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# Neutrino masses and mixings in a minimal $S_3$-invariant extension of the standard model

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The mass matrices of the charged leptons and neutrinos, that had been derived in the framework of a Minimal $S_3$-invariant Extension of the Standard Model, are reparametrized in terms of the masses of the charged leptons and neutrinos. Then, the neutrino mixing matrix $V_{PMNS}$ is computed and we obtain explicit, analytic and exact expressions for the neutrino mixing angles as function of the masses of the charged leptons and neutrinos. By comparison with the experimental data on neutrino mixings, we obtain numerical values for the neutrino masses in good agreement with the experimental bounds extracted from neutrinoless double beta decay and precision cosmological observations.

*Keywords:* Flavor symmetries; quark and lepton masses and mixings; neutrino masses and mixings.

Las matrices de masas de los leptones cargados y los neutrinos, que previamente habíamos obtenido en una extensión Mínima $S_3$-invariante del Modelo Standard, se reparametrizan en términos de las masas de los leptones cargados y los neutrinos. A partir de aquí, se calcula la matriz de mezclas de los neutrinos, $V_{PMNS}$, y se obtienen expresiones analíticas, explícitas y exactas para los ángulos de mezcla y las fases de Majorana como función de las masas de los leptones cargados y los neutrinos. Por comparación con los datos experimentales sobre mezclas de neutrinos, obtenemos valores numéricos de las masas de los neutrinos en buen acuerdo con las cotas experimentales extraídas de las medidas de la desintegración beta, la desintegración doble beta sin neutrinos y las observaciones cosmológicas de precisión.

*Descriptores:* Simetrías del sabor; masas y mezclas de quarks y leptones; masas y mezclas de neutrinos.

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## 1. Introduction

The discovery of neutrino masses and mixings marked a turning point in our understanding of nature and brought neutrino physics to the focus of attention of the particle, nuclear and astrophysics communities [1]. Recent neutrino oscillation observations and experiments have allowed the determination of the differences of the neutrino masses squared and the flavour mixing angles in the leptonic sector. The solar [2-5], atmospheric [6,7] and reactor [8,9] experiments produced the following results:

$$7.1 \times 10^{-5}(eV)^2 \leq \Delta^2 m_{12} \leq 8.9 \times 10^{-5}(eV)^2, \quad (1)$$

$$0.29 \leq sin^2\theta_{12} \leq 0.40, \quad (2)$$

$$1.4 \times 10^{-3}(eV)^2 \leq \Delta^2 m_{13} \leq 3.3 \times 10^{-3}(eV)^2, \quad (3)$$

$$0.34 \leq sin^2\theta_{23} \leq 0.68, \quad (4)$$

at $90\%$ confidence level [10, 11]. The CHOOZ experiment [12] determined an upper bound for the flavour mixing angle between the first and the third generations:

$$sin^2\theta_{13} \leq 0.046. \quad (5)$$

Neutrino oscillation data are insensitive to the absolute value of neutrino masses and also to the fundamental issue of whether neutrinos are Dirac or Majorana particles. Hence, the importance of the upper bounds on neutrino masses provided by the searches that probe the neutrino mass values at rest: beta decay experiments [13], neutrinoless double beta decay [14] and precision cosmology [15].

On the theoretical side, the discovery of neutrino masses and mixings has also brought about important changes. In the Standard Model, the Higgs and Yukawa sectors, which are responsible for the generation of the masses of quarks and charged leptons, do not give mass to the neutrinos. Furthermore, the Yukawa sector of the Standard Model already has too many parameters whose values can not be determined from experiment. These two facts, taken together, point to the necessity and convenience of eliminating parameters and systematizing the observed hierarchies of masses and mixings, as well as the presence or absence of CP violating phases by means of a flavour or family symmetry under which the families transform in a non-trivial fashion. Such a flavour symmetry might be a continuous group or, more economically, a finite group.

In a recent paper, we argued that such a flavour symmetry, unbroken at the Fermi scale, is the permutational symmetry of three objects, $S_3$, and introduced a Minimal $S_3$-invariant Extension of the Standard Model [16]. In this model, we imposed $S_3$ as a fundamental symmetry in the matter sector. This assumption led us necessarily to extend the concept of flavour and generations to the Higgs sector. Hence, going to the irreducible representations of $S_3$, we added to the Higgs $SU(2)_L$ doublet in the $S_3$-singlet representation, two more Higgs $SU(2)_L$ doublets which can only belong to the two components of the $S_3$-doublet representation. In this way, all the matter fields in the Minimal $S_3$-invariant Extension of the Standard Model - Higgs, quark and lepton fields, including the right handed neutrino fields- belong to the three dimensional representation $\mathbf{1} \oplus \mathbf{2}$ of the permutational group

$S_3$. The leptonic sector of the model was further constrained by an Abelian $Z_2$ symmetry.

The group $S_3$ [17-25] and the product groups $S_3 \times S_3$ [25-28] and $S_3 \times S_3 \times S_3$ [29,30] have been considered by many authors to explain successfully the hierarchical structure of quark masses and mixings in the Standard Model. However, in these works, the $S_3$, $S_3 \times S_3$ and $S_3 \times S_3 \times S_3$ symmetries are explicitly broken at the Fermi scale to give mass to the lighter quarks and charged leptons, neutrinos are left massless. Some other interesting models based on the $S_3$, $S_4$ and $A_4$ flavour symmetry groups, unbroken at the Fermi scale, have also been proposed [31-36], but in those models, equality of the number of fields and the irreducible representations is not obtained.

In this paper, we derive exact, explicit, analytic expression for the elements of the leptonic mixing matrix, $V_{PMNS}$, as functions of the masses of the charged leptons and the neutrinos. By comparison with the latest experimental data on neutrino mixings, we obtain numerical values for the neutrino masses in good agreement with the experimental bounds extracted from the precision cosmological observations and the neutrinoless double beta decay.

## 2. The Minimal $S_3$-invariant Extension of the Standard Model

In the Standard Model analogous fermions in different generations have completely identical couplings to all gauge bosons of the strong, weak and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to permutations of the left and right fermionic fields.

The six possible permutations of three objects $(f_1, f_2, f_3)$ are elements of the permutational group $S_3$. This is the discrete, non-Abelian group with the smallest number of elements. The three-dimensional real representation is not an irreducible representation of $S_3$. It can be decomposed into the direct sum of a doublet $f_D$ and a singlet $f_s$, where

$$\begin{aligned} f_s &= \frac{1}{\sqrt{3}}(f_1 + f_2 + f_3), \\ f_D^T &= \left(\frac{1}{\sqrt{2}}(f_1 - f_2), \frac{1}{\sqrt{6}}(f_1 + f_2 - 2f_3)\right). \end{aligned} \tag{6}$$

The direct product of two doublets $\mathbf{p_D}^T = (p_{D1}, p_{D2})$ and $\mathbf{q_D}^T = (q_{D1}, q_{D2})$ may be decomposed into the direct sum of two singlets $\mathbf{r_s}$ and $\mathbf{r_{s'}}$, and one doublet $\mathbf{r_D}^T$ where

$$\mathbf{r_s} = p_{D1}q_{D1} + p_{D2}q_{D2}, \mathbf{r_{s'}} = p_{D1}q_{D2} - p_{D2}q_{D1}, \tag{7}$$

$$\begin{aligned} \mathbf{r_D}^T &= (r_{D1}, r_{D2}) \\ &= (p_{D1}q_{D2} + p_{D2}q_{D1}, p_{D1}q_{D1} - p_{D2}q_{D2}). \end{aligned} \tag{8}$$

The antisymmetric singlet $\mathbf{r_{s'}}$ is not invariant under $S_3$.

Since the Standard Model has only one Higgs $SU(2)_L$ doublet, which can only be an $S_3$ singlet, it can only give mass to the quark or charged lepton in the $S_3$ singlet representation, one in each family, without breaking the $S_3$ symmetry.

Hence, in order to impose $S_3$ as a fundamental symmetry, unbroken at the Fermi scale, we are led to extend the Higgs sector of the theory. The quark, lepton and Higgs fields are

$$\begin{aligned} Q^T &= (u_L, d_L)\ ,\ u_R\ ,\ d_R\ , \\ L^T &= (\nu_L, e_L)\ ,\ e_R\ ,\ \nu_R \ \text{and}\ H, \end{aligned} \tag{9}$$

in an obvious notation. All of these fields have three species, and we assume that each forms a reducible representation $\mathbf{1}_S \oplus \mathbf{2}$. The doublets carry capital indices $I$ and $J$, which run from 1 to 2, and the singlets are denoted by $Q_3, u_{3R}, d_{3R}, L_3,\ e_{3R},\ \nu_{3R}$ and $H_S$.Note that the subscript 3 denotes the singlet representation and not the third generation. The most general renormalizable Yukawa interactions are given by

$$\mathcal{L}_Y = \mathcal{L}_{Y_D} + \mathcal{L}_{Y_U} + \mathcal{L}_{Y_E} + \mathcal{L}_{Y_\nu}, \tag{10}$$

where

$$\begin{aligned} \mathcal{L}_{Y_D} = &- Y_1^d \overline{Q}_I H_S d_{IR} - Y_3^d \overline{Q}_3 H_S d_{3R} \\ &- Y_2^d [\, \overline{Q}_I \kappa_{IJ} H_1 d_{JR} + \overline{Q}_I \eta_{IJ} H_2 d_{JR}\, ] \\ &- Y_4^d \overline{Q}_3 H_I d_{IR} - Y_5^d \overline{Q}_I H_I d_{3R} + \text{ h.c.}, \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{L}_{Y_U} = &- Y_1^u \overline{Q}_I (i\sigma_2) H_S^* u_{IR} - Y_3^u \overline{Q}_3 (i\sigma_2) H_S^* u_{3R} \\ &- Y_2^u [\, \overline{Q}_I \kappa_{IJ} (i\sigma_2) H_1^* u_{JR} + \eta \overline{Q}_I \eta_{IJ} (i\sigma_2) H_2^* u_{JR}\, ] \\ &- Y_4^u \overline{Q}_3 (i\sigma_2) H_I^* u_{IR} - Y_5^u \overline{Q}_I (i\sigma_2) H_I^* u_{3R} + \text{h.c.}, \end{aligned} \tag{12}$$

$$\begin{aligned} \mathcal{L}_{Y_E} = &- Y_1^e \overline{L}_I H_S e_{IR} - Y_3^e \overline{L}_3 H_S e_{3R} \\ &- Y_2^e [\, \overline{L}_I \kappa_{IJ} H_1 e_{JR} + \overline{L}_I \eta_{IJ} H_2 e_{JR}\, ] \\ &- Y_4^e \overline{L}_3 H_I e_{IR} - Y_5^e \overline{L}_I H_I e_{3R} + \text{ h.c.}, \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{L}_{Y_\nu} = &- Y_1^\nu \overline{L}_I (i\sigma_2) H_S^* \nu_{IR} - Y_3^\nu \overline{L}_3 (i\sigma_2) H_S^* \nu_{3R} \\ &- Y_2^\nu [\, \overline{L}_I \kappa_{IJ} (i\sigma_2) H_1^* \nu_{JR} + \overline{L}_I \eta_{IJ} (i\sigma_2) H_2^* \nu_{JR}\, ] \\ &- Y_4^\nu \overline{L}_3 (i\sigma_2) H_I^* \nu_{IR} - Y_5^\nu \overline{L}_I (i\sigma_2) H_I^* \nu_{3R} + \text{ h.c.}, \end{aligned} \tag{14}$$

and

$$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos

$$\mathcal{L}_M = -M_1 \nu_{IR}^T C \nu_{IR} - M_3 \nu_{3R}^T C \nu_{3R}. \tag{16}$$

Due to the presence of three Higgs fields, the Higgs potential $V_H(H_S, H_D)$ is more complicated than that of the Standard Model. This potential was analyzed by Pakvasa and Sugawara [18] who found that in addition to the $S_3$ symmetry, it has a permutational symmetry $S_2$: $H_1 \leftrightarrow H_2$, which is not a subgroup of the flavour group $S_3$ and an Abelian discrete symmetry that we will use for selection rules of the

Yukawa couplings in the leptonic sector. In this communication, we will assume that the vacuum respects the accidental $S_2$ symmetry of the Higgs potential and

$$\langle H_1 \rangle = \langle H_2 \rangle. \tag{17}$$

With these assumptions, the Yukawa interactions, Eqs. (11)-(14) yield mass matrices, for all fermions in the theory, of the general form

$$\mathbf{M} = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}. \tag{18}$$

The Majorana mass for the left neutrinos $\nu_L$ will be obtained from the see-saw mechanism. The corresponding mass matrix is given by

$$\mathbf{M}_\nu = \mathbf{M}_{\nu_\mathbf{D}} \tilde{\mathbf{M}}^{-1} (\mathbf{M}_{\nu_\mathbf{D}})^T \tag{19}$$

where $\tilde{\mathbf{M}} = \text{diag}(M_1, M_1, M_3)$.

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry $S_3$.

The mass matrices are diagonalized by bi-unitary transformations as

$$U^\dagger_{d(u,e)L} \mathbf{M}_{d(u,e)} U_{d(u,e)R} = \text{diag}(m_{d(u,e)}, m_{s(c,\mu)}, m_{b(t,\tau)}),$$
$$U^T_\nu \mathbf{M}_\nu U_\nu = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}). \tag{20}$$

The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The mixing matrices are, by definition,

$$V_{CKM} = U^\dagger_{uL} U_{dL}, \quad V_{PMNS} = U^\dagger_{eL} U_\nu K, \tag{21}$$

where $K$ is the diagonal matrix of the Majorana phases.

## 3. The mass matrices in the leptonic sector and $Z_2$ symmetry

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian $Z_2$ symmetry. A possible set of charge assignments of $Z_2$, compatible with the experimental data on masses and mixings in the leptonic sector is given in Table I

These $Z_2$ assignments forbid certain Yukawa couplings,

$$Y^e_1 = Y^e_3 = Y^\nu_1 = Y^\nu_5 = 0. \tag{22}$$

Therefore, the corresponding entries in the mass matrices vanish,*i.e.*, $\mu^e_1 = \mu^e_3 = 0$ and $\mu^\nu_1 = \mu^\nu_5 = 0$.

TABLE I. $Z_2$ assignment in the leptonic sector.

| $-$ | $+$ |
|---|---|
| $H_S$, $\nu_{3R}$ | $H_I$, $L_3$, $L_I$, $e_{3R}$, $e_{IR}$, $\nu_{IR}$ |

### 3.1. The mass matrix of the charged leptons

The mass matrix of the charged leptons takes the form

$$M_e = m_\tau \begin{pmatrix} \tilde{\mu}_2 & \tilde{\mu}_2 & \tilde{\mu}_5 \\ \tilde{\mu}_2 & -\tilde{\mu}_2 & \tilde{\mu}_5 \\ \tilde{\mu}_4 & \tilde{\mu}_4 & 0 \end{pmatrix}. \tag{23}$$

The unitary matrix $U_{eL}$ that enters in the definition of the mixing matrix, $V_{PMNS}$, is calculated from

$$U^\dagger_{eL} M_e M^\dagger_e U_{eL} = \text{diag}(m^2_e, m^2_\mu, m^2_\tau), \tag{24}$$

where $m_e$, $m_\mu$ and $m_\tau$ are the masses of the charged leptons, and

$$M_e M^\dagger_e = m^2_\tau \times \begin{pmatrix} 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & |\tilde{\mu}_5|^2 & 2|\tilde{\mu}_2||\tilde{\mu}_4|e^{-i\delta_e} \\ |\tilde{\mu}_5|^2 & 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & 0 \\ 2|\tilde{\mu}_2||\tilde{\mu}_4|e^{i\delta_e} & 0 & 2\,|\tilde{\mu}_4|^2 \end{pmatrix}, \tag{25}$$

Notice that this matrix has only one non-ignorable phase factor.

The parameters $|\tilde{\mu}_2|$, $|\tilde{\mu}_4|$ and $|\tilde{\mu}_5|$ may readily be expressed in terms of the charged lepton masses. From the invariants of $M_e M^\dagger_e$, we get the set of equations

$$Tr(M_e M^\dagger_e) = m^2_e + m^2_\mu + m^2_\tau = m^2_\tau \left[4|\tilde{\mu}_2|^2 + 2\left(|\tilde{\mu}_4|^2 + |\tilde{\mu}_5|^2\right)\right], \tag{26}$$

$$\chi(M_e M^\dagger_e) = m^2_\tau(m^2_e + m^2_\mu) + m^2_e m^2_\mu = 4m^4_\tau \left[|\tilde{\mu}_2|^4 + |\tilde{\mu}_2|^2\left(|\tilde{\mu}_4|^2 + |\tilde{\mu}_5|^2\right) + |\tilde{\mu}_4|^2|\tilde{\mu}_5|^2\right], \tag{27}$$

$$det(M_e M^\dagger_e) = m^2_e m^2_\mu m^2_\tau = 4m^6_\tau |\tilde{\mu}_2|^2|\tilde{\mu}_4|^2|\tilde{\mu}_5|^2, \tag{28}$$

where $\chi(M_e M^\dagger_e) = \frac{1}{2}\left[(Tr(M_e M^\dagger_e))^2 - Tr(M_e M^\dagger_e)^2\right]$.

Solving these equations for $|\tilde{\mu}_2|^2$, $|\tilde{\mu}_4|^2$ and $|\tilde{\mu}_5|^2$, we obtain

$$|\tilde{\mu}_2|^2 = \frac{1}{2}\frac{m^2_e + m^2_\mu}{m^2_\tau} - \frac{m^2_e m^2_\mu}{m^2_\tau(m^2_e + m^2_\mu)} + \beta, \tag{29}$$

in this expression, $\beta$ is the smallest solution of the equation

$$\beta^3 - \frac{1}{2}\left(1 - 2y + 6\frac{z}{y}\right)\beta^2 - \frac{1}{4}\left(y - y^2 - 4\frac{z}{y} + 7z - 12\frac{z^2}{y^2}\right)\beta - \frac{1}{8}yz - \frac{1}{2}\frac{z^2}{y^2} + \frac{3}{4}\frac{z^2}{y} - \frac{z^3}{y^3} = 0. \tag{30}$$

where $y = (m^2_e + m^2_\mu)/m^2_\tau$ and $z = m^2_\mu m^2_e/m^4_\tau$.

A good, order of magnitude, estimation for $\beta$ is obtained from (30)

$$\beta \approx -\frac{m^2_\mu m^2_e}{2m^2_\tau(m^2_\tau - (m^2_\mu + m^2_e))}. \tag{31}$$

The parameters $|\tilde{\mu}_4|^2$ and $|\tilde{\mu}_5|^2$ are, then, readily expressed in terms of $|\tilde{\mu}_2|^2$,

$$|\tilde{\mu}_{4,5}|^2=\frac{1}{4}\left(1-\frac{m_\mu^2+m_e^2}{m_\tau^2}+4\frac{m_e^2m_\mu^2}{m_\tau^2(m_e^2+m_\mu^2)}-4\beta\right)\pm\sqrt{(1-\frac{m_\mu^2+m_e^2}{m_\tau^2}+4\frac{m_e^2m_\mu^2}{m_\tau^2(m_e^2+m_\mu^2)}-4\beta)^2-\frac{m_\mu^2m_e^2}{m_\tau^4}\frac{1}{|\tilde{\mu}_2|^2}}. \quad (32)$$

Once $M_eM_e^\dagger$ has been reparametrized in terms of the charged lepton masses, it is straightforward to compute $U_{eL}$ also as a function of the lepton masses. Here, in order to avoid a clumsy notation, we will write the result to order of $\left(m_\mu m_e/m_\tau^2\right)^2$.

$$M_e\approx m_\tau\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sqrt{m_\mu^2+m_e^2}}{m_\tau}\sqrt{1-2\frac{m_\mu^2m_e^2}{(m_\mu^2+m_e^2)m_\tau^2}} & \frac{\sqrt{m_\mu^2+m_e^2}}{m_\tau}\sqrt{1-2\frac{m_\mu^2m_e^2}{(m_\mu^2+m_e^2)m_\tau^2}} & \sqrt{1-\frac{m_\mu^2+m_e^2}{m_\tau^2}+4\frac{m_e^2m_\mu^2}{m_\tau^2(m_\mu^2+m_e^2)}} \\ \frac{\sqrt{m_\mu^2+m_e^2}}{m_\tau}\sqrt{1-2\frac{m_\mu^2m_e^2}{(m_\mu^2+m_e^2)m_\tau^2}} & -\frac{\sqrt{m_\mu^2+m_e^2}}{m_\tau}\sqrt{1-2\frac{m_\mu^2m_e^2}{(m_\mu^2+m_e^2)m_\tau^2}} & \sqrt{1-\frac{m_\mu^2+m_e^2}{m_\tau^2}+4\frac{m_e^2m_\mu^2}{m_\tau^2(m_\mu^2+m_e^2)}} \\ \frac{m_\mu m_e}{m_\tau\sqrt{m_\mu^2-m_e^2}}e^{i\delta_e} & \frac{m_\mu m_e}{m_\tau\sqrt{m_\mu^2-m_e^2}}e^{i\delta_e} & 0 \end{pmatrix} \quad (33)$$

and

$$U_{eL}\approx\begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\frac{m_e}{m_\mu}}{\sqrt{1-\left(\frac{m_e}{m_\mu}\right)^2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\left(\frac{m_e}{m_\mu}\right)^2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\frac{m_em_\mu}{m_\tau^2}}} \\ -\frac{1}{\sqrt{2}}\frac{\frac{m_e}{m_\mu}}{\sqrt{1-\left(\frac{m_e}{m_\mu}\right)^2}} & -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\left(\frac{m_e}{m_\mu}\right)^2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\frac{m_em_\mu}{m_\tau^2}}} \\ \frac{\sqrt{1-2\left(\frac{m_e}{m_\mu}\right)^2}e^{i\delta_e}}{\sqrt{1-\left(\frac{m_e}{m_\mu}\right)^2}} & \frac{\frac{m_e}{m_\mu}e^{i\delta_e}}{\sqrt{1+\left(\frac{m_e}{m_\mu}\right)^2}} & \frac{\frac{m_em_\mu}{m_\tau^2}e^{i\delta_e}}{\sqrt{1+\frac{m_em_\mu}{m_\tau^2}}} \end{pmatrix} \quad (34)$$

## 4. The mass matrix of the neutrinos

According with the $Z_2$ selection rule eq. (22), the mass matrix of the Dirac neutrino takes the form

$$\mathbf{M}=\begin{pmatrix} \mu_2^\nu & \mu_2^\nu & 0 \\ \mu_2^\nu & -\mu_2^\nu & 0 \\ \mu_4^\nu & \mu_4^\nu & \mu_3^\nu \end{pmatrix}. \quad (35)$$

Then, the mass matrix for the left-handed Majorana neutrinos obtained from the see-saw mechanism as

$$\mathbf{M}_\nu=\mathbf{M}_{\nu_\mathbf{D}}\tilde{\mathbf{M}}^{-1}(\mathbf{M}_{\nu_\mathbf{D}})^T \quad (36)$$

$$=\begin{pmatrix} 2(\rho_2^\nu)^2 & 0 & 2\rho_2^\nu\rho_4^\nu \\ 0 & 2(\rho_2^\nu)^2 & 0 \\ 2\rho_2^\nu\rho_4^\nu & 0 & 2(\rho_4^\nu)^2+(\rho_3^\nu)^2 \end{pmatrix}, \quad (37)$$

where $\rho_2^\nu=(\mu_2^\nu)/M_1^{1/2}$, $\rho_4^\nu=(\mu_4^\nu)/M_1^{1/2}$ and $\rho_3^\nu=(\mu_3^\nu)/M_3^{1/2}$; $M_1$ and $M_3$ are the masses of the right handed neutrinos appearing in (16).

The non-Hermitian, complex, symmetric matrix $M_\nu$ may be brought to a diagonal form by a bi-unitary transformation, as

$$U_\nu^TM_\nu U_\nu=\text{diag}\left(|m_{\nu_1}|e^{i\phi_1},|m_{\nu_2}|e^{i\phi_2},|m_{\nu_3}|e^{i\phi_\nu}\right). \quad (38)$$

In order to compute $U_\nu$, we notice that $M_\nu$ may be brought to a block diagonal form by a permutation of the second and third rows and columns

$$S_{23}M_\nu S_{23}=\begin{pmatrix} 2(\rho_2^\nu)^2 & 2\rho_2^\nu\rho_4^\nu & 0 \\ 2\rho_2^\nu\rho_4^\nu & 2(\rho_4^\nu)^2+(\rho_3^\nu)^2 & 0 \\ 0 & 0 & 2(\rho_2^\nu)^2 \end{pmatrix}, \quad (39)$$

where

$$S_{23}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The entry in the lower right corner of the matrix in the right hand side of eq. (39), is already diagonal and may be identified with $m_{\nu_3}e^{i\phi_\nu}$; $|m_{\nu_3}|$ is the physical mass.

Then, the $2\times2$ block in the upper left hand corner may be diagonalized by a $2\times2$ bi-unitary transformation as

$$U_{2\times2}^TM_{2\times2}U_{2\times2}=\text{diag}\left(|m_{\nu_1}|e^{i\phi_1},|m_{\nu_2}|e^{i\phi_2}\right).$$

$U_{2\times2}$ is computed from

$$U_{2\times2}^\dagger M_{2\times2}M_{2\times2}^\dagger U_{2\times2}=\text{diag}\left(|m_{\nu_1}|^2,|m_{\nu_2}|^2\right).$$

Since the $2 \times 2$ Hermitian matrix $M_{2\times 2}M_{2\times 2}^{\dagger}$ has only one phase factor, the unitary $U_{2\times 2}$ matrix may be written as

$$U_{2\times 2} = \begin{pmatrix} \sin\eta & \cos\eta \\ -\cos\eta e^{i\delta_\nu} & \sin\eta e^{i\delta_\nu} \end{pmatrix}. \tag{40}$$

Explicit expressions for the entries of $M_{2\times 2}$ and $\sin\eta$, $\cos\eta$ as functions of the complex masses $m_{\nu_1}$, $m_{\nu_2}$ and $m_{\nu_3}$ are obtained when the expression (40) for $U_{2\times 2}$ is substituted in eq (38) and we require that this equation be satisfied as an identity. The result is

$$\cos\eta = \sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}}}, \quad \sin\eta = \sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} - m_{\nu_1}}} \tag{41}$$

The unitarity of $U_{2\times 2}$ constrains $\sin\eta$ to be real and $|\sin\eta| \leq 1$. This condition fixes the phases $\phi_1$ and $\phi_2$ as

$$|m_{\nu_1}|\sin\phi_1 = |m_{\nu_2}|\sin\phi_2 = |m_{\nu_3}|\sin\phi_\nu. \tag{42}$$

The real phase $\delta_\nu$ appearing in eq. (40) is not constrained by the unitarity of $U_\nu$.

By doing a second permutation of the second and third rows and columns in eq. (39) and writing the entries in $M_\nu$ as functions of the complex masses $m_{\nu_1}$, $m_{\nu_2}$ and $m_{\nu_3}$, we find

$$M_\nu = \begin{pmatrix} m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})}e^{-i\delta_\nu} \\ 0 & m_{\nu_3} & 0 \\ \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})}e^{-i\delta_\nu} & 0 & (m_{\nu_1} + m_{\nu_2} - m_{\nu_3})e^{-2i\delta_\nu} \end{pmatrix}. \tag{43}$$

and

$$U_\nu = \begin{pmatrix} \sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} - m_{\nu_1}}} & \sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}}} & 0 \\ 0 & 0 & 1 \\ -\sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}}}e^{i\delta_\nu} & \sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} - m_{\nu_1}}}e^{i\delta_\nu} & 0 \end{pmatrix}. \tag{44}$$

The only free parameters in these matrices, other than the neutrino masses, are the phase $\phi_\nu$, implicit in $m_{\nu_1}$, $m_{\nu_2}$ and $m_{\nu_3}$, and the Dirac phase $\delta_\nu$.

### 4.1. The neutrino mixing matrix

The neutrino mixing matrix $V_{PMNS}$, in the standard form advocated by the $PDG$ [37], is obtained by taking the product $U_{eL}^{\dagger}U_\nu K$ and making an appropriate transformation of phases. Writing the resulting expression to the same approximation as in eq. (33), we get

$$V_{PMNS} \approx \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{x}{\sqrt{1-x^2}}\sin\eta + \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}\cos\eta & \frac{1}{\sqrt{2}}\frac{x}{\sqrt{1-x^2}}\cos\eta - \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}\sin\eta & -\frac{1}{\sqrt{2}}\frac{x}{\sqrt{1-x^2}}e^{-i\delta} \\ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+x^2}}\sin\eta - \frac{x}{\sqrt{1-x^2}}\cos\eta e^{i\delta} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+x^2}}\cos\eta + \frac{x}{\sqrt{1-x^2}}\sin\eta e^{i\delta} & -\frac{1}{\sqrt{2}}\frac{1+2\frac{z}{y(1-y)}}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\sqrt{z}}}\sin\eta - \frac{\sqrt{z}}{\sqrt{1+\sqrt{z}}}\cos\eta e^{i\delta} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\sqrt{z}}}\cos\eta + \frac{\sqrt{z}}{\sqrt{1+\sqrt{z}}}\sin\eta e^{i\delta} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\sqrt{z}}} \end{pmatrix} K. \tag{45}$$

where $\cos\eta$ and $\sin\eta$ are given in eq. (41), $x = m_e/m_\mu$, $\delta = \delta_\nu - \delta_e$ and $K$ is the diagonal matrix of the Majorana phases, $K = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$.

A comparison of this expression with the standard parametrization allowed us to derive expressions for the mixing angles in terms of the charged lepton and neutrino masses

$$\sin\theta_{13} \approx \frac{\frac{1}{\sqrt{2}}\frac{m_e}{m_\mu}}{\sqrt{1 - \left(\frac{m_e}{m_\mu}\right)^2}},$$

$$\sin\theta_{23} \approx -\frac{1}{\sqrt{2}}\frac{\sqrt{1 - \left(\frac{m_e}{m_\mu}\right)^2}}{\sqrt{1 - \frac{1}{2}\left(\frac{m_e}{m_\mu}\right)^2}} \tag{46}$$

and

$$\tan\theta_{12} = -\sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}}} \times \left(\frac{\sqrt{1 - 2x^2} - \frac{1}{\sqrt{2}}x\sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}}}{\sqrt{1 - 2x^2} + \frac{1}{\sqrt{2}}x\sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}}}}\right) \tag{47}$$

The dependence of $\tan\theta_{12}$ on the phase $\phi_\nu$ and the physical masses of the neutrinos is made explicit with the help of the unitarity constraint on $U_\nu$, eq (42), we get

$$\tan^2\eta = \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}} = -\frac{(|m_{\nu_2}|^2 - |m_{\nu_3}|^2\sin^2\phi_\nu)^{1/2} - |m_{\nu_3}||\cos\phi_\nu|}{(|m_{\nu_1}|^2 - |m_{\nu_3}|^2\sin^2\phi_\nu)^{1/2} + |m_{\nu_3}||\cos\phi_\nu|}. \tag{48}$$

Similarly, the Majorana phases are given by

$$\sin 2\alpha = \sin(\phi_1 - \phi_2) = \pm \frac{|m_{\nu_3}| \sin\phi_\nu}{|m_{\nu_1}||m_{\nu_2}|} \times \left( \sqrt{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2\phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2\phi_\nu} \right)$$

$$\sin 2\beta = \sin(\phi_1 - \phi_\nu) = \pm \frac{\sin\phi_\nu}{|m_{\nu_1}|} \left( |m_{\nu_3}| \sqrt{1 - \sin^2\phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2\phi_\nu} \right) \quad (49)$$

A more complete and detailed discussion of the Majorana phases in the neutrino mixing matrix $V_{PMNS}$ obtained in our model is given in J. Kubo [38].

In this model, $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ are determined by the masses of the charged leptons in very good agreement with the experimental values [10, 11, 39],

$$(\sin^2\theta_{13})^{th} = 1.1 \times 10^{-5}, \quad (\sin^2\theta_{13})^{exp} \leq 0.046,$$

and

$$(\sin^2\theta_{23})^{th} = 0.49 \times 10^{-5}, \quad (\sin^2\theta_{23})^{exp} = 0.5^{+0.06}_{-0.05}.$$

In the present model, the experimental restriction $|\Delta m^2_{12}| < |\Delta m^2_{13}|$ implies an inverted neutrino mass spectrum, $|m_{\nu_3}| < |m_{\nu_1}| < |m_{\nu_2}|$ [16].

As seen from Eqs. (47) and (48), the solar mixing angle is sensitive to the neutrino mass differences and the phase $\phi_\nu$ but is only very weakly sensitive to the charged lepton masses. Writing the neutrino mass differences $m_{\nu_i} - m_{\nu_j}$ in terms of the differences of the mass squared and one of the neutrino masses, say $|m_{\nu_2}|$, from our previous expressions, (47) and (48), we obtain

$$\frac{m^2_{\nu_2}}{\Delta m^2_{13}} = \frac{1 + 2t^2_{12} + t^4_{12} - rt_{12}}{4t^2_{12}(1 + t^2_{12})(1 + t^2_{12} - rt^2_{12})cos^2\phi_\nu} - \tan^2\phi_\nu + O(x^2)$$
$$\approx \frac{1}{\sin^2 2\theta_{12} \cos^2\phi_\nu} - \tan^2\phi_\nu \text{ for } r \ll 1, \quad (50)$$

where $t_{12} = \tan\theta_{12}$ and $r = \Delta m^2_{21}/\Delta m^2_{13}$.

The mass $|m_{\nu_2}|$ assumes its minimal value when $\sin\phi_\nu$ vanishes, then

$$|m_{\nu_2}| \approx \frac{\sqrt{\Delta m^2_{13}}}{\sin 2\theta_{12}}. \quad (51)$$

Hence, we find

$$|m_{\nu_2}| \approx 0.0507eV,$$
$$|m_{\nu_1}| \approx 0.0499eV \quad (52)$$
$$|m_{\nu_3}| \approx 0.0193eV$$

where we used the values $\Delta m^2_{13} = 2.2^{+0.37}_{-0.27} \times 10^{-3} eV^2$ and $\sin^2\theta_{12} = 0.31^{+0.02}_{-0.03}$ taken from M. Maltoni *et al.* [10], T. Schwetz [11] and G.L. Fogli *et al.* [39].

With those values for the neutrino masses we compute the effective electron neutrino mass $m_\beta$,

$$m_\beta = \left[ \sum_i |U_{ei}|^2 m^2_{\nu_i} \right]^{\frac{1}{2}} = 0.0502eV, \quad (53)$$

well below the upper bound $m_\beta < 1.8eV$ coming from the tritium $\beta$-decay experiments [13, 39, 40].

## 5. Conclusions

By introducing three Higgs fields that are $SU(2)_L$ doublets in the theory, we extended the concept of flavour and generations to the Higgs sector and formulated a Minimal $S_3$-invariant Extension of the Standard Model [16]. A defined structure of the Yukawa couplings is obtained which permits the calculation of mass and mixing matrices for quarks and leptons in a unified way. A further reduction of redundant parameters is achieved in the leptonic sector by introducing a $Z_2$ symmetry. The flavour symmetry group $Z_2 \times S_3$ relates the mass spectrum and mixings. This allowed us to compute the neutrino mixing matrix explicitly in terms of the masses of charged leptons and neutrinos. In this model, the magnitudes of the three mixing angles are determined by the interplay of the flavour $S_3 \times Z_2$ symmetry, the see-saw mechanism and the lepton mass hierarchy. We also found that $V_{PMNS}$ has one Dirac and two Majorana CP violating phases. The numerical values of the mixing angles are determined by the masses of the charged leptons only in very good agreement with the experiment. The solar mixing angle $\theta_{12}$ is almost insensitive to the values of the masses of the charged leptons, but its experimental value allowed us to fix the scale and origin of the neutrino mass spectrum which has an inverted hierarchy with the values $|m_{\nu_2}| = 0.0507eV$, $|m_{\nu_1}| = 0.0499eV$ and $|m_{\nu_3}| = 0.0193eV$.

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