# Exact spectrum and wave functions of the hyperbolic Scarf potential in terms of finite Romanovski polynomials

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The Schrödinger equation with the hyperbolic Scarf potential reported so far in the literature is somewhat artificially manipulated into the form of the Jacobi equation with an imaginary argument and parameters that are complex conjugate to each other. Instead we here solve the former equation anew and make the case that it reduces straightforward to a particular form of the generalized real hypergeometric equation whose solutions are referred to in the mathematics literature as the finite Romanovski polynomials, in reference to the observation that for any parameter set only a finite number of such polynomials appear to be orthogonal. This is a qualitatively new integral property that does not copy any of the features of the Jacobi polynomials. In this manner, the finite number of bound states within the hyperbolic Scarf potential is brought into correspondence with a finite system of a new class of orthogonal polynomials. This work adds a new example to the circle of the problems on the Schrödinger equation. The techniques used by us extend the teachings on the Sturm-Liouville theory of ordinary differential equations beyond their standard presentation in the textbooks on mathematical methods in physics.

*Keywords:* Schrodinger equation; Scarf potentials; Romanovski polynomials.

La solución a la ecuación de Schrödinger con el potencial de Scarf hiperbólico reportada hasta ahora en la literatura física está manipulada artificialmente para obtenerla en la forma de los polinomios de Jacobi con argumentos imaginarios y parámetros que son complejos conjugados entre ellos. En lugar de eso, nosotros resolvimos la nueva ecuación obtenida y desarrollamos el caso en el que realmente se reduce a una forma particular de la ecuación hipergeométrica generalizada real, cuyas soluciones se refieren en la literatura matemática como los polinomios finitos de Romanovski. La notación de finito se refiere a que, para cualquier parámetro fijo, solo un número finito de dichos polinomios son ortogonales. Esta es una nueva propiedad cualitativa de la integral que no surge como copia de ninguna de las características de los polinomios de Jacobi. De esta manera, el número finito de estados en el potencial de Scarf hiperbólico es consistente en correspondencia a un sistema finito de polinomios ortogonales de una nueva clase.

*Descriptores:* Ecuación de Schrodinger; potenciales de Scarf; polinomios de Romanovski.

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## 1. Introduction

The exactly solvable Schrödinger equations occupy a pole position in quantum mechanics insofar as most of them relate directly to physical systems. Suffices to mention as prominent examples the quantum Kepler or Coulomb problem and its importance in the description of the discrete spectrum of the hydrogen atom [1], the harmonic-oscillator, the Hulthen, and the Morse potentials with their relevance to vibrational spectra [2, 3]. Another good example is given by the Pöschl-Teller potential [4] which appears as an effective mean field in many-body systems with $\delta$-interactions [5]. In terms of path integrals, the criteria for exact resolvability of the Schrödinger equation is the existence of exactly solvable corresponding path integrals [6].

There are various methods of finding the exact solutions of the Schrödinger equation (SE) for the bound states, an issue on which we shall focus in the present work. The traditional method, to be pursued by us here, consists in reducing SE by an appropriate change of the variables to that very form of the generalized hypergeometric equation [7] whose solutions are polynomials, the majority of them being well known. The second method suggests to first unveiling the dynamical symmetry of the potential problem and then employing the relevant group algebra in order to construct the solutions as the group representation spaces [8, 9]. Finally, there is also the most recent and powerful method of super-symmetric quantum mechanics (SUSYQM) which considers the special class of Schrödinger equations (in units of $\hbar = 1 = 2m$) that allow for a factorization according to [10-12],

$$(H(z) - e_n)\,\psi_n(z) = \left(-\frac{d^2}{dz^2} + v(z) - e_n\right)\psi_n(z) = 0\,,$$
$$H(z) = A^+(z)A^-(z) + e_0\,,$$
$$A^\pm(z) = \left(\pm\frac{d}{dz} + U(z)\right)\,. \quad (1)$$

Here, $H(z)$ stands for the (one-dimensional) Hamiltonian, $U(z)$ is the so called super-potential, and $A^\pm(z)$ are the ladder operators connecting neighboring solutions. The super-potential allows us to recover the ground state wave function, $\psi_{\rm gst}(z)$, as

$$\psi_{\rm gst}(z) \sim e^{-\int^z U(y)dy}\,. \quad (2)$$

The excited states are then built up on top of $\psi_{\rm gst}(z)$ through the repeated action of the $A^+(z)$ operators.

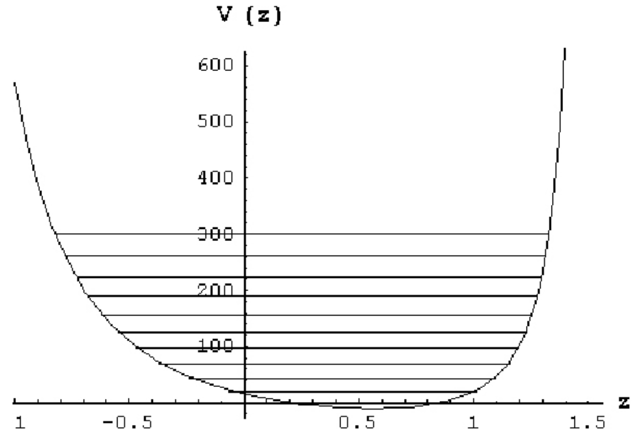


FIGURE 1. The trigonometric Scarf potential (Scarf I) for the set of parameters, $a = 10$, $b = 5$, and $\alpha = 1$. The horizontal lines represent the discrete levels.

## 1.1. The trigonometric Scarf potential

Super-symmetric quantum mechanics manages a family of exactly solvable potentials (see Refs. 11 to 13 for details), one of which is the so called trigonometric Scarf potential (Scarf I) [14], here denoted by $v_t(z)$ and given by

$$v_t^{(a,b)}(z) = -\,a^2 + (a^2 + b^2 - a\alpha)\sec^2 \alpha z \\ - b(2a+\alpha)\tan\alpha z \sec\alpha z. \tag{3}$$

It has been used in the construction of a periodic potential and employed in one-dimensional crystal models in solid state physics.

The exact solution of the Schrödinger equation with the trigonometric Scarf potential (displayed in Fig. 1) is well known [11, 13] and given in terms of the Jacobi polynomials, $P_n^{\beta,\alpha}(x)$, as

$$\psi_n(x) = \sqrt{(1-x)^\gamma(1+x)^\delta}\,P_n^{\gamma-\frac{1}{2},\delta-\frac{1}{2}}(x), \\ x = \sin\alpha z, \\ w^{\gamma-\frac{1}{2},\delta-\frac{1}{2}}(x) = (1-x)^{\gamma-\frac{1}{2}}(1+x)^{\delta-\frac{1}{2}}, \\ \gamma = \frac{1}{\alpha}(a-b), \quad \delta = \frac{1}{\alpha}(a+b)\,. \tag{4}$$

Here, $w^{\gamma-\frac{1}{2},\delta-\frac{1}{2}}(x)$ stands for the weight function from which the Jacobi polynomials $P_n^{\gamma-\frac{1}{2},\delta-\frac{1}{2}}(x)$ are obtained via the Rodrigues formula.

The corresponding energy spectrum is obtained as

$$\epsilon_n = e_n + a^2 = (a+n\alpha)^2\,. \tag{5}$$

## 1.2. The hyperbolic Scarf potential

By means of the substitutions

$$a \longrightarrow ia\,, \qquad \alpha \longrightarrow -i\alpha\,, \quad b \longrightarrow b\,, \tag{6}$$

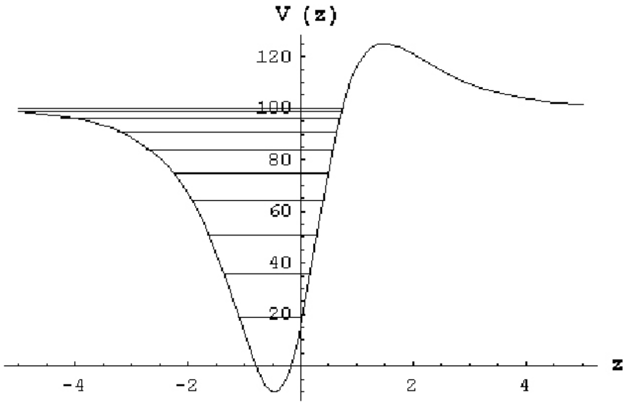


FIGURE 2. The hyperbolic Scarf potential (Scarf II) for the set of parameters, $a = 10$, $b = 5$, and $\alpha = 1$. The horizontal lines represent the energies, $e_n$, of the bound states.

Scarf I is transformed into the so-called *hyperbolic* Scarf potential (Scarf II), here denoted by $v_h^{(a,b)}(z)$ and displayed in Fig. 2,

$$v_h^{(a,b)}(z) = a^2 + (b^2 - a^2 - a\alpha)\mathrm{sech}^2\alpha z \\ + b(2a+\alpha)\mathrm{sech}\alpha z\tanh\alpha z\,. \tag{7}$$

The latter potential has also been found independently within the framework of super-symmetric quantum mechanics while exploring the super-potential [11, 13, 15]:

$$U(z) = a\tanh\alpha z + b\mathrm{sech}\alpha z\,. \tag{8}$$

Upon the above substitutions, and taking $\alpha = 1$ for simplicity, the energy changes to

$$\epsilon_n = e_n - a^2 = -(a-n)^2\,, \quad n = 0, 1, 2, ... < a\,. \tag{9}$$

It is important to notice that, while the trigonometric Scarf potential allows for an infinite number of bound states, the number of discrete levels within the hyperbolic Scarf potential is finite, a difference that will be explained in Sec. 3 below. Yet the most violent changes seem to be suffered by the Jacobi weight function in Eq. (4) and are due to the opening of the finite interval $[-1,+1]$ toward infinity

$$x{=}\sin\alpha z \in [-1,1] \longrightarrow x{=}\sinh\alpha z \in [-\infty,+\infty]\,. \tag{10}$$

In this case, the wave functions become [11, 16, 17],

$$\psi_n(-ix) = (1+x^2)^{-\frac{a}{2}}e^{-b\tan^{-1}x}c_n P_n^{\eta^*,\eta}(-ix)\,, \\ \eta = ib - a - \frac{1}{2}\,. \tag{11}$$

Here, $c_n$ is some state dependent complex phase to be fixed later on. The latter equation gives the impression that the exact solutions of the hyperbolic Scarf potential rely exclusively upon those peculiar Jacobi polynomials with imaginary arguments and complex indices. We here draw attention to the fact that this need not be so.

### 1.3. The goal

> The goal of this work is to solve the Schrödinger equation with the hyperbolic Scarf potential anew and to make the case that it reduces in a straightforward manner to a particular form of the generalized real hypergeometric equation whose solutions are given by a finite set of real orthogonal polynomials. In this manner, the finite number of bound states within the hyperbolic Scarf potential is brought in correspondence with a finite system of orthogonal polynomials of a new class.

These polynomials were discovered in 1884 by the English mathematician Sir Edward John Routh [18] and rediscovered 45 years later by the Russian mathematician Vsevolod Ivanovich Romanovski in 1929 [19] within the context of probability distributions. Though they have been studied on few occasions in the current mathematical literature where they are termed to as "finite Romanovski" [20-23] or "Romanovski-Pseudo-Jacobi" polynomials [24], they have been completely ignored by the textbooks on mathematical methods in physics and, surprisingly enough, by the standard mathematics textbooks as well [7,25-28]. The notion "finite" refers to the observation that, for any given set of parameters (*i.e.* in any potential), only a finite number of polynomials appear orthogonal.

The Romanovski polynomials happen to be equal (up to a phase factor) to Jacobi polynomials with imaginary arguments and parameters that are complex conjugate to each other, much like the $\sinh z = i \sin iz$ relationship. Although one may (but does not have to) deduce the local characteristics of the latter, such as generating function and recurrence relations, from those of the former, the finite orthogonality theorem is qualitatively new. It does not copy any of the properties of the Jacobi polynomials, but requires an independent proof.

Our work adds a new example to the circle of typical quantum mechanical problems [29]. The techniques used by us here extend the study of the Sturm-Liouville theory of ordinary differential equations beyond that of the usual textbooks.

A final comment on the importance of the potential in Eq. (7). The hyperbolic Scarf potential finds various applications in physics ranging from electrodynamics and solid state physics to particle theory. In solid state, physics Scarf II is used in the construction of more realistic periodic potentials in crystals [30] than those built from the trigonometric Scarf potential. In electrodynamics, Scarf II appears in a class of problems with non-central potentials (see Sec. 4). In particle physics, Scarf II finds application in studies of the non-perturbative sector of gauge theories by means of toy models such as the scalar field theory in (1+1) space-time dimensions. Here, one encounters the so called "kink-like" solutions which are nothing more than static solitons. The spatial derivative of the kink-like solution is viewed as the ground state wave function of an appropriately constructed Schrödinger equation, which is then employed in the calculation of the quantum corrections to the first order. In Ref. 17 it was shown that specifically Scarf II is amenable to a stable renormalizable scalar field theory.

The paper is organized as follows. In the next section we first highlight in brief the basics of the generalized hypergeometric equation and then relate it to the Schrödinger equation with the hyperbolic Scarf potential. The solutions are obtained in terms of finite Romanovski polynomials and are presented in Sec. 3. Section 4 is devoted to the disguise of the Romanovski polynomials as non–spherical angular functions. The paper ends with a brief summary.

## 2. Master formulas for the polynomial solutions to the generalized hypergeometric equation

All classical orthogonal polynomials appear as solutions of the so called generalized hypergeometric equation (the presentation in this section closely follows Ref. 22):

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0, \tag{12}$$

$$\sigma(x) = ax^2 + bx + c, \quad \tau(x) = xd + e\,,$$
$$\lambda_n = n(n-1)a + nd. \tag{13}$$

There are various methods for finding the solution, here denoted by

$$y_n(x) \quad \equiv \quad P_n\left(\begin{array}{ccc|c} d & & e & \\ a & b & c & x \end{array}\right), \tag{14}$$

with the symbol

$$P_n\left(\begin{array}{ccc|c} d & & e & \\ a & b & c & x \end{array}\right),$$

in which the equation parameters have been made explicit, standing for a polynomial of degree $n$, $\lambda_n$ being the eigenvalue parameter, and $n = 0, 1, 2, \ldots$. In Ref. 22 a master formula for the (monic, $\bar{P}_n$) polynomial solutions has been derived by Koepf and Masjed-Jamei; according to them, one finds

$$\bar{P}_n\left(\begin{array}{ccc|c} d & & e & \\ a & b & c & x \end{array}\right) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(n)}(a,b,c,d,e)x^k\,,$$

$$G_k^{(n)} = \left(\frac{2a}{b+\sqrt{b^2-4ac}}\right)^n {}_2F_1\left(\begin{array}{c} (k-n), \\ 2-\frac{d}{a}-2n \end{array} \quad \left(\frac{2ae-bd}{2a\sqrt{b^2-4ac}}+1-\frac{d}{2a}-n\right) \middle| \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right). \tag{15}$$

Though the formal proof of this relation is a bit lengthy, its verification with symbolic mathematical software such as Maple is straightforward. Notice that the $G_k^{(n)}$ are not normalized. On the other side, Eq. (12) can be treated alternatively as described in the textbook by Nikiforov and Uvarov in Ref. 7, where it is cast into a self-adjoint form and its weight function, $w(x)$, satisfies the so called Pearson differential equation,

$$\frac{\partial}{dx}\left(\sigma(x)w(x)\right) = \tau(x)w(x)\,. \tag{16}$$

The Pearson equation is solved by

$$w(x) \equiv \mathcal{W}\left(\begin{array}{ccc} d & & e \\ a & b & c \end{array}\middle| x\right) = \exp\left(\int \frac{(d-2a)x+(e-b)}{ax^2+bx+c}dx\right), \tag{17}$$

and shows how one can calculate any weight function associated with any parameter set of interest (we again used a symbol for the weight function that makes explicit the parameters of the equation). The corresponding polynomials are now classified according to the weight function, and are built up from the Rodrigues representation as

$$P_n\left(\begin{array}{ccc} d & & e \\ a & b & c \end{array}\middle| x\right) = \Pi_{k=1}^{k=n}(d+(n+k-2)a)\bar{P}_n\left(\begin{array}{ccc} d & & e \\ a & b & c \end{array}\middle| x\right) = \frac{1}{\mathcal{W}\left(\begin{array}{ccc} d & & e \\ a & b & c \end{array}\middle| x\right)} \times \frac{d^n}{dx^n}\left((ax^2+bx+c)^n\mathcal{W}\left(\begin{array}{ccc} d & & e \\ a & b & c \end{array}\middle| x\right)\right). \tag{18}$$

The master formulas in the respective Eqs. (15) and (18), allow for the construction of all the polynomial solutions to the generalized hypergeometric equation. One identifies as special cases the canonical parameterizations known as

- the Jacobi polynomials with $a=-1$, $b=0$, $c=1$, $d=-\gamma-\delta-2$, and $e=-\gamma+\delta$,
- the Laguerre polynomials with $a=0$, $b=1$, $c=0$, $d=-1$, and $e=\alpha+1$,
- the Hermite polynomials with $a=b=0$, $c=1$, $d=-2$, and $e=0$,
- the Romanovski polynomials with $a=1$, $b=0$, $c=1$, $d=2(1-p)$, and $e=q$ with $p>0$,
- the Bessel polynomials with $a=1$, $b=0$, $c=0$, $d=\alpha+2$, and $e=\beta$.

All other parameterizations can be reduced to one of the above five by an appropriate shift of the variables. The first three polynomials are the only ones that are traditionally presented in the standard textbooks on mathematical methods in physics such as [25-28], while the fourth and fifth seem to have escaped due attention. Notice that the Legendre, Gegenbauer, and Chebychev polynomials appear as particular cases of the Jacobi polynomials. The Bessel polynomials are not orthogonal in the conventional sense, *i.e.* within a real interval, and will be left out of consideration.

Some of the properties of the fourth polynomials have been studied in the specialized mathematics literature such as Refs. 20, 21, and 23. Their weight function is calculated from Eq. (17) as

$$w^{(p,q)}(x) = (x^2+1)^{-p}e^{q\tan^{-1}x}\,. \tag{19}$$

This weight function was first reported by Routh [18], then and independently by Romanovski [19]. The polynomials associated with Eq. (19) are named after Romanovski, and will be denoted by $R_m^{(p,q)}(x)$. They have non-trivial orthogonality properties over the infinite interval $[-\infty,+\infty]$. Indeed, as long as the weight function decreases as $x^{-2p}$, integrals of the type

$$\int_{-\infty}^{+\infty} w^{(p,q)}(x)R_m^{(p,q)}(x)R_{m'}^{(p,q)}(x)dx \tag{20}$$

will be convergent only if

$$m+m' < 2p-1\,, \tag{21}$$

meaning that only a finite number of Romanovski polynomials are orthogonal. This is the reason for the term "finite"Romanovski polynomials (details are given in Ref. 31). The differential equation satisfied by the Romanovski polynomials reads as

$$(1+x^2)\frac{d^2R_n^{(p,q)}(x)}{d^2x}+(2(-p+1)x+q)\,\frac{dR_n^{(p,q)}(x)}{dx} - (n(n-1)+2n(1-p))R_n^{(p,q)}(x)=0\,. \quad (22)$$

In the next section we shall show that the Schrödinger equation with the hyperbolic Scarf potential reduces precisely to that very Eq. (22).

### 2.1. The real polynomial equation associated with the hyperbolic Scarf potential

The Schrödinger equation for the potential of interest when rewritten in a new variable, $x$, introduced via an appropriate point canonical transformation [32, 33], taken by us as $z=f(x)=\sinh^{-1}x$, is obtained as:

$$(1+x^2)\frac{d^2g_n(x)}{dx^2}+x\frac{dg_n(x)}{dx} + \left(\frac{-b^2+a(a+1)}{1+x^2}-\frac{b(2a+1)}{1+x^2}x+\epsilon_n\right)g_n(x)=0, \quad (23)$$

with $g_n(x)=\psi_n(\sinh^{-1}x)$, and $\epsilon_n=e_n-a^2$. Equation (19) suggests the following substitution in Eq. (23):

$$g_n(x)=(1+x^2)^{\frac{\beta}{2}}e^{-\frac{\alpha}{2}\tan^{-1}x}D_n^{(\beta,\alpha)}(x)\,,\quad x=\sinh z\,,\quad -\infty<x<+\infty. \quad (24)$$

In effect, Eq. (23) reduces to the following equation for $D_n^{(\beta,\alpha)}(x)$,

$$(1+x^2)\frac{d^2D_n^{(\beta,\alpha)}(x)}{dx^2}+((2\beta+1)x-\alpha)\frac{dD_n^{(\beta,\alpha)}(x)}{dx} + \left(\beta^2+\epsilon_n+\frac{(a+a^2+\beta-\beta^2-b^2+\frac{\alpha^2}{4})+x(-b-2ab+\frac{\alpha}{2}-\alpha\beta)}{1+x^2}\right)D_n^{(\beta,\alpha)}(x)=0\,. \quad (25)$$

If the potential equation (25) is to coincide with the Romanovski equation (22), then

- first the coefficient in front of the $1/(x^2+1)$ term in (25) must vanish,
- the coefficients in front of the first derivatives must be equal, *i.e.* $2(-p+1)+q=(2\beta+1)x-\alpha$,
- the eigenvalue constants should also be equal, *i.e.* $\epsilon_n+\beta^2=-n((n-1)+2(1-p))$.

The first condition restricts the parameters of the $D_n^{(\beta,\alpha)}(x)$ polynomials to

$$a+a^2-b^2+\frac{\alpha^2}{4}+\beta-\beta^2=0\,, \quad (26)$$

$$-b-2ab+\frac{\alpha}{2}-\alpha\beta=0\,. \quad (27)$$

Solving Eqs. (26) and (27) for $\alpha$ and $\beta$ results in

$$\beta=-a\,,\qquad \alpha=2b\,. \quad (28)$$

The second condition relates the parameters $\alpha$ and $\beta$ to $p$ and $q$, and amounts to

$$\beta=-a=-p+\frac{1}{2},\quad -\alpha=q=-2b. \quad (29)$$

Finally, the third restriction leads to a condition that fixes the Scarf II energy spectrum as

$$\epsilon_n=-(a-n)^2\,. \quad (30)$$

In this way, the polynomials that enter the solution of the Schrödinger equation will be

$$D_n^{(\beta=-a,\alpha=2b)}(x)\equiv R_n^{(p=a+\frac{1}{2},q=-2b)}(x). \quad (31)$$

They are obtained by means of the Rodrigues formula from the weight function $w^{(a+\frac{1}{2},-2b)}(x)$ as

$$R_n^{(a+\frac{1}{2},-2b)}(x)=\frac{1}{w^{(a+\frac{1}{2},-2b)}(x)}\frac{d^m}{dx^m} \times(1+x^2)^m w^{(a+\frac{1}{2},-2b)}(x),$$
$$w^{(a+\frac{1}{2},-2b)}(x)=(1+x^2)^{-a-\frac{1}{2}}e^{-2b\tan^{-1}x}. \quad (32)$$

As a result, the wave function of the $n$th level, $\psi_n$, takes the form

$$\psi_n(z=\sinh^{-1}x)\stackrel{\text{def}}{:=}g_n(x)=\frac{1}{\sqrt{\frac{d\sinh^{-1}x}{dx}}}\sqrt{(1+x^2)^{-\left(a+\frac{1}{2}\right)}e^{-2b\tan^{-1}x}}R_n^{\left(a+\frac{1}{2},-2b\right)}(x),$$
$$d\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}dx\,. \quad (33)$$

The orthogonality integral of the Schrödinger wave functions gives rise to the following orthogonality integral of the Romanovski polynomials:

$$\int_{-\infty}^{+\infty} \psi_n(z)\psi_{n'}(z)dz = \int_{-\infty}^{+\infty} (1+x^2)^{-\left(a+\frac{1}{2}\right)} e^{-2b\tan^{-1}x} R_n^{(a+\frac{1}{2},-2b)}(x) R_{n'}^{(a+\frac{1}{2},-2b)}(x)dx\,, \tag{34}$$

which coincides in form with the integral in Eq. (20) and is convergent for $n < a$. The fact that only a finite number of Romanovski polynomials are orthogonal is reflected by the finite number of bound states within the potential of interest, a number that depends on the potential parameters, in accord with Eq. (21).

As to the complete Scarf II spectrum, it was constructed in Ref. 9 using the dynamical symmetry approach [8]. There, the Scarf II potential was found to possess $SU(1,1)$ as a symmetry group. The bound states have been assigned to the discrete unitary irreducible representations of $SU(1,1)$. The scattering and resonant states (which are beyond the scope of the present study) have been related to the continuous unitary and non-unitary representations of $SU(1,1)$, respectively.

A comment is in order on the relation between the Romanovski polynomials and the Jacobi polynomials of imaginary arguments and parameters that are complex conjugate to each other. Recall the real Jacobi equation,

$$(1-x^2)\frac{d^2P_n^{\gamma,\delta}(x)}{dx^2} + (\gamma-\delta-(\gamma+\delta+2)x)\frac{dP_n^{\gamma,\delta}(x)}{dx} - n(n+\gamma+\delta+1)P_n^{\gamma,\delta}(x) = 0\,. \tag{35}$$

As mentioned above, the real Jacobi polynomials are orthogonal within the $[-1,1]$ interval with respect to the weight-function in Eq. (4). Transforming to a complex argument, $x \to ix$, and parameters, $\gamma = \delta^* = c + id$, Eq. (35) transforms into

$$(1+x^2)\frac{d^2P_n^{c+id,c-id}(ix)}{dx^2} + (-2d+2(c+1)x)\frac{dP_n^{c+id,c-id}(ix)}{dx} + n(n+2c+1)P_n^{c+id,c-id}(ix) = 0\,. \tag{36}$$

Correspondingly, the weight function turns out to be

$$w^{c+id,c-id}(ix) = (1+x^2)^c e^{-2d\tan^{-1}x}\,, \tag{37}$$

and it coincides with the weight function of the Romanovski polynomials in Eq. (19) upon the identifications $c = -p$, and $q = -2d$. This means that $P_n^{c+id,c-id}(ix)$ will differ from the Romanovski polynomials by a phase factor found as $i^n$ in Ref. 35:

$$i^n P_n\left(\begin{array}{ccc} 2(1-p) & & iq \\ -1 & 0 & 1 \end{array}\middle| ix\right) = P_n\left(\begin{array}{ccc} 2(1-p) & & q \\ 1 & 0 & 1 \end{array}\middle| x\right)\,. \tag{38}$$

Because of this relationship, the Romanovski polynomials have been termed to as “Romanovski-Pseudo-Jacobi” by Lesky [24]. The relationship in Eq. (38) tells that the $R_n^{(p,q)}(x)$ properties translate into those of $P^{-p-i\frac{q}{2},-p+i\frac{q}{2}}(ix)$ and visa versa, and that it is a matter of convenience to prefer one polynomial over the other. When it comes to recurrence relations, generating functions, etc., it is perhaps more convenient to favor the Jacobi polynomials, although the generating function of the Romanovski polynomials is equally well calculated directly from the corresponding Taylor series expansion [31]. However, concerning the orthogonality integrals, the advantage is clearly on the side of the real Romanovski polynomials. This is so because the complex Jacobi polynomials are known for their highly non-trivial orthogonality properties, which depend on the interplay between integration contour and parameter values [36]. For this reason, in random matrix theory [37], the problem on the gap probabilities in the spectrum of the circular Jacobi ensemble is treated in terms of the Cauchy random ensemble, a venue that heads one again to the Romanovski polynomials (notice that for $p = 1, q = 0$ the weight function in Eq. (19) reduces to the Cauchy distribution).

In summary, for all the reasons given above, the Romanovski polynomials qualify as the most adequate real degrees of freedom in the mathematics of the hyperbolic Scarf potential.

## 3. The polynomial construction

The construction of the $R_n^{(a+\frac{1}{2},-2b)}(x)$ polynomials is now straightforward and based upon the Rodrigues representation in Eq. (18), where we plug in the weight function from

Eq. (19). In carrying out the differentiations, we find the lowest four (unnormalized) polynomials to be

$$R_0^{(a+\frac{1}{2},-2b)}(x) = 1, \tag{39}$$

$$R_1^{(a+\frac{1}{2},-2b)}(x) = -2b + (1-2a)x, \tag{40}$$

$$R_2^{(a+\frac{1}{2},-2b)}(x) = 3 - 2a + 4b^2 - 8b(1-a)x + (6 - 10a + 4a^2)x^2, \tag{41}$$

$$\begin{aligned} R_3^{(a+\frac{1}{2},-2b)}(x) &= -266 + 12ab - 8b^3 + [-3(-15+16a-4a^2)+12(3-2a)b^2]x \\ &\quad + (-72b + 84ab - 24a^2 b)x^2 + 2(-2+a)(-15+16a-4a^2)x^3. \end{aligned} \tag{42}$$

As illustration, in Fig. 3 we show the Scarf II wave functions of the first and third levels.

The finite orthogonality of the Romanovski polynomials becomes especially transparent in the interesting limiting case of the $\mathrm{sech}^2 z$ potential (it appears in the non-relativistic reduction of the sine-Gordon equation), where one easily finds that the normalization constants $N_n^{(a+\frac{1}{2},0)}$ are given by the following expressions:

$$\begin{aligned} \left(N_1^{(a+\frac{1}{2},0)}\right)^2 &= \frac{(2a-1)^2\sqrt{\pi}\,\Gamma(a-1)}{2\Gamma(a+\frac{1}{2})}, \quad a>1\,, \\ \left(N_2^{(a+\frac{1}{2},0)}\right)^2 &= \frac{2\sqrt{\pi}(a-1)\Gamma(a-2)}{\Gamma(a-\frac{1}{2})}(3-2a)^2, \quad a>2\,, \\ \left(N_3^{(a+\frac{1}{2},0)}\right)^2 &= \frac{3\sqrt{\pi}(a-2)\Gamma(a-3)}{\Gamma(a-\frac{1}{2})}(4a^2-16a+15)^2, \quad a>3 \text{ etc.} \end{aligned} \tag{43}$$

Software like Maple or Mathematica are quite useful for the graphical study of these functions. The latter expressions show that, for positive integer values of the $a$ parameter, $a = n$, only the first $(n-1)$ Romanovski polynomials are orthogonal (the convergence of the integrals requires $n < a$), as it should be in accordance with Eqs. (21), (9). The general expressions for the normalization constants of any Romanovski polynomial are defined by integrals of the type

$$\int_{-\infty}^{+\infty} (1+x^2)^{-(p-n)} e^{q\tan^{-1}x} dx$$

and are analytic for $(p-n)$ integer or semi-integer.

## 4. Romanovski polynomials and non-spherical angular functions in electrodynamics with non-central potentials

In recent years, there have been several studies of the bound states of an electron within a compound Coulomb- and non-central potential (see Refs. 38 and 39, and references therein). Let us assume the following potential:

$$V(r,\theta) = V_{\mathcal{C}}(r) + \frac{V_2(\theta)}{r^2}, \quad V_2(\theta) = -c\cot\theta\,, \tag{44}$$

where $V_{\mathcal{C}}(r)$ denotes the Coulomb potential and $\theta$ is the polar angle (see Fig. 4). The corresponding Schrödinger equation reads

$$\left[-\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r,\theta)\right]\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi)\,, \tag{45}$$

and is solved as usual by separating the variables:

$$\Psi(r,\theta,\phi) = \mathcal{R}(r)\Theta(\theta)\Phi(\varphi)\,. \tag{46}$$

The radial and angular differential equations for $\mathcal{R}(r)$ and $\Theta(\theta)$ are then found to be

$$\frac{d^2\mathcal{R}(r)}{dr^2} + \frac{2}{r}\frac{d\mathcal{R}(r)}{dr} + \left[(V_{\mathcal{C}}(r)+E) - \frac{l(l+1)}{r^2})\right]\mathcal{R}(r) = 0, \tag{47}$$

and

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot(\theta)\frac{d\Theta(\theta)}{d\theta} + \left[l(l+1) - V_2(\theta) - \frac{m^2}{\sin^2\theta}\right]\Theta(\theta) = 0\,, \tag{48}$$

with $l(l+1)$ being the separation constant. From now on we shall focus our attention on Eq. (48). It is obvious that for $V_2(\theta) = 0$, and upon changing variables from $\theta$ to $\cos\theta$, it

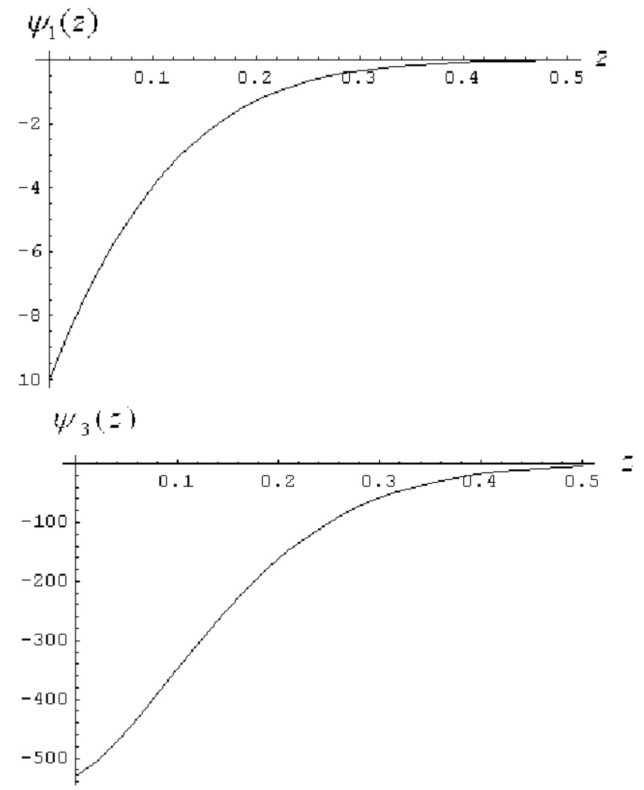


FIGURA 3. Wave functions for the first and third levels within the hyperbolic Scarf potencial.

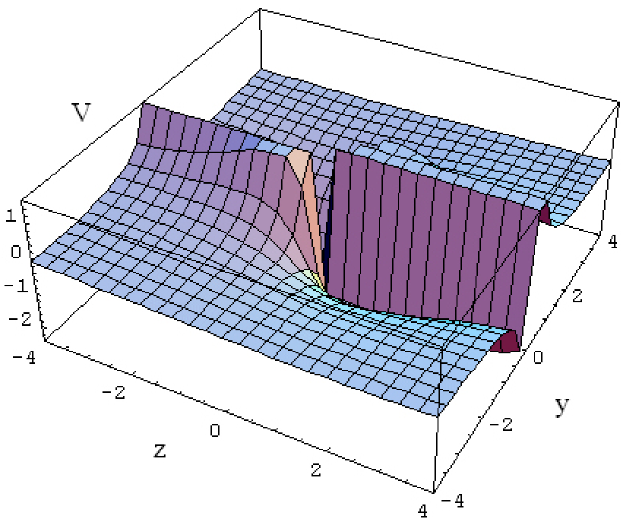


FIGURE 4. The non-central potential $V(r,\theta)$, displayed here in its intersection with the $x=0$ plane, *i.e.* for $r=\sqrt{y^2+z^2}$, and $\theta=\tan^{-1}y/z$. The polar angle part of its exact solutions is expressed in terms of the Romanovski polynomials.

becomes the associated Legendre equation. Correspondingly, $\Theta(\theta)$ approaches the associated Legendre functions,

$$\Theta(\theta)\ \stackrel{V_2(\theta)\to 0}{\longrightarrow}\ P_l^m(\cos\theta)\,, \tag{49}$$

an observation that will become important below.

In order to solve Eq. (48) we follow the prescription given in Ref. 38 and begin by substituting the polar angle by the new variable, $z$, introduced via $\theta\to f(z)$. This transformation leads to the new equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[-\frac{f''(z)}{f'(z)}+f'(z)\cot f(z)\right]\frac{\mathrm{d}}{\mathrm{d}z}+\left[-V_2(f(z)) + l(l+1)-\frac{m^2}{\sin^2 f(z)}\right]f'^{\,2}(z)\right]\psi(z)\ =0, \tag{50}$$

with $f'(z)\equiv\mathrm{d}f(z)/\mathrm{d}z$, and $\psi(z)$ defined as $\psi(z)=\Theta(f(z))$. Next, one requires the coefficient in front of the first derivative to vanish, which transforms Eq. (50) into a $1d$ Schrödinger equation. This restricts $f(z)$ to satisfying the differential equation:

$$\frac{f''(z)}{f'(z)}=f'(z)\cot f(z)\,, \tag{51}$$

which is solved by $f(z)=2\tan^{-1}e^z$. With this relation and after some algebraic manipulations, one finds that

$$\sin\theta=\frac{1}{\cosh z},\quad \cos\theta=-\tanh z\,, \tag{52}$$

and consequently,

$$f'(z)=\sin f(z)=\mathrm{sech}\, z. \tag{53}$$

Equation (52) implies $\sinh z\ =\ -\cot\theta$, or, equivalently, $\theta\ =\ \cot^{-1}(-\sinh z)$. Upon substituting Eq. (53) into Eqs. (44), and (50), one arrives at

$$\begin{aligned}&\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2}+\left[l(l+1)\frac{1}{\cosh^2 z} + c\tanh z\frac{1}{\cosh z}-m^2\right]\psi(z)=0\,,\\ &\psi(z)\stackrel{\mathrm{def}}{:=}\Theta\left(\theta=\cot^{-1}(-\sinh z)\right)\\ &\Theta(\theta)\stackrel{\mathrm{def}}{:=}\psi\left(z=\sinh^{-1}(-\cot\theta)\right)\,.\end{aligned} \tag{54}$$

Taking into consideration Eqs. (7),(44), and (52), one realizes that the latter equation is precisely the one-dimensional Schrödinger equation with the hyperbolic Scarf potential where

$$\begin{aligned}&l(l+1)=-(b^2-a(a+1))\,,\quad m^2=-\epsilon_n=(a-n)^2\,,\\ &m>0\,,\quad c=-b(2a+1).\end{aligned} \tag{55}$$

Taking into account Eq. (9) above, the first Eqs. (55) is satisfied by

$$a\ \ =\ \ b=l(l+1)\,. \tag{56}$$

As long as the integer number $m$ satisfies

$$m=a-n=l(l+1)-n>0,$$

then $1 \leq m \leq l(l+1)$. In this way, the solution for $\Theta$ becomes

$$\begin{aligned}\Theta(\theta) &= \psi_{n=l(l+1)-m}\left(z=\sinh^{-1}(-\cot\theta)\right) \\ &= (1+\cot^2\theta)^{-\frac{l(l+1)}{2}} e^{-l(l+1)\tan^{-1}(-\cot\theta)} \\ &\times R^{(l(l+1)+\frac{1}{2},-2l(l+1))}_{l(l+1)-m}(-\cot\theta)\,. \end{aligned} \tag{57}$$

Therefore, the complete angular part of the solution is given by

$$\Theta(\theta)\Phi(\varphi)=\psi_{n=l(l+1)-m}\big(z=\sinh^{-1}(-\cot\theta)\big)\, e^{i\epsilon_n\varphi}. \tag{58}$$

From now on, we shall introduce $Z_l^m(\theta,\varphi)$, a new notation, according to

$$Z_l^m(\theta,\varphi)=\psi_{n=l(l+1)-m}\big(z=\sinh^{-1}(-\cot\theta)\big)\, e^{i\epsilon_n\varphi}, \tag{59}$$

and refer to $Z_l^m(\theta,\varphi)$ as non-spherical angular functions. In Fig. 5, we display two of the lowest $|Z_l^m(\theta,\varphi)|$ functions for illustrative purposes. A more extended sampler can be found in Ref. 42. A comment is in order on $|Z_l^m(\theta,\varphi)|$. In that regard, it is important to become aware of the fact already mentioned above that the Scarf II potential possesses $su(1,1)$ as a potential algebra, a result reported by Refs. 9 and 34 among others. There, it was pointed out that the respective Hamiltonian, $H$, equals $H=-C-1/4$, with $C$ being the $su(1,1)$ Casimir operator, whose eigenvalues in our convention are $j(j-1)$ with $j>0$ versus $j(j+1)$ and $j<0$ in the convention of [9, 34]. As a consequence, the bound state solutions to Scarf II are assigned to infinite discrete, irreducible unitary representations, $\{D_j^{+\,(m')}(\theta,\varphi)\}$, of the $SU(1,1)$ group. The $SU(1,1)$ labels $m'$, and $j$ are mapped onto ours via

$$\begin{aligned} m' &= a+\frac{1}{2} = l(l+1)+\frac{1}{2}, \\ j &= m'-n\,, \quad m'=j, j+1, j+2, .... \end{aligned} \tag{60}$$

meaning that both $j$ and $m'$ are half-integers. The representations are infinite because, for a fixed $j$ value, $m'$ is bounded from below to $m'_{\min}=j$, but it is not bound from above.

In terms of the $SU(1,1)$ labels the energy rewrites as $\epsilon_n=-(j-1/2)^2$. The condition $a>n$ translates now as $j>1/2$. As a result, $\Theta(\theta)$ becomes

$$\begin{aligned}\Theta(\theta) &= \psi_{n=m'-j}\left(\sinh^{-1}(-\cot\theta)\right) \\ &= \sqrt{(1+\cot^2\theta)^{-m'+1/2}e^{-2b\tan^{-1}(-\cot\theta)}} \\ &\times R^{(m',-2b)}_{m'-j}(-\cot\theta) \\ &= D^{+\;(m'=l(l+1)+1/2)}_{j=m+1/2}(\theta,\varphi)e^{-im'\varphi}. \end{aligned} \tag{61}$$

Here we kept the parameter $b$ general because its value does not affect the $SU(1,1)$ symmetry. Within this context, $|\psi_{m'-j}\left(\sinh^{-1}(-\cot\theta\right)|$ can be viewed as the absolute value of a component of an irreducible $SU(1,1)$ representation [40, 41] in terms of the Romanovski polynomials. The $|Z_l^m(\theta,\varphi)|$ functions are then images in polar coordinate space of the $|D^{+\;(m'=l(l+1)+1/2)}_{j=m+1/2}|$ components.

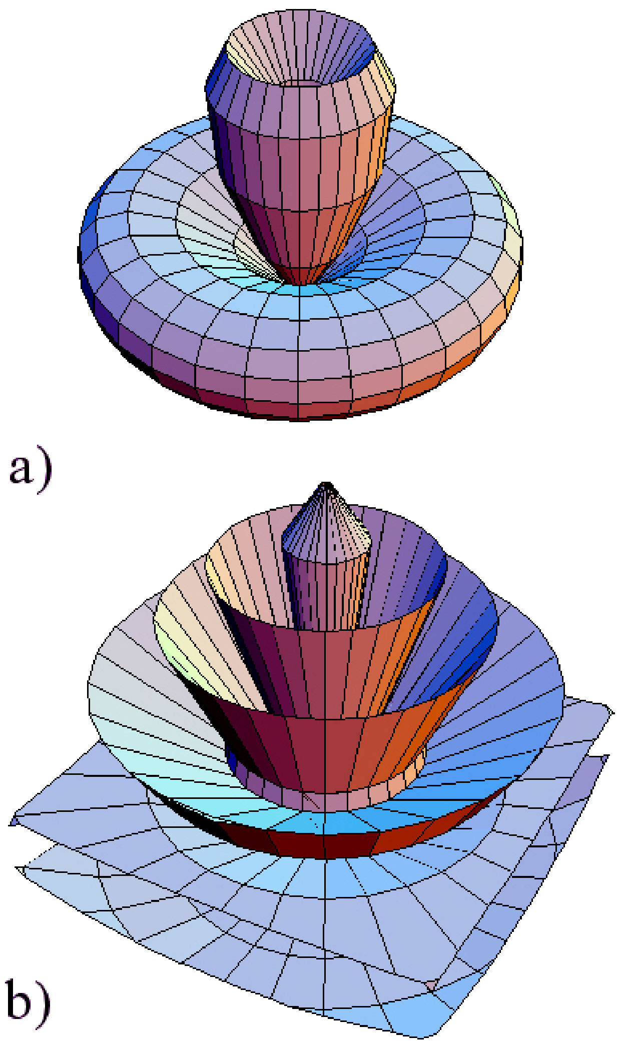


FIGURE 5. Graphical presentation of the non-spherical-angular functions $|Z_1^1(\theta,\varphi)|$ (left), and $|Z_2^1(\theta,\varphi)|$ (right) to the $V_2(r,\theta)$ potential in Eq. (44). They portray in polar coordinate space in turn the components $|D^{+\;(m'=5/2)}_{j=3/2}(\theta,\varphi)|$, and $|D^{+\;(m'=13/2)}_{j=3/2}(\theta,\varphi)|$ of the respective infinite discrete unitary $SU(1,1)$ representation.

### 4.1. Romanovski polynomials and associated Legendre functions.

It is quite instructive to consider the case of a vanishing $V_2(\theta)$, *i.e.* $c=0$, and compare Eq. (54) to Eq. (7) for $b=0$. In this case

$$l=a\,, \qquad m^2=(l-n)^2\,, \tag{62}$$

which allows one to relate $n$ to $l$ and $m$ as $m=l-n$. As long as the two equations are equivalent, their solutions differ at most by a constant factor. This allows us to establish a relationship between the associated Legendre functions and

TABLE I. Characteristics of the orthogonal polynomial solutions to the generalized hypergeometric equation.

| Notion | Symbol | $w(x)$ | Interval | Number of orth. polynomials |
|---|---|---|---|---|
| Jacobi | $P_n^{\nu,\mu}(x)$ | $(1-x)^\nu(1+x)^\mu$ | $[-1,1]$ | infinite |
| Hermite | $H(x)$ | $e^{-x^2}$ | $[-\infty,\infty]$ | infinite |
| Laguerre | $L^{\alpha,\beta}(x)$ | $x^\beta e^{-\alpha x^2}$ | $[0,\infty]$ | infinite |
| Romanovski | $R_n^{(p,q)}(x)$ | $(1+x^2)^{-p}e^{q\tan^{-1}x}$ | $[-\infty,\infty]$ | finite |

the Scarf II wave functions. Considering Eqs. (24), and (33) together with Eqs. (52), one finds $\cot\theta = -\sinh z$ which produces the following intriguing relationship between the associated Legendre functions and the Romanovski polynomials:

$$P_l^m(\cos\theta) \sim (1+\cot^2\theta)^{-\frac{l}{2}} R_{l-m}^{(l+\frac{1}{2},0)}(-\cot\theta)\,, \qquad l-m=n=0,1,2,...l. \tag{63}$$

Substituting the latter expression into the orthogonality integral between the associated Legendre functions,

$$\int_0^\pi P_l^m(\cos\theta)P_{l'}^m(\cos\theta)\mathrm{d}\cos\theta = 0\,, \quad l\neq l', \tag{64}$$

results in the following integral:

$$\int_0^\pi (1+\cot^2\theta)^{-\frac{l+l'}{2}} R_{l-m}^{(l+\frac{1}{2},0)}(-\cot\theta) R_{l'-m}^{(l'+\frac{1}{2},0)} \times(-\cot\theta)\mathrm{d}\cos\theta = 0\,, \quad l\neq l'. \tag{65}$$

Rewriting in conventional notations, the latter expression becomes

$$\int_{-\infty}^{+\infty} \sqrt{w^{(l+\frac{1}{2},0)}(x)}R_{n=l-m}^{(l+\frac{1}{2},0)}(x) \times \sqrt{w^{(l'+\frac{1}{2},0)}(x)}R_{n'=l'-m}^{(l'+\frac{1}{2},0)}(x)\frac{\mathrm{d}x}{1+x^2} = 0\,, \quad l\neq l'\,,$$
$$x=\sinh z\,, \quad l-n=l'-n'=m\geq 0\,. \tag{66}$$

This integral describes orthogonality between an *infinite* set of Romanovski polynomials with *different polynomial parameters* (they would define wave functions of states bound in *different potentials*). This new orthogonality relationship does not contradict the finite orthogonality in Eq. (21), which is valid for states belonging to *same potential* (*equal polynomial parameters*). Rather, for different potentials, Eq. (21) can be fulfilled for an infinite number of states. To see this, let us consider for simplicity $n=n'=l-m$, *i.e.*, $l=l'$. Given $p=l+1/2$, the condition in Eq. (21) defines normalizability and takes the form

$$2(l-m) < 2\left(l+\frac{1}{2}\right)-1 = 2l\,, \tag{67}$$

which is automatically fulfilled for any $m>0$. The presence of the additional factor of $(1+x^2)^{-1}$ guarantees convergence also for $m=0$. Equation (66) reveals that, for parameters attached to the degree of the polynomial, an infinite number of Romanovski polynomials can appear orthogonal, although not precisely with respect to the weight function that defines their Rodrigues representation. The study presented here is similar to Ref. 43. There, the exact solutions of the Schrödinger equation with the trigonometric Rosen-Morse potential have been expressed in terms of Romanovski polynomials (not recognized as such at that time), and also with parameters that depended on the degree of the polynomial. Also in this case, the $n$-dependence of the parameters, and the corresponding varying weight function make it possible to satisfy Eq. (21) for an infinitely many polynomials.

## 5. Summary

In this work we presented the classification of the orthogonal polynomial solutions to the generalized hypergeometric equation in the schemes of Koepf–Masjed-Jamei [22], on the one hand, and Nikiforov-Uvarov [7], on the other. We found among them the real polynomials that define the solutions of the bound states within the hyperbolic Scarf potential. These so called Romanovski polynomials have the remarkable property that, for any given set of parameters, only a finite number of them are orthogonal. In such a manner, the finite number of bound states within Scarf II were mapped onto a finite set of orthogonal polynomials of a new type.

We showed that the Romanovski polynomials also define the angular part of the wave function of the non-central potential considered in Sec. 4. Yet, in this case, the polynomial parameters turned out to be dependent on the polynomial degree. We identified these non-spherical angular solutions to the non-central potential under investigation as images in polar coordinate space of components of infinite discrete unitary $SU(1,1)$ representations. In the limit of the vanishing non-central piece of the potential, we established a non-linear relationship between the Romanovski polynomials and the associated Legendre functions. On the basis of the orthogonality integral for the latter, we derived a new such integral for the former.

The presentation contains all the details which to our understanding are essential for reproducing our results. By this means, we worked out two problems which could be used in the class on quantum mechanics and on mathematical methods in physics as well and which allow us to practice performing with symbolic software. The appeal of the two examples is that they simultaneously relate to relevant peer research.

The hyperbolic Scarf potential and its exact solutions are interesting mathematical entities on their own, with several applications in physics, ranging from super-symmetric quantum mechanics over soliton physics to field theory. We expect future research to reveal additional, interesting properties and problems related to the hyperbolic Scarf potential and its exact real polynomial solutions.

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