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RESUMEN

Se modela la propagación de petróleo derramado como una mancha bidimensional en la superficie de una área limitada del mar, cuando hay un flujo no nulo del petróleo a través de la frontera abierta. Se sugieren las condiciones para las partes de entrada y salida de la frontera del dominio, y se demuestra la existencia y la unicidad de la solución del problema de transporte de petróleo y de su adjunto en las clases de funciones generalizadas. También se muestra que las soluciones de cada uno de los problemas son estables en la presencia de errores en las condiciones iniciales, y en la tasa de emisión de petróleo desde el buque accidentado. Se sugieren las estimaciones directas y adjuntas de la concentración de petróleo para evaluar las consecuencias de un accidente que involucre el derrame de petróleo.

PALABRAS CLAVE: derrame y transporte de petróleo, estimaciones de contaminación, existencia, unicidad y estabilidad de soluciones.

ABSTRACT

An oil spill is modeled as a two-dimensional slick on a limited sea area when there is a nonzero oil flux through the open boundary. Conditions are suggested for the input and output parts of the domain boundary, and the unique solvability of the oil transport problem and its adjoint is proved for classes of generalized functions. It is also shown that solutions of the problem are stable in the presence of errors in the initial condition and in the oil emission rate from a damaged tanker. Direct and adjoint oil concentration estimates are suggested to evaluate the consequences of an accident involving oil spillage.

KEY WORDS: oil spill and transport, pollution estimates, existence, uniqueness and stability of solutions.

INTRODUCTION

Many studies have been devoted lately to the numerical modeling of oil slick spreading (Elliott, 1986; Elliott *et al*., 1992; Proctor *et al*., 1994, Kennicutt *et al*., 1992). It is a rather complicated problem, since the oil being released into marine environment is subjected to various weathering processes such as spreading and drift, advection and dissolution, evaporation and sinking, etc. Models have continued to play an increasingly important part in the study of the role of these processes and in the improvement of their parameterizations. Moreover, the oil spill forecast and effective strategies to monitor and control the oil pollution cannot be developed without using adequate oil transport model. However any model result makes sense only if the model is mathematically well posed, that is, if it has a unique solution which is stable to errors in the initial and boundary conditions and in the model forcing.

We formulate a simple two-dimensional mathematical model of oil transport in a limited area in the case of an accident with an oil tanker. Even for a simplified limited-area model, it is not trivial to pose the problem well. Since the oil flux through the open boundary is unknown, the boundary errors will propagate inside the domain by advection and diffusion, and perturb or destroy the exact solution. In addition, errors in the initial conditions and the oil emission rate from the damaged tanker can also distort the solution. Thus it is important to select boundary conditions that are corrected, both physically and mathematically. Such conditions are suggested below. We prove that with such conditions, the oil transport problem has a unique solution for certain classes of generalized functions, and that any solution is stable to perturbations in the forcing and initial condition. The unique solvability and stability conditions establish restrictions on the magnitude and smoothness of the model coefficients. These restrictions should be taken into account when different parameterizations of physical and chemical processes are incorporated in the model.

Direct and adjoint estimates of the oil concentration in ecologically significant zones are given here. We use the approach earlier developed by Marchuk and Skiba (1976, 1990) for evaluating average temperature anomalies in the atmosphere. While the direct estimates require the solution of the oil transport problem and provide a comprehensive analysis of the oil spill consequences, the adjoint estimates use the adjoint transport model solutions and are effective and economical in the model sensitivity study (Skiba, 1995a; 1996a; 1997, 1998b). On the whole, the direct and adjoint approaches

complement each other nicely in studying the consequences of the oil spill. We also show the unique solvability of the adjoint transport problem and the stability of its solution with respect to perturbations in the forcing and initial condition.

THE TWO-DIMENSIONAL OIL SPILL PROBLEM

We now summarize the main results obtained in Skiba (1996a). Let $r_0 = (x_0, y_0)$ be the site of an accident with an oil tanker in a two-dimensional open sea domain D with a smooth boundary S, and let $t=0$ be the time of the accident. Let $F(t)$ be the rate of oil spilling in unit time from the damaged tanker, and $\phi(r,t)$ the oil slick thickness on the sea surface at a point $r = (x,y)$ at time $t>0$. The oil slick propagation in D and time interval $(0,T)$ is described by the transport equation

$$\frac{\partial}{\partial t}\phi + U \cdot \nabla\phi + \sigma\phi - \nabla \cdot \mu\nabla\phi = f(r,t) \quad , \qquad (1)$$

with the forcing function

$$f(r,t) \equiv F(t)\delta(r - r_0) \quad , \qquad (2)$$

where $\mu(r,t)$ is the diffusion coefficient, ∇ is the two-dimensional gradient, and $\delta(r-r_0)$ is the Dirac mass at the accident point r_0. The parameter σ characterizes the decay of $\phi(r,t)$ because of evaporation. The velocity $U(r,t)=\{u(r,t),v(r,t)\}$ of the oil propagation is assumed to be known and to satisfy the continuity equation

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad . \qquad (3)$$

This vector can be calculated by using the climatic (seasonal or monthly) sea surface currents and winds (Doerffer, 1992), or the real currents and winds from dynamic models (Zalesny, 1986; Bulgakov *et al.*, 1999). The initial condition at $t=0$ is the absence of oil on the sea surface:

$$\phi(r,0) = 0 \quad . \qquad (4)$$

To obtain a well-posed problem according to Hadamard (1923), care is required in setting conditions at the boundaries (Marchuk, 1986; Poinsot and Lele, 1992; Skiba, 1996b). Let U_n be the projection of the velocity U on the outward normal n to the boundary S. We divide S into the outflow part S^+ where $U_n \geq 0$ (oil flows out of the domain D), and the inflow part S^- where $U_n < 0$ (oil flows into D). The boundary conditions for Eq.(1) are

$$\mu\frac{\partial}{\partial n}\phi - U_n\phi = 0 \ \text{ at } S^- \quad , \qquad (5)$$

$$\mu\frac{\partial}{\partial n}\phi = 0 \ \text{ at } S^+ \quad . \qquad (6)$$

By (5), the combined diffusive plus advective oil flow is absent at the inflow part S^- as no oil flows into D from the

outside where water is free of oil. Condition (6) means that at the boundary S^+, the diffusive oil flow is negligible as compared with the advective oil outflow $\mu\partial\phi/\partial n$ from D. In the non-diffusion limit ($\mu=0$), condition (5) is reduced to $\phi=0$ (there is no oil on the inflow boundary), while (6) vanishes, as it must. Indeed, the pure advection problem ($\mu=\sigma=0$) does not require conditions at the outflow boundary, since its solution is predetermined by the method of the characteristic lines (Godunov, 1971). Note that condition (6) includes the coastline where $U_n=0$. In particular, for a closed basin D everywhere bounded by the coastline, S^- is empty and $S=S^+$. Thus equations (5) and (6) include the coastline condition and approach the correct boundary conditions of the pure advection problem in the non-diffusion limit.

It is easy to show that any solution of problem (1)-(6) satisfies the oil balance equation

$$\frac{\partial}{\partial t}\int_D \phi dr = F(t) - \int_D \sigma\phi dr - \int_{S^+} U_n\phi dS \qquad (7)$$

and the integral equation

$$\frac{\partial}{\partial t}\int_D \phi^2 dr + 2\int_D (\sigma\phi^2 + \mu|\nabla\phi|^2)dr + \int_{S^+} U_n\phi^2 dS -$$

$$\int_{S^-} U_n\phi^2 dS = 2F(t)\phi(r_0,t) \qquad (8)$$

where $\int_D \phi^2 dr$ is the norm squared in Hilbert space $L_2(D)$ of square-integrable functions in the domain D. Due to (7), the total oil concentration $\int_D \phi dr$ in domain D increases because of the oil spill ($F>0$), and decreases by reason of both dissipation ($\sigma > 0$, $\mu > 0$) and advective oil outflow across S^+. However, since the oil spill from the damaged tanker is over ($F=0$), both $\int_D \phi dr$ and $\int_D \phi^2 dr$ decrease with time. If, in addition to $F=0$, the dissipation is absent ($\sigma = 0$, $\mu = 0$) and if $U_n = 0$ everywhere at the boundary S, both integrals are conserved over time.

DIRECT AND ADJOINT ESTIMATES

Let us consider in the same domain D and time interval $(0,T)$ an adjoint transport problem

$$-\frac{\partial}{\partial t}g - U \cdot \nabla g + \sigma g - \nabla \cdot \mu\nabla g = P(r,t) \qquad (9)$$

with the boundary conditions

$$\mu\frac{\partial}{\partial n}g = 0 \ \text{ at } \ S^-, \quad \mu\frac{\partial}{\partial n}g + U_n g = 0 \ \text{ at } \ S^+ \ , \qquad (10)$$

with the condition

$$g(r,T) = 0 \quad \text{at } t = T \ , \tag{11}$$

and the velocity vector $U(r,t)$ as in (1). The problem (9)-(11) can be introduced by using the concept of the adjoint operator in Hilbert space (Lyusternik and Sobolev, 1964; Marchuk, 1995; Marchuk *et al.*, 1993). We will show that this problem is only well posed according to Hadamard if it is solved in D backward from $t=T$ to $t=0$. The value

$$J(\phi) = \frac{1}{\tau|\Omega|} \int_{T-\tau}^{T} \int_{\Omega} \phi(r,t) dr dt \ , \tag{12}$$

where $|\Omega|$ is the area of a zone Ω, represents the average oil concentration in Ω in a time interval $(T-\tau,T)$. It will be called a direct estimate. If we solve the adjoint problem (9)-(11) with the forcing

$$P(r,t) = \begin{cases} 1/(\tau|\Omega|), & \text{if } (r,t) \text{ belongs to } \Omega \times [T-\tau,T] \\ 0 \ , & \text{otherwise} \end{cases} \tag{13}$$

we can obtain the adjoint estimate

$$J(\phi) = \int_{0}^{T} g(r_0,t) F(t) dt \ , \tag{14}$$

where $g(r_0,t)$ is the adjoint problem solution at the accident point r_0 (Skiba, 1995b).

The direct estimate (12) and the adjoint estimate (14) are equivalent and complement each other in accident studies. Sometimes one or the other of these estimates may be preferred. The direct estimate (12) utilizes the solution $\phi(r,t)$ of the problem (1)-(6); thus it depends on the two main parameters: the oil spill rate $F(t)$ and the accident site r_0. It is to be preferred when the oil concentrations are required in the whole domain D, or if the time available for counter-measures is assessed (Skiba, 1996a; Example 3). However, such comprehensive information is rather costly and often unnecessary. Sometimes it may be sufficient to obtain estimate (12) in some ecologically important zones Ω of domain D. In this case, benefit can be gained from the adjoint estimate (14) that uses the solution $g(r_0, t)$ of equation (9) at the accident point r_0, rather than problem (1)-(6). It should be noted that the adjoint problem (9)-(11) may be solved for each zone Ω irrespective of a specific accident with an oil tanker. This approach is convenient and economical for model sensitivity studies when the r_0-dependence (or $F(t)$-dependence) of the oil concentration $J(\phi)$ is analyzed (Skiba, 1996a; Examples 4,5).

Given some ecologically significant zone Ω, let us find the critical point on the tanker route where the oil spill maxi-

mizes the average value (12) (or (14)) in Ω. Integral (14), calculated for any point $r=(x,y)$ substituted for r_0, determines in D a two-dimensional function $G(r) = \int_{0}^{T} g(r,t) F(t) dt$. Then the tanker-route point r where $G(r)$ peaks is the critical point, where spillage hazard is highest.

The adjoint method is particularly efficient when the oil transport problem is studied with climatic (seasonal or monthly mean) winds and currents (Skiba, 1996a; Examples 1,2). Then the adjoint transport solution can be calculated for each ecologically significant zone in advance, and fed into a computer. Estimate (14) uses the adjoint solution values at the precise accident site; thus only the adjoint solution values at the grid points along the tanker route need to be stored in the computer. Any of these points is a possible site of an oil spill. When an oil tanker has an accident, and the site and oil spill rate are approximately known, a preliminary estimate of the average oil concentration may be given for any zone, by selecting the corresponding solution and taking the time integral (14).

For the solution of the main and the adjoint oil transport problems, balanced, absolutely stable and compatible numerical schemes and algorithms were suggested by Marchuk and Skiba (1978, 1992), Skiba (1993, 1998a), Skiba and Adem (1995), and Skiba *et al.* (1996). The splitting method and Crank-Nicolson schemes are used to discretize the two-, and three-dimensional problems in time. As a result, the numerical solutions of two-, or three-dimensional problems can be found without iterations by factorization.

SOME FUNCTIONAL SETS AND SPACES

In order to show that the oil spill problem (1)-(6) and its adjoint (9)-(11) are both well posed, let us prove the existence, uniqueness and stability of their solutions. To this end, we introduce some functional sets, spaces and estimates. The solutions are defined in a three-dimensional time-space domain $Q = D \times (0,T)$. We introduce a Hilbert space $L_2(Q)$ of real-valued functions in Q with the inner product

$$\langle f,g \rangle = \int_{Q} f(r,t) g(r,t) dr dt \tag{15}$$

and the norm

$$\|g\| = \langle g,g \rangle^{1/2} \ . \tag{16}$$

Let M be a set of functions $\phi(r,t)$ belonging to the class $C^2(\overline{Q})$ of twice continuously differentiable functions in the closed domain $\overline{Q}$ ($\overline{Q} = \overline{D} \times [0,T]$, $\overline{D} = D + S$) that satisfy condition (4) at the initial moment and have a finite norm

$$\|\phi\|_H = \left(\|\phi_t\|^2 + \|\phi_x\|^2 + \|\phi_y\|^2 + \|\phi\|_S^2 \right)^{1/2} . \quad (17)$$

Hereafter ϕ_t, ϕ_x, and ϕ_y denote the partial derivatives of function $\phi(x,y,t)$ with respect to t, x and y, and

$$\|\phi\|_S = \left\{ \int_0^T \int_{S^+} |\phi|^2 \, dS dt \right\}^{1/2} \quad (18)$$

is the L_2-norm of the trace of function $\phi(x,y,t)$ on the boundary S^+. We introduce in M the inner product

$$\langle \phi, \varphi \rangle_H = \langle \phi_t, \varphi_t \rangle + \langle \phi_x, \varphi_x \rangle + \langle \phi_y, \varphi_y \rangle + \langle \phi, \varphi \rangle_S , \quad (19)$$

where

$$\langle \phi, \varphi \rangle_S = \int_0^T \int_{S^+} \phi \varphi \, dS dt \quad (20)$$

and we denote by $H(Q)$ the Hilbert space obtained by closing the set M in the norm (17).

Lemma 1. *For any function $g(r,t)$ of $H(Q)$, the norm $\|g\|_H$ is equivalent to the norm $\left(\|g\|^2 + \|g\|_H^2 \right)^{1/2}$.*

Proof. It is sufficient to show that

$$\left(\|g\|^2 + \|g\|_H^2 \right)^{1/2} \le C \|g\|_H . \quad (21)$$

Because of (4),

$$g(r,t) = \int_0^t \frac{\partial g(r,\tau)}{\partial \tau} d\tau ,$$

and hence,

$$\|g\|^2 = \int_Q \left(\int_0^t \frac{\partial g}{\partial \tau} d\tau \right)^2 dr dt \le T^2 \|g_t\|^2 \le T^2 \|g\|_H^2 .$$

Thus the inequality (21) is satisfied with $C = \sqrt{1+T^2}$. The lemma is proved.

We now define a functional set V as the totality of the functions

$$g(r,t) = \int_0^t \psi(r,t) d\tau \quad (22)$$

where $\psi(r,t)$ is any function of $H(Q)$. The set V is dense in $H(Q)$, since M is dense in $H(Q)$, and the closure $\overline{V}$ of the set V in norm (17) contains M.

EXISTENCE, UNIQUENESS AND STABILITY OF THE OIL TRANSPORT SOLUTIONS

Definition. A function $\phi(r,t)$ in space $H(Q)$ is said to be a *generalized solution* of the oil transport problem (1)-(6) if it satisfies the identity

$$\langle \phi_t, g_t e^{-\delta t} \rangle + \langle \mu \phi_x, g_{tx} e^{-\delta t} \rangle + \langle \mu \phi_y, g_{ty} e^{-\delta t} \rangle$$

$$+ \langle \sigma \phi, g_t e^{-\delta t} \rangle + \int_0^T \int_{S^+} U_n \phi g_t e^{-\delta t} \, dS dt \quad (23)$$

$$+ \langle u \phi_x, g_t e^{-\delta t} \rangle + \langle v \phi_y, g_t e^{-\delta t} \rangle = \langle f, g_t e^{-\delta t} \rangle$$

for any function $g(r,t)$ of the set V and some $\delta > 0$.

Since $\phi(r,t)$ belongs to $H(Q)$, and $g(r,t)$ belongs to V, all terms of the identity (23) are meaningful. Formally, it may be derived by multiplying the equation (1) by the function $g_t \, e^{-\delta t}$ and integrating by parts with boundary conditions (5) and (6). The solution defined in such a manner is a generalization of the classical solution of the oil transport problem (1)-(6). Indeed, if a generalized solution is sufficiently smooth (i.e., if all its spatial derivatives up to the second order are continuous) in a smooth domain D, then it is the classical one. Note that if $g_t \, e^{-\delta t} \equiv 1$ (or $g_t \, e^{-\delta t} = 2\phi$), then identity (23) coincides with balance equation (7) (or (8)) integrated over the interval $(0,T)$.

Theorem 1. *Let $F(t) \in L_2(0,T)$. Let the oil velocity U satisfy continuity equation (3), and let*

$$\max_Q \left\{ \mu, |\mu_t|, \sigma, |\sigma_t|, |U|, |(U_n)_t| \right\} = \beta < \infty, \qquad \min_Q \{\mu, \sigma\} = \alpha > 0 .$$
$$(24)$$

Then the oil transport problem (1)-(6) has a unique generalized solution $\phi(r,t)$ in the space $H(Q)$, that is stable to variations in the forcing and initial conditions.

Proof. *1. Existence.* We use the functional method. Denote the ith term of the left-hand side of the identity (23) by , $a_i(\phi,g)$, $i=1,2,\ldots,7$, and the right-hand side term of (23) by $a_8(f,g)$. For a given g of the set V, each left-hand term $a_i(\phi,g)$ is a linear bounded functional (of ϕ) in $H(Q)$, $i=1,2,\ldots,7$. Thus, for example,

$$|a_1(\phi,g)| \equiv \left| \langle \phi_t, g_t e^{-\delta t} \rangle \right| \le \|g_t e^{-\delta t}\| \cdot \|\phi_t\| \le C_1(g) \cdot \|\phi\|_H , \quad (25)$$

$$|a_2(\phi,g)| \equiv \left| \langle \mu \phi_x, g_{tx} e^{-\delta t} \rangle \right| \le \beta \|g_{tx} e^{-\delta t}\| \cdot \|\phi_x\| = \beta \|\psi_x e^{-\delta t}\| \cdot \|\phi_x\|$$

$$\le \beta \|\psi_x\| \cdot \|\phi_x\| \le \beta \|\psi\|_H \cdot \|\phi\|_H \le C_2(g) \cdot \|\phi\|_H , \quad (26)$$

and

$$|a_5(\phi,g)| \equiv \left| \int_0^T \int_{S^+} U_n \phi g_t e^{-\delta t} dSdt \right| \le \beta \|\phi\|_S \|\psi\|_S \le \beta \|\psi\|_H \|\phi\|_H \le$$

$$C_5(g)\|\phi\|_H \qquad (27)$$

where $\psi = g_t \in H(Q)$ because of (22). The boundedness of the forms $a_i(\phi,g)$ for $i=3,4,6,7$ may be shown in a similar manner. Therefore, by Riesz theorem (Kolmogorov and Fomin, 1968), the ith functional is represented as the inner product

$$a_i(\phi,g) = \langle \phi, g_i \rangle_H \qquad (28)$$

with some uniquely determined element g_i of $H(Q)$ $(i=1,2,\ldots,7)$. The one-to-one correspondence $g \to g_i$ defines a linear operator A_i acting from V to $H(Q)$: $g_i = A_i g$ $(i=1,2,\ldots,7)$.

Further, according to (2), the condition $F(t) \in L_2(0,T)$ implies $f(r,t) \in L_2(Q)$, and hence, due to Lemma 1, $f(r,t) \in H(Q)$. Thus for a fixed $F(t) \in L_2(0,T)$ (i.e., for a fixed f) the term $a_8(f,g)$ is also a linear bounded functional (of ϕ) over $H(Q)$:

$$|a_8(f,g)| \equiv \left| \langle f, g_t e^{-\delta t} \rangle \right| \le \|f\| \|g_t\| \le C_8(f) \|g\|_H \qquad (29)$$

and, by Riesz' theorem, may be represented as

$$a_8(f,g) = \langle R, g \rangle_H \qquad (30)$$

with some uniquely determined element R of $H(Q)$. Let $A = \sum_{i=1}^{7} A_i$. Operator A acts from V to $H(Q)$. Then identity (23) can be written as

$$\langle \phi, Ag \rangle_H = \langle R, g \rangle_H \quad . \qquad (31)$$

To complete the proof of the existence part of the theorem we will need the following statement.

Lemma 2. *Let $G \subseteq H(Q)$ be the domain of values of the operator A, that is, A acts from V to G. Then the inverse operator A^{-1} defined on the domain G exists and is bounded.*

Proof. We prove Lemma 2 if we show that

$$\langle g, Ag \rangle_H \ge \frac{1}{2} e^{-\delta t} \langle g, g \rangle_H \qquad (32)$$

for each g of the set V, and some sufficiently large $\delta > 0$. Indeed, by (32), $Ag=0$ implies $g=0$, that is, A^{-1} exists and is bounded, since

$$\|A^{-1}\|_H = \underset{w \in G}{\mathrm{Sup}} \frac{\|A^{-1}w\|_H}{\|w\|_H} = \underset{\phi \in V}{\mathrm{Sup}} \frac{\|g\|_H^2}{\|g\|_H \|Ag\|_H} \le$$

$$\underset{\phi \in V}{\mathrm{Sup}} \frac{\langle g,g \rangle_H}{\langle g, Ag \rangle_H} \le 2e^{\delta T} \qquad (33)$$

where $w=Ag$. Let us now prove inequality (32). Let g belong to V, and hence, $g(r,0)=0$. Then

$$a_1(g,g) = \left\langle g_t, g_t e^{-\delta t} \right\rangle \ge \left\| g_t e^{-\frac{\delta}{2}t} \right\|^2 , \qquad (34)$$

$$a_2(g,g) = \left\langle \mu g_x, g_{tx} e^{-\delta t} \right\rangle = \frac{1}{2} e^{-\delta t} \int_D \mu g_x^2(r,T) dr$$

$$+ \frac{\delta}{2} \int_Q \mu g_x^2 e^{-\delta t} drdt - \frac{1}{2} \int_Q \mu_t g_x^2 e^{-\delta t} drdt \ge \frac{1}{2}(\delta\alpha - \beta) \left\| g_x e^{-\frac{\delta}{2}t} \right\|^2 ,$$

$$(35)$$

$$a_3(g,g) = \left\langle \mu g_y, g_{ty} e^{-\delta t} \right\rangle \ge \frac{\delta\alpha}{2} \left\| g_y e^{-\frac{\delta}{2}t} \right\|^2 , \qquad (36)$$

$$a_4(g,g) = \left\langle \sigma g, g_t e^{-\delta t} \right\rangle = \frac{1}{2} e^{-\delta t} \int_D \sigma g^2(r,T) dr$$

$$+ \frac{\delta}{2} \int_Q \sigma g^2 e^{-\delta t} drdt - \frac{1}{2} \int_Q \sigma_t g^2 e^{-\delta t} drdt \ge \frac{1}{2}(\alpha\delta - \beta) \left\| g e^{-\frac{\delta}{2}t} \right\|^2 , (37)$$

$$a_5(g,g) \equiv \int_0^T \int_{S^+} U_n g g_t e^{-\delta t} dSdt = \frac{1}{2} e^{-\delta t} \int_{S^+} U_n g^2(r,T) dS$$

$$+ \frac{\delta}{2} \int_0^T \int_{S^+} U_n \left(g e^{-\frac{\delta}{2}t} \right)^2 dS - \frac{1}{2} \int_0^T \int_{S^+} (U_n)_t \left(g e^{-\frac{\delta}{2}t} \right)^2 dS$$

$$\ge \frac{1}{2}(\delta\alpha_0 - \beta) \left\| g e^{-\frac{\delta}{2}t} \right\|_S^2 . \qquad (38)$$

In the last estimate we used the mean-value theorem

$$\int_0^T \int_{S^+} U_n \left(g e^{-\frac{\delta}{2}t} \right)^2 dS = \alpha_0 \left\| g e^{-\frac{\delta}{2}t} \right\|_S^2 \qquad (39)$$

where $a_0 > 0$, since U_n is non-negative on S^+. Note that if S^+ is a coastline ($U_n=0$) then the fifth left-hand term in (23) is absent, and hence, (38) is also absent. To estimate $a_6(g,g)$ and $a_7(g,g)$ we will use the ε-inequality:

$$a_6(g,g) \equiv \left\langle u g_x, g_t e^{-\delta t} \right\rangle \le \varepsilon \left\| g_t e^{-\frac{\delta}{2}t} \right\|^2 + \frac{\beta}{4\varepsilon} \left\| g_x e^{-\frac{\delta}{2}t} \right\|^2 , \qquad (40)$$

$$a_7(g,g) \equiv \left\langle v g_y, g_t e^{-\delta t} \right\rangle \leq \varepsilon \left\| g_t e^{-\frac{\delta}{2}t} \right\|^2 + \frac{\beta}{4\varepsilon} \left\| g_y e^{-\frac{\delta}{2}t} \right\|^2. \quad (41)$$

Let us select ε so that $1 - 2\varepsilon \geq \frac{1}{2}$, and take δ large enough so that $\delta\alpha - \beta(1 + \frac{1}{4\varepsilon}) \geq 1$ and $\delta\alpha_0 - \beta \geq 1$ are satisfied. Then we obtain

$$\left\langle g, Ag \right\rangle_H = \sum_{i=1}^{7} a_i(g,g) \geq \sum_{i=1}^{5} a_i(g,g) - \sum_{i=6}^{7} \left| a_i(g,g) \right|$$

$$\geq (1 - 2\varepsilon) \left\| g_t e^{-\frac{\delta}{2}t} \right\|^2 + \frac{1}{2} \left\{ \delta\alpha - \beta \left(1 + \frac{1}{4\varepsilon} \right) \right\} \left\{ \left\| g_x e^{-\frac{\delta}{2}t} \right\|^2 + \left\| g_y e^{-\frac{\delta}{2}t} \right\|^2 \right\}$$

$$+ \frac{1}{2} (\delta\alpha_0 - \beta) \left\| g e^{-\frac{\delta}{2}t} \right\|_S^2 + \frac{1}{2} (\delta\alpha - \beta) \left\| g e^{-\frac{\delta}{2}t} \right\|^2 \geq \frac{1}{2} e^{-\delta T} \left\langle g, g \right\rangle_H. \quad (42)$$

To derive the last inequality, we used definition (17) and Lemma 1. Thus Lemma 2 is proved.

Let us now proceed with the proof of the theorem. By continuity, operator A^{-1} is defined in $\overline{G}$ where $\overline{G}$ is the closure of G in the norm (17). Let L be the orthogonal complement of the set G in $H(Q)$. We extend A^{-1} from $\overline{G}$ to $H(Q)$ by defining $A^{-1} w = 0$ for any w of L. As a result we obtain a bounded operator $\tilde{A}^{-1}$ defined on the whole space $H(Q)$: $\left\| \tilde{A}^{-1} \right\|_H \leq 2e^{\delta T}$. The operator $(\tilde{A}^{-1})^*$ adjoint to $\tilde{A}^{-1}$ is uniquely defined and has the same norm as $\tilde{A}^{-1}$. Let $Ag = w$, where $w \in G$. Then, because of (31),

$$\left\langle \phi, w \right\rangle_H = \left\langle \phi, Ag \right\rangle_H = \left\langle R, g \right\rangle_H = \left\langle R, A^{-1} w \right\rangle_H \equiv \left\langle R, \tilde{A}^{-1} w \right\rangle_H. \quad (43)$$

Instead of searching for a solution ϕ from (30) when $w \in G$, let us impose a more restrictive condition. The equation

$$\left\langle \phi, w \right\rangle_H = \left\langle R, \left(\tilde{A}^{-1} \right) w \right\rangle_H = \left\langle \left(\tilde{A}^{-1} \right)^* R, w \right\rangle_H \quad (44)$$

is satisfied for any $w \in H(Q)$. This is possible if and only if $\phi = \left(\tilde{A}^{-1} \right)^* R$. With such a ϕ, (43) and the equivalent original identity (23) are also satisfied. Thus the existence of a generalized solution is proved.

2. _Uniqueness and stability._ Let $\phi(r,t) \in H(Q)$ be a generalized solution of problem (1)-(6). Then

$$g = \int_0^t \phi(r,\tau) e^{\delta\tau} d\tau \quad (45)$$

belongs to set V. Substituting (45) into (23), we obtain after some simple transformations that the generalized solution satisfies the inequality

$$\frac{1}{2} \left\| \phi(r,T) \right\|_{L_2(D)}^2 + \alpha \left(\left\| \phi_x \right\|^2 + \left\| \phi_y \right\|^2 \right) \leq \frac{1}{2} \left\| \phi(r,0) \right\|_{L_2(D)}^2 + \left\langle f, \phi \right\rangle \quad (46)$$

where

$$\left\| \phi(r,t) \right\|_{L_2(D)}^2 = \int_D \phi^2(r,t) dr. \quad (47)$$

Inequality (46) leads to

$$\frac{1}{2} \left\| \phi(r,T) \right\|_{L_2(D)}^2 + \alpha \left(\left\| \phi_x \right\|^2 + \left\| \phi_y \right\|^2 \right) \leq \max_{0 \leq \tau \leq T} \left\| \phi(r,\tau) \right\|_{L_2(D)} \times$$

$$\left(\sqrt{T} \| f \| + \frac{1}{2} \left\| \phi(r,0) \right\|_{L_2(D)} \right) \leq \left| \phi \right|_Q \left(\sqrt{T} \| f \| + \frac{1}{2} \left\| \phi(r,0) \right\|_{L_2(D)} \right) (48)$$

where

$$\left| \phi \right|_Q \equiv \max_{0 \leq \tau \leq T} \left\| \phi(r,\tau) \right\|_{L_2(D)} + \left\| \phi_x \right\| + \left\| \phi_y \right\| \quad (49)$$

is the energetic norm of a Banach space (Ladyzhenskaya, 1973). Using the fact that each term of the left-hand side of (48) is not larger than the right-hand side, it is easy to get the estimate

$$\left| \phi \right|_Q \leq \left(\frac{2}{\sqrt{\alpha}} + \sqrt{2} \right)^2 \left(\sqrt{T} \| f \| + \frac{1}{2} \left\| \phi(r,0) \right\|_{L_2(D)} \right) \quad (50)$$

that relates the solution norm (49) to the norms of the forcing and initial condition. Note that due to (4), the last term in (50) is zero. Since problem (1)-(6) is linear, estimate (50) implies the uniqueness of the generalized solution as well as its stability to variations $\delta f(r,t)$ and $\delta\phi(r,0)$ in the forcing and initial condition, respectively:

$$\left| \delta\phi \right|_Q \leq \left(\frac{2}{\sqrt{\alpha}} + \sqrt{2} \right)^2 \left(\sqrt{T} \| \delta f \| + \frac{1}{2} \left\| \delta\phi(r,0) \right\|_{L_2(D)} \right). \quad (51)$$

The theorem is proved.

Corollary. *Let $P(r,t) \in L_2(Q)$, and the coefficients of the adjoint problem (9)-(11) satisfy all the requirements of Theorem 1. Then in the space $H(Q)$, the adjoint problem has the only generalized solution stable to variations (errors) in the forcing and the initial conditions.*

CONCLUSIONS

The spreading of oil spilling from a damaged oil tanker is considered in a limited sea area when there is oil flowing across the liquid boundaries. We formulate a two-dimensional oil transport-diffusion problem and its adjoint with the bound-

ary conditions that are correct both mathematically and physically. The existence, uniqueness and stability of a generalized solution of the oil transport problem and its adjoint are proved for certain functional classes.

Equivalent direct and adjoint estimates of the average oil concentration in ecologically important zones are given for studying the consequences of the oil spill. The direct estimate (12) based on the oil transport problem solution is preferable when a comprehensive oil information is required in the whole domain *D*. On the other hand, the adjoint estimate (14) explicitly relates the average oil concentration in a zone to the oil spill rate through the adjoint solution at the accident site. This estimate is convenient for studying oil concentration variations caused by variations in the oil spill rate or/and the accident point. Several examples given in Skiba (1996a) show how to decide between dual estimates in various situations, or to modify the adjoint estimate for prediction purposes. These estimates can also be applied if oil enters the marine environment from other sources (offshore production, liquid waste, etc.).

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