partially-loaded circular arch, whose physical and geometric properties are given in Figure 1a. In this figure, E corresponds to elastic modulus, I to moment of inertia, R to radius, and A to the cross-sectional area. The arch modeling used twenty beam-column elements. As nonlinear solution methodology control, a convergence tolerance ζ equal to 10^{-6} and the modified Newton Raphson were adopted. To begin the analysis, the uniformly-distributed load intensity p for the first increment was considered to be equal to 100 N/m. Be aware that this value is corrected during the iterative process of this first increment. For the rest of the load steps, the definition of the load intensity occurs automatically by using some load increment strategies (definition of $\Delta\lambda^0$).

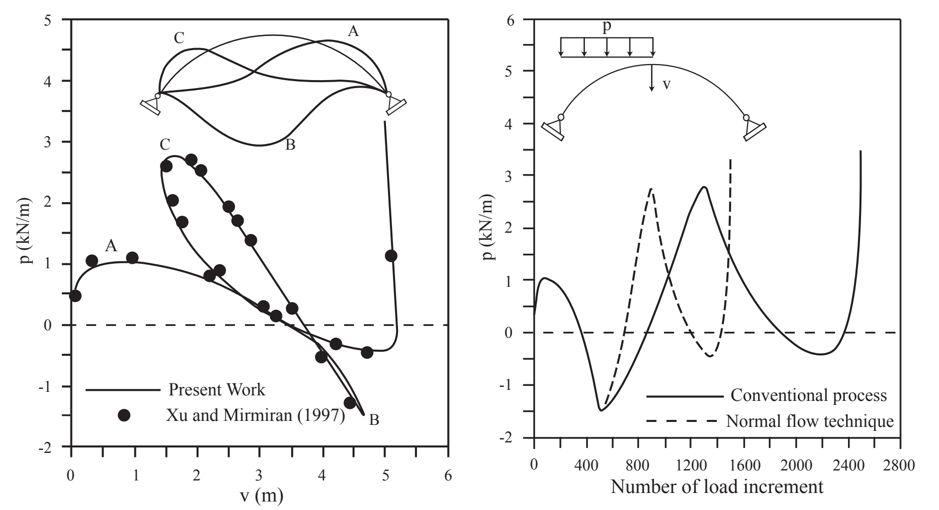


(a) Partially-loaded circular arch

(b) Circular arch with central load

Figure 1
Arches analyzed: geometry and loads.

Figure 2a shows the equilibrium paths obtained by controlling the vertical displacement, v, at the center of the arch. The generalized displacement strategy was used associated to the normal flow technique. This structure was studied by Xu and Mirmiram (1997) using a co-rotational beam formulation and the non-linear solution strategy proposed by Riks (1979). Note the good agreement between the response obtained by these authors and achieved in this work. Figure 2b shows the deformed arch configurations referring to the three load limit points, whose positions are indicated by the letters *A*, *B* and *C*.



(a) Equilibrium path

(b) Variation of *p* during the incremental process

Figure 2
Results obtained for the partially-loaded circular arch.

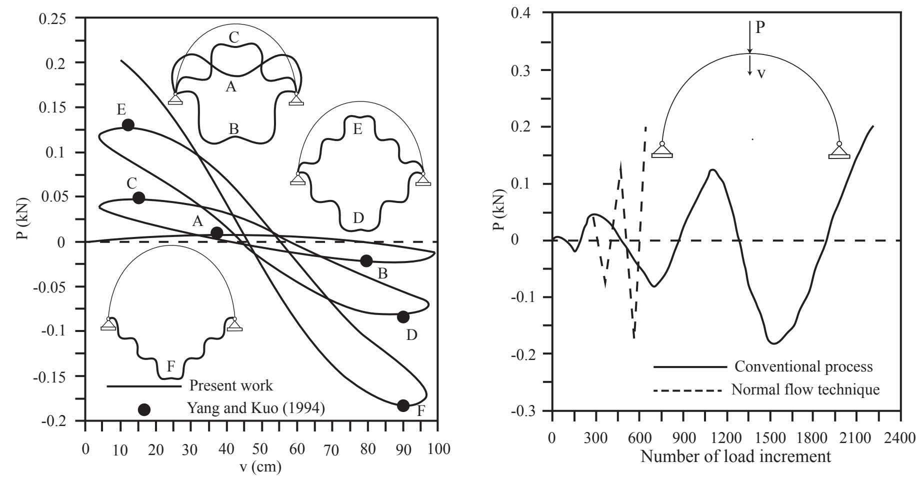
Table 2 presents the values for the parameters N_{tot}, I_{tot}, I_{avg}, *CPU* and *Rest* obtained by using the three iterations strategies: AL, MD and GD, when associated to the normal flow technique. The results for the conventional process are also indicated in this table. When comparing these parameters for AL and MD, it is possible to observe the similarity between the computational performance obtained with the conventional process and with the normal flow technique. Differently, the GD strategy was more efficient when associated with the normal flow technique. It was observed that a smaller number of load increments and total iterations were necessary and consequently, the analysis was concluded in a shorter processing time. The best performance of the normal flow technique is also graphically illustrated in Fig. 2b. Notice that after reaching the load limit point *B*, a significant difference occurs between these two forms to correct the nodal displacement in the iterative process.

Strategies	CONVENTIONAL PROCESS					NORMAL FLOW TECHNIQUE				
	N_{tot}	I_{tot}	I_{avg}	Rest	CPU(s)	N_{tot}	I_{tot}	I_{avg}	Rest	CPU(s)
AL	1702	10251	6	0	7.61	1705	10269	6	0	7.75
MD	1500	9412	6	2	7.46	1500	9412	6	2	7.60
GD	2497	13187	5	3	11.15	1500	9412	6	2	7.46

Table 2
Evaluation of the computation efficiency of the adopted strategies - Partially loaded circular arch.

3.2 Circular arch with central load

The circular arch submitted to a concentrated load of intensity P, as showed in Fig. 1(b), is the other structure chosen to evaluate the efficiency of different iterative strategies when associated with the normal flow technique. Taking into consideration the symmetry of the model, only half of the arch was discretized and 35 finite elements were used. Load P was presumed to be equal to 0.4 N to initiate the analysis. This value, as discussed in the previous example, is corrected during the iterative process of the first increment. A tolerance of $\zeta = 10^{-3}$ for the convergence of the iterative cycle was adopted. The study was performed utilizing the standard Newton-Raphson method. In this situation, the obtained load x vertical displacement curve at the point of load application is exhibited in Fig. 3(a). The load limit points in the analysis obtained numerically by Yang and Kuo (1994) are indicated by the positions: *A, B, C, D, E*, and *F* of this same figure as well as the arch deformation configuration when these points are reached. These authors studied this structure under a central load for validation of generalized displacement strategy (DG) and adopted 26 finite elements. Again, note the consistency of this paper's results with those found in literature.



(a) Equilibrium path

(b) Variation of p during the incremental process

Figure 3
Results obtained for the circular arch with central load.

Table 3 presents the values encountered for the parameters N_{tot}, I_{tot}, I_{avg}, *CPU* and *Rest* obtained using the three iteration strategies: AL, MD and GD, all associated with the normal flow technique. The conventional solution process was adopted for comparison. Therefore, as in the previous example, a significant difference between these two approaches could be observed when the GD strategy was used. The results showed a number of smaller load increments, less total iterations and no restarting occurrences, insuring substantially reduced processing time.

Figure 3(b) presents the variation for load *P* during the incremental process. Note that the results differ around the limit point C, and from this point on the efficiency of the normal flow technique can be perceived. Returning to Table 3, see that the arc-length strategy was not able to trace the equilibrium path, diverging at the very beginning of the analysis.

Table 3
Evaluation of the computational efficiency of the adopted strategies - Circular arch with central load.

Strategies	CONVENTIONAL PROCESS					NORMAL FLOW TECHNIQUE				
	N_{tot}	I_{tot}	I_{avg}	Rest	CPU(s)	N_{tot}	I_{tot}	I_{avg}	Rest	CPU(s)
AL*	400	862	2	2	9.36	400	859	2	2	9.66
MD	649	1451	2	0	15.17	649	1451	2	0	15.81
GD	2220	3224	1	3	39.92	649	1451	2	0	15.34

*It was not able to go beyond the load limit point B

4. Discussion

This study investigated the computational performance of three iterative strategies associated with the normal flow technique when used for the nonlinear static response of structural systems. The normal flow technique is related to a modification of the Newton-Raphson iterative scheme so as to overcome convergence problems and/or accelerate the solution process. The path-following strategies used in the study were: cylindrical arc-length control, minimum residual displacement norm control, and generalized displacement control.

The nonlinear analysis of two slender arches was performed and the obtained results were compared to the solutions found in literature. When utilizing the strategy based on the generalized displacement control, the responses obtained in both analyses showed that the normal flow technique significantly contributed to improving the computational performance of the adopted nonlinear solution methodology. This combination permitted the complete tracing of the equilibrium trajectory for a smaller number of load increments and iterations, and as such, diminished the computer-processing time over that of conventional processes. In this case, larger load increments were maintained even when the curve nonlinearity was accentuated, as was also observed by Ragon *et al*. (2002). In addition, the solution process was restarted less often than in the conventional process. The same efficiency was not observed when the normal flow technique was combined with the arc-length or the minimum residual displacement norm strategy.

Finally, it is important to point out that among the strategies used in this study, the minimum residual displacement norm excelled in the two analyses when the conventional process was used.

5. Conclusions

Based on the results obtained in the present work, it can be concluded that the normal flow technique can significantly influence the computational performance of the adopted nonlinear solution methodology. As discussed above, the normal flow technique significantly contributed to improving the computational performance of the adopted nonlinear solution methodology when combined with the generalized displacement-based strategy. However, the same efficiency was not observed, at least regarding the two structures analyzed, when this technique was combined with the minimum residual displacement control. Thus, under these conditions, this technique cannot be justified.

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