RevMexAA (Serie de Conferencias), **34**, 83–86 (2008)

DYNAMICAL FRICTION FORCE EXERTED ON SPHERICAL BODIES

O. Esquivel[1,2] and B. Fuchs[1]

RESUMEN

Siguiendo un enfoque mecánico-ondular calculamos la fuerza de arrastre ejercida por un sistema homogéneo e infinito de estrellas de fondo sobre un pertubador mientras éste se mueve a través del sistema. Recuperamos la fórmula clásica para la fuerza de fricción (FF) derivada por Chandrasekhar, pero con un logaritmo de Coulomb modificado. Al estimar la FF ejercida sobre una esfera de Plummer y un perturbador que posee un perfil tipo Hernquist, consideramos un intervalo de modelos que abarca toda distribución plausible de satélites galácticos. Se muestra que la configuración del perturbador afecta únicamente la forma exacta del logaritmo de Coulomb. Tal logaritmo converge a pequeñas escalas porque los encuentros entre la partícula de prueba y las estrellas de fondo, cuyos parámetros de impacto son inferiores al tamaño del perturbador masivo, resultan ineficientes. Comprobamos así los resultados previos basados en la aproximación de impulso de pequeñas deflecciones angulares.

ABSTRACT

Following a wave-mechanical treatment we calculate the drag force exerted by an infinite homogeneous background of stars on a perturber as it makes its way through the system. We recover Chandrasekhar's classical dynamical friction (DF) law with a modified Coulomb logarithm. We take into account a range of models that encompasses all plausible density distributions for satellite galaxies by considering the DF exerted on a Plummer sphere and a perturber having a Hernquist profile. It is shown that the shape of the perturber affects only the exact form of the Coulomb logarithm. The latter converges on small scales, because encounters of the test and field stars with impact parameters less than the size of the massive perturber become inefficient. We confirm this way earlier results based on the impulse approximation of small angle scatterings.

Key Words: galaxies: kinematics and dynamics — methods: analytical

1. INTRODUCTION

The process of dynamical friction (DF) is one of the most classical and fundamental problems encountered in the description of the evolution of almost all astrophysical systems. From the critical momentum exchange in a protoplanet-protoplanetary disk set up, passing through the problem of satellites in galaxies to galaxies in large clusters, proper understanding of DF is a prerequisite to more ambitious attempts at constructing physically justified models.

In his seminal paper Chandrasekhar (1943) envisaged the scenario of a sequence of consecutive gravitational two-body encounters of test and field stars in order to calculate the drag force (cf. Hénon 1973 for a modern presentation). In particular, application of DF to calculate the rate of a sinking satellite has received special attention, and efforts have been made to include more general background distributions. However, as for the perturber itself, White (1976) has been the only one to consider a more realistic finite-size perturber to compute analytically the DF based on an impulse-approximation approach. His main result was a modification of the Coulomb logarithm so that it does not diverge anymore at small scales, because gravitational encounters at impact parameters smaller than the size of the perturbing body become ineffective. Here we rigorously calculate the drag force exerted on different bodies following the approach of both Marochnik (1968) and Kalnajs (1972) who determined the "polarization cloud" created in the background medium as a massive object was making its way through the system. This method can be simply understood as linear and angular momentum exchange in stellar systems (Lynden-Bell & Kalnajs 1972; Dekker 1976; Tremaine & Weinberg 1984; Fuchs 2004), and has been extensively used in plasma physics (cf. Stix 1992). It is shown in § 4 below that the shape of the perturber affects only the exact form of the Coulomb logarithm. As concrete examples we calculate the drag force exerted on a Plummer sphere and on a

[1]Astronomisches Rechen-Institut am Zentrum für Astronomie der Universität Heidelberg, Mönchhofstraße 12-14, 69120 Heidelberg, Germany.

[2]Fellow of International Max-Planck Research School for Astronomy and Cosmic Physics, Heidelberg (esquivel@ari.uni-heidelberg.de).

sphere with the density distribution of a Hernquist (1990) profile, respectively, and compare them with the drag force exerted on a point mass.

2. A WAVE-MECHANICAL TREATMENT

We assume an infinite homogenous distribution of field stars on isotropic straight-line orbits. The response of the system of background stars to the perturbation due to a massive perturber is determined by solving the linearized Boltzmann equation

$$\frac{\partial f_1}{\partial t} + \sum_{i=1}^{3} \mathrm{v_i} \frac{\partial f_1}{\partial \mathrm{x_i}} - \frac{\partial \Phi_1}{\partial \mathrm{x_i}} \frac{\partial f_0}{\partial \mathrm{v_i}} = 0\,, \qquad (1)$$

where Φ_1 denotes the gravitational potential of the perturber and f_1 and f_0 are the perturbed and unperturbed distribution functions of the field stars in phase space. By Fourier-transforming both the perturbations f_1 and the potential Φ_1 (whose forms are $f_{\omega,\mathbf{k}}$; $\Phi_{\omega,\mathbf{k}} \exp\, i[\omega t + \mathbf{k}\cdot\mathbf{x}]$; where ω and $\mathbf{k}$ denote the frequency and wave vector of the Fourier components) the solution of Boltzmann equation is greatly facilitated. Without loss of generality the spatial coordinates $\mathrm{x_i}$ and the corresponding velocity components $\mathrm{v_i}$ can be oriented with one axis parallel to the direction of the wave vector. The Boltzmann equation (1) takes then the form

$$\omega f_{\omega,\mathbf{k}} + \upsilon k f_{\omega,\mathbf{k}} - k\Phi_{\omega,\mathbf{k}} \frac{\partial f_0}{\partial \upsilon} = 0\,, \qquad (2)$$

with $k = |\mathbf{k}|$ and υ denoting the velocity component parallel to $\mathbf{k}$. Equation (2) has been integrated over the two velocity components perpendicular to $\mathbf{k}$. In the following we assume for the field stars always a Gaussian velocity distribution function, to find the solution

$$f_{\omega,\mathbf{k}} = -\frac{k\upsilon}{\omega + k\upsilon} \frac{n_\mathrm{b}}{\sqrt{2\pi}\sigma^3} e^{-\frac{\upsilon^2}{2\sigma^2}} \Phi_{\omega,\mathbf{k}}\,, \qquad (3)$$

where n_b denotes the spatial density of the field stars. Integrating equation (3) over the υ-velocity leads to the density distribution of the induced polarization cloud. This has been calculated here without taking into account the self-gravity of the background medium. However, Fuchs (2004) has shown that in linear approximation the effects of self-gravity are not important for the dynamics of the polarization cloud.

3. POTENTIALS OF THE PERTURBING BODIES

To start with, we consider in our analysis the potential of a point mass, which moves with the velocity $\mathrm{v_0}$ along the y-axis,

$$\Phi_1 = -\frac{Gm}{\sqrt{\mathrm{x}^2 + (\mathrm{y} - \mathrm{v_0}t)^2 + \mathrm{z}^2}}\,. \qquad (4)$$

Its Fourier-Transform can be calculated using formulae (3.754) and (6.561) of Gradshteyn & Rhyzik (2000) as

$$\Phi_\mathbf{k} = -\frac{Gm}{2\pi^2}\frac{1}{k^2} e^{-i\mathrm{k_y v_0}t}\,. \qquad (5)$$

Next, the potential (4) is generalized to

$$\Phi_1 = -\frac{Gm}{\sqrt{r_0^2 + \mathrm{x}^2 + (\mathrm{y} - \mathrm{v_0}t)^2 + \mathrm{z}^2}}\,, \qquad (6)$$

which corresponds to an extended body with the mass distribution of a Plummer sphere ($\rho \propto 1/r_0^3(1+r^2/r_0^2)^{-\frac{5}{2}}$ (Binney & Tremaine 1987). The Fourier transform of a moving Plummer sphere is given by

$$\Phi_\mathbf{k} = -\frac{Gm}{2\pi^2}\frac{r_0}{k} K_1(kr_0) e^{-i\mathrm{k_y v_0}t}\,, \qquad (7)$$

where K_1 denotes the modified Bessel function of the second kind. As third example we consider a perturber which has the mass density distribution of a Hernquist profile ($\rho \propto (r_0/r)(r_0 + r)^{-3}$ (Hernquist 1990). Its gravitational potential is given by

$$\Phi_1 = -\frac{Gm}{r_0 + r}\,. \qquad (8)$$

The Fourier transform of a moving Hernquist sphere can be calculated using equation (3.722) of Gradshteyn & Rhyzik (2000)[3] leading to

$$\Phi_\mathbf{k} = -\frac{Gm}{2\pi^2}\frac{1}{k^2}[1 + kr_0 \cos\,(kr_0)\mathrm{si}\,(kr_0) - kr_0 \sin\,(kr_0)\mathrm{ci}\,(kr_0)] e^{-i\mathrm{k_y v_0}t} \quad , \qquad (9)$$

where si and ci denote the sine- and cosine-integrals, respectively. The model of a Plummer sphere has often been used in numerical simulations of the accretion and their eventual disruption of satellite galaxies in massive parent galaxies. Plummer spheres have constant density cores, whereas numerical simulations of the formation of galactic haloes in cold dark matter cosmology show that dark haloes may have a central density cusp (Navarro, Frenk, & White 1997). Thus models of a Plummer or a Hernquist sphere should encompass the range of plausible models for satellite galaxies. The density in both models falls off radially steeper than found in the cold dark matter galaxy cosmogony simulations. This mimics the tidal truncation of satellite galaxies in the gravitational field of their parent galaxies.

[3] We use the identity $\sin\, kr = -\frac{1}{r}\frac{\partial}{\partial k}\cos\, kr$ in equation (3.722).

4. DYNAMICAL FRICTION

The ensemble of stars is accelerated by the moving perturber as

$$< \dot{\mathbf{v}} >= - \int d^3\mathrm{x} \int d^3\mathrm{v} f(\mathbf{x}, \mathbf{v}) \nabla \Phi_1 \,, \tag{10}$$

where f denotes the full distribution function $f = f_0 + f_1$. The contribution from f_0 cancels out, and introducing the Fourier transforms we find

$$\begin{aligned} < \dot{\mathbf{v}} > &= - \int d^3\mathrm{x} \int d^3\mathrm{v} \int d^3\mathrm{k}\, i\mathbf{k}\Phi_{\mathbf{k}} \\ &\times \; e^{i[\omega t + \mathbf{k}\cdot\mathbf{x})]} \int d^3\mathrm{k}' f_{\mathbf{k}'} e^{i[\omega' t + \mathbf{k}'\cdot\mathbf{x}]} \,. \end{aligned} \tag{11}$$

From symmetry reasons the acceleration vector $<\dot{\mathbf{v}}>$ is expected to be oriented along the y-axis. In equation (11) the frequency ω, and similarly ω', is given according to equations (5), (7) and (9) by $\omega = -\mathrm{k_y v_0} - i\lambda$ where we have introduced, following Landau's rule, a negative imaginary part, which we will let go to zero in the following. Moreover, $\Phi^*_{\mathbf{k}} = \Phi_{-\mathbf{k}}$ so that the potential is a real quantity. Equation (11) simplifies to

$$< \dot{\mathbf{v}} >= (2\pi)^3 \int d^3\mathrm{v} \int d^3\mathrm{k}\, i\mathbf{k}\Phi_{-\mathbf{k}} f_{\mathbf{k}} e^{2\lambda t} \,. \tag{12}$$

Using expression (3) and taking the limit $\lambda \to 0$ we get

$$< \dot{\mathbf{v}} >= \frac{(2\pi)^{5/2}\pi n_\mathrm{b}}{\sigma^3} \int d^3\mathrm{k}\, |\Phi_{\mathbf{k}}|^2 \frac{\mathbf{k}}{k}\, \mathrm{k_y v_0} e^{-\frac{(\mathrm{k_y v_0})^2}{2k^2\sigma^2}} \,. \tag{13}$$

The Fourier transform of any potential with spherical symmetry depends only on $k = |\mathbf{k}|$. Thus it follows immediately from equation (13) that indeed the two acceleration components $<\dot{\mathrm{v}}_\mathrm{x}>=<\dot{\mathrm{v}}_\mathrm{z}>= 0$ as anticipated. Only in the direction of motion of the perturber is there a net effect. According to Newton's third law the drag force exerted on the perturber is given by $m\dot{\mathrm{v}} = -m_\mathrm{b}<\dot{\mathrm{v}}_\mathrm{y}>$ where m_b is the mass of a background particle, so that the drag force is anti-parallel to the velocity of the perturber. In order to evaluate the integrals over the wave numbers in equation (13) it is advantageous to switch from Cartesian form $\mathrm{k_x, k_y, k_z}$ to a mixed representation $\mathrm{k_y}$, $k = \sqrt{\mathrm{k_x^2 + k_y^2 + k_z^2}}$, $\arctan(\mathrm{k_x/k_z})$, and we obtain for the deceleration the general result

$$\dot{\mathrm{v}} = -\frac{4\pi G^2 m m_\mathrm{b} n_\mathrm{b}}{\mathrm{v}_0^2} \left[\mathrm{erf}\left(\frac{\mathrm{v}_0}{\sqrt{2}\sigma}\right) - \sqrt{\frac{2}{\pi}} \frac{\mathrm{v}_0}{\sigma} e^{-\frac{\mathrm{v}_0^2}{2\sigma^2}} \right] \ln\Lambda \tag{14}$$

where erf denotes the usual error function. The Coulomb logarithm is defined as

$$\ln\Lambda = \frac{4\pi^4}{G^2 m^2} \int_{k_\mathrm{min}}^{k_\mathrm{max}} dk\; k^3 |\Phi_\mathrm{k}|^2 \,. \tag{15}$$

In the case of a point mass formula (5) implies $\Lambda = k_\mathrm{max}/k_\mathrm{min}$. This result was first obtained in this form by Kalnajs (1972) and is identical to Chandrasekhar's (1943) formula, if $m + m_b \approx m$. The Coulomb logarithm diverges in the familiar way both on small and large scales, i.e., at k_max^{-1} and k_min^{-1}, respectively. The Coulomb logarithm of the dynamical friction force exerted on a Plummer sphere can be calculated by inserting equation (7) into (15) leading to

$$\ln\Lambda = r_0^2 \int_{k_\mathrm{min}}^{\infty} dk\; k\, K_1^2(r_0 k) \,. \tag{16}$$

If the Plummer radius r_0 shrinks to zero, expression (16) changes smoothly into the Coulomb logarithm of a point mass, because $\lim_{r_0\to 0}\; r_0 K_1(k_0 k) = k^{-1}$. The integral over the square of the Bessel functions in equation (16) can be evaluated using formula (5.54) of Gradshteyn & Rhyzik (2000),

$$\ln\Lambda = -\frac{r_0^2 k_\mathrm{min}^2}{2} \left[K_1^2(r_0 k_\mathrm{min}) - K_0(r_0 k_\mathrm{min}) K_2(r_0 k_\mathrm{min}) \right] \,, \tag{17}$$

which is approximately

$$\ln\Lambda \approx -1/2 - \ln(r_0 k_\mathrm{min}) \,, \tag{18}$$

in the limit of $r_0 k_\mathrm{min} \ll 1$. This modified Coulomb logarithm converges on small scales precisely as found by White (1976), but still diverges on large scales. A natural cut-off will be then the size of the stellar system under consideration. For a perturber with the density distribution of a Hernquist profile we find a Coulomb logarithm of the form

$$\begin{aligned} \ln\Lambda = \int_{k_\mathrm{min}}^{\infty} dk\; \frac{1}{k} [1 &+ r_0 k \cos(r_0 k) \mathrm{si}(r_0 k) \\ &- r_0 k \sin(r_0 k) \mathrm{ci}(r_0 k)]^2 \,. \end{aligned} \tag{19}$$

It can be shown using the asymptotic expansions of the sine- and cosine-integrals given by Abramowitz & Stegun (1972) that the integrand in expression (19) falls off at large k as $4r_0(r_0 k)^{-5}$. Thus the Coulomb logarithm converges at small scales. This is expected because, although the density distribution has an inner density cusp, the deflecting mass 'seen' by a field star with a small impact parameter scales with the square of the impact parameter. At small wave numbers a Taylor expansion shows that the square bracket in expression (19) approaches 1 so that we find a logarithmic divergence of the Coulomb logarithm as in the case of the Plummer sphere. In Figure 1 we illustrate the Coulomb logarithms of the Plummer sphere and of a sphere with a Hernquist density distribution according to equations (17) and (19) as function of $r_0 k_{\min}$. Figure 1 shows clearly that at a given mass small-sized perturbers experience a stronger dynamical friction force than larger ones. Since the cut-off of the wave number at $k_{\min}$ is determined by the radial extent of the stellar system, which corresponds roughly to one half of the largest subtended wave length $\lambda_{\max} = 2\pi/k_{\min}$, we use actually $r_0/(\lambda_{\max}/4)$ as abscissa in Figure 1. For comparison we have also drawn $-\ln\,(r_0 k_{\min})$ in Figure 1. The insert shows the cumulative mass distributions of both mass models. The half mass radius of the Hernquist model measured in units of r_0 is about twice that of the Plummer sphere. Thus for a proper comparison of the drag forces exerted on a Plummer sphere and a sphere with a Hernquist density profile the dashed line in Figure 1 should be stretched by a factor of about 2 towards the right. But it is clear from Figure 1 that the drag force exerted on a Plummer is always larger that the drag on a sphere with a Hernquist density profile. This is to be expected because of its shallower density profile. The logarithm $-\ln\,(r_0 k_{\min})$, although it is the asymptotic expansion of the Coulomb logarithms (17) and (19) for $r_0 k_{\min} \to 0$, is not a good approximation at larger $r_0 k_{\min}$. There is a systematic off-set relative to the true Coulomb logarithms which is given explicitly in equation (18) for the case of the Plummer sphere. It could be argued that perturbers can be deformed by tidal fields. However, this effect is expected to be small (a higher order effect). There is a further effect if the perturbers are gravitationally bound systems themselves like globular clusters or dwarf satellite galaxies. Such objects can and do lose mass due to tidal shocking and both effects can even be of comparable magnitude. Finally, we want to mention that our analysis can be extended, in a straightforward way, to anisotropic velocity distributions of the field stars. Fuchs & Athanassoula (2005) have shown that the velocity dispersion in the solution of the Boltzmann equation (3) is replaced by an effective velocity dispersion which depends on the semi-axes of the velocity ellipsoid and its orientation relative to the wave vector **k**. We intend to present our results in a forthcoming paper (Esquivel & Fuchs, in preparation).

O.E. gratefully acknowledges financial support by the International-Max-Planck-Research-School for Astronomy and Cosmic Physics at the University of Heidelberg.

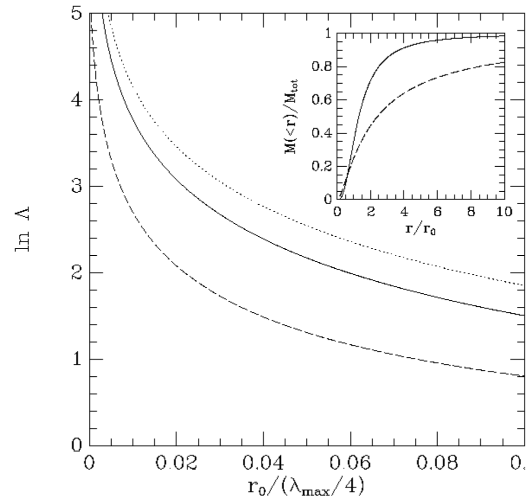

Fig. 1. Coulomb logarithms of the Plummer sphere (solid line), and of a sphere with a Hernquist density profile (dashed line). The dotted line indicates $-\ln\,(2\pi r_0/\lambda_{\max})$. r_0 is the radial scale length of the spheres and $\lambda_{\max}$ the upper cut-off of the wavelength of the density perturbations (see text). The inset shows the cumulative mass distributions of the Plummer and Hernquist models.

REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (New York: Dover)
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)
Chandrasekhar, S. 1943, ApJ, 97, 255
Dekker, E. 1976, Phys. Rev. D, 24, 315
Fuchs, B. 2004, A&A, 419, 941
Fuchs, B., & Athanassoula, E. 2005, A&A, 444, 455
Gradshteyn, I. S., & Rhyzik, I. M. 2000, Table of Integrals, Series, and Products (New York: Academic Press)
Hénon, M. 1973, in Dynamical Structure and Evolution of Stellar Systems, ed. G. Contopoulos, M. Hénon, & D. Lynden-Bell, Lectures of the 3rd Advance Course of the Swiss Society of Astronomy and Astrophysics (Sauverny: Geneva Observatory), 182
Hernquist, L. 1990, ApJ, 356, 359
Kalnajs, A. J. 1972, in Gravitational N-Body Problem, ed. M. Lecar, (Dordrecht: Reidel), 13
Lynden-Bell, D., & Kalnajs, A. J. 1972, MNRAS, 157, 1
Marochnik, L. S. 1968, Soviet Ast., 11, 873
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1997, ApJ, 490, 493
Stix, T. H. 1992, Waves in Plasmas (New York: AIP)
Tremaine, S., & Weinberg, M. D. 1984, MNRAS, 209, 729
White, S. D. M. 1976, MNRAS, 174, 19