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*The concept of exploitation in a
general linear model with heterogeneous labour*

TAKAO FUJIMOTO*

INTRODUCTION

The Fundamental Marxian Theorem (FMT), as formulated by Okishio (1963), asserts that in a capitalist economy exploitation is necessary to realize profits. After the original contribution by Okishio, there have appeared many contributions which generalize or criticize his result, notably by Morishima (1973, 1974) and Roemer (1981, 1982). In these it is assumed the economy is in equilibrium, or that labour is homogeneous. Some have claimed that the economy should be in a sort of equilibrium in order to reproduce the present system, *i.e.*, production and class relationships. If,

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however, the FMT is valid only in equilibrium, its scope is so limited that the proposition becomes uninteresting. It has been also argued that Walrasian general equilibrium or micro foundation for the FMT is desirable. Again if the FMT is valid only when all the people maximize their utility function under some constraints, it is not applicable to the real world. We had better construct the robust concept of exploitation and consider the validity of the claim made by FMT in a general disequilibrium state for a snap-shot period, say this year.

Our framework is similar to that of Morishima (1974). It is, however, considered independently of equilibrium conditions, and we allow for the existence of heterogeneous labour, and workers may save and have properties, which will bring them extra incomes. Besides proper joint production, thus durable capital goods, as well as durable consumption goods are admitted. The essence of our method is that various types of labour and normal goods are treated in a symmetric way, as expounded in Fujimoto and Opocher (2007).¹ As in Morishima model, capitalists are not explicit, or simply they do not need to exist as those who are eager to exploit others. In our model there are only various types of workers, who render their services such as lathing, educational or managerial ones. When prices and wage rates are specified, it is possible to tell which types of labour are exploited, and some workers may well feel themselves as non-workers, or even as celebrated capitalists. Our model does not specify how a particular type of labour is created from other types. Various types are just given at the beginning of a production period, and at the end of this period they may change to other types through on-job-training or educational processes, all in the background. We simply need processes which reproduce each type of labour by consuming certain sets of commodities and services. These processes are called household activities in this paper, and we allow for the existence of alternative household activities to reproduce each type of labour. Our definition of values

¹ For simplicity each type of labour is assumed to be abundantly available, though heterogeneous labour can be dealt with in the same manner as various vintages of capital goods.

needs no price, and the propositions make clear some basic properties of the values defined. Our definition of exploitation, on the other hand, surely needs prices including wage rates. We argue that exploitation, or more precisely the exploiters and the exploited can in general exist among workers themselves, in models with heterogeneous labour even without typical capitalists.

The structure of this paper is as follows. Second section tells the reader a short history of the FMT and the merits of our model. In third section, in order to make this note easier to understand, we explain our definition of labour values and our approach, using a simple Leontief model, and then fourth section describes our model and defines the values of goods (and services) and various types of labour in terms of a particular type of labour. Fifth section defines the concept of exploitation. The main definitions and propositions are presented in fourth and fifth sections. Sixth section gives three numerical examples in order to show how our definitions work in a concrete way, though in abstract models. Finally concluding remarks, especially those concerning the links between the real world capitalist system and the new definitions as well as a new analytical method discussed in this paper, are stated in seventh section.

A SHORT HISTORY OF THE FUNDAMENTAL MARXIAN THEOREM

Okishio (1963) first established the FMT in a Leontief model of circulating capital and homogeneous labour. What he showed is that the existence of positive profits is equivalent to the existence of exploitation. Exploitation in Okishio (1963) means that the labour value of the consumption basket for one unit of labour force is less than unity. After the first contribution, Okishio (1977) continued to generalize his FMT to linear models with durable capital goods together with his students.

Morishima (1973, 1974), on the other hand, extended the FMT to a von Neumann model of joint production, thus incorporating durable capital goods. In these Morishima's contributions, exploitation exists when the minimum amount of labour force required to produce what has been

consumed by the working class in a given production period is less than the actual amount of labour force used in the period. There appeared some papers claiming the finding of counter-examples: Steedman (1975) and Petri (1980). The former counter-example is related to whether one adopts the equality system or the inequality system in a model of joint production, while the latter is concerned with a model with so low a level of technology that no positive growth is possible.

Fujimoto (1978) established the FMT in disequilibrium employing a von Neumann model, and using the social average profit rate as well as the social average growth rate. Until this paper, the FMT had been dealt with in equilibrium only. Fujimori (1982) tried to generalize the FMT to models with heterogeneous labour by reducing various types of labour to a particular class of labour, *e.g.*, unskilled one. Krause (1981) gave a method of reduction from various types of labour to a particular type. Heterogeneous labour has also been dealt with, *e.g.*, in Bowles and Gintis (1977), Okishio (1977), and Steedman (1977). In their models, either single production is assumed or reduction to a particular type of labour is pursued. Moreover, workers' consumption basket is fixed depending on their type.

Then appeared a seemingly fundamental criticism against the FMT, *i.e.*, the commodity exploitation theorem. This commodity exploitation theorem by Bowles and Gintis (1981) and Roemer (1982, 1986) attacks the FMT by insisting that only labour is given a special role in determining values. When commodities and labour are dealt with in a symmetrical way, any commodity would be exploited alike. The cause of this confusion comes from the misunderstanding that the existence of exploitation can be determined within the value world without introducing prices and wages. Even in the paper by Okishio (1963), it is implicitly assumed that prices are such that one unit of labour can obtain the designated basket of consumption goods by spending his/her wages. We cannot discuss on the existence of exploitation without knowing how much workers can buy using their wages. This is the point made explicit in Fujimoto and Fujita (2008).

In a recent paper, Matsuo (2008) has introduced utility functions for workers, in order to avoid the counter-example by Petri (1980) and to

renounce the commodity exploitation theorem. It is desirable, however, to establish the FMT without borrowing the concept of individual optimization from the neoclassical world. Exploitation should be defined and grasped irrespective of the existence of utility functions, and, if at all, irrespective of whether workers maximize their utility or not.

Intuitively, in a model with homogeneous labour, if we choose that homogeneous labour as the standard of value, the value of the total net output of a period is the amount of labour rendered by the working class in the period. Then, it is natural to define the existence of exploitation to be the case where not all the total net products come into the workers' hands. Therefore, when there are positive profits, all or some portions of which escape from workers and are used to purchase some of net products, we can declare exploitation is going on. This natural way is in the background in fifth section (more precisely subsection 5.1) of this paper so long as a model with homogeneous labour is concerned.

In addition to the above intuitive approach to the phenomenon of exploitation, our model below has other merits. In the literature so far published on the FMT, none allow for durable consumption goods, nor for the existence of alternative household activities to reproduce labour. None treated values in a model with heterogeneous labour without reducing various types to simple labour, except for Bowles and Gintis (1977). Our model in section 4 can include these elements in a natural way, with no worry about the reduction problem. Besides, workers can save and have properties which bring them incomes other than wages. In a sense, some workers may be 'have-a little's or simply 'haves': this fact has been neglected in the literature on the topic.

LABOUR VALUES FOR A LEONTIEF MODEL²

We first explain our definition of values and our method by employing a Leontief model of circulating capital. Let us assume there are n kinds of

² This section is a simpler version of what was presented in Fujimoto and Opocher (2007).

commodities [goods and services (other than direct labour)] and one type of homogeneous labour. Let A be the $n \times n$ material (and service) input coefficient matrix, I the $n \times n$ or $(n + 1) \times (n + 1)$ identity matrix depending on the context. The symbol, ℓ , means the row n -vector of labour input coefficients, and c the column n -vector of workers' consumption basket which enables a household activity to reproduce one unit of labour in a production period. The row n -vector of labour values are written as λ . The symbol R_+^n means the nonnegative orthant of the n -dimensional Euclidean space, R^n . A prime to a vector indicates its transposition. A subscript attached to a vector, as in x_i or $(Ax)_i$, means the i -th element of the vector. Following Okishio (1963), the labour values of normal commodities are determined by

$$\lambda = \lambda A + \ell \quad [1]$$

while the labour value of one unit of labour (force), λ_ℓ , is calculated as the inner product

$$\lambda_\ell = \lambda \cdot c \quad [2]$$

Now we define an $(n + 1) \times (n + 1)$ matrix A as

$$A \equiv \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix}$$

which is termed the 'complete matrix' in Bródy (1970, p. 23). The last n -th column of A may be called the input of the household activity to reproduce one unit of labour force. The above two equations [1] and [2] can be combined to lead to

$$(\lambda, \lambda_\ell) = (\lambda, 1)A \quad [3]$$

It is important to note here that in the above equation, the direct input of labour is given unity as its value on the RHS, while the value of one unit of labour is calculated as λ_ℓ on the LHS. Then equation [3] is rewritten as

$$(\lambda, 1) = (\lambda, 1)A + (0_n, 1 - \lambda_\ell) \quad [3']$$

where 0_n is the row n -vector whose elements are all zero. This leads to

$$(v\lambda, v) = (v\lambda, v)A + [0_n, v(1 - \lambda_\ell)]$$

where v is a positive scalar. Supposing $(1 - \lambda_\ell) > 0$, we normalize v so that $v(1 - \lambda_\ell) = 1$. By putting $q \equiv (v\lambda, v)$, we finally obtain the equation

$$q = q \cdot A + (0_n, 1) \quad [4]$$

We make the following assumption.

Assumption A1 (productiveness assumption). There exists a nonnegative column $(n + 1)$ -vector x such that $x > Ax$. Thus, $(I - A)^{-1} \geq 0$.³

Now equation [4] is solved for q as

$$q = (0_n, 1) (I - A)^{-1} \quad [5]$$

On the other hand, from equation [3']

$$(\lambda, 1) = (0_n, 1 - \lambda_\ell)(I - A)^{-1}$$

It is clear from equation [5] that q is the last row of the inverse $(I - A)^{-1}$, thus the above equation gives

$$1 = q_{n+1}(1 - \lambda_\ell), \text{ that is, } \lambda_\ell = \frac{q_{n+1} - 1}{q_{n+1}} \quad [6]$$

³ See Hawkins and Simon (1949) for their conditions concerning A .

$$\lambda_j = \frac{q_j}{q_{n+1}} \text{ for } j = 1, \dots, n \quad [7]$$

It naturally follows that $0 \leq \lambda_\ell < 1$ and $\lambda_j \geq 0$ for $j = 1, \dots, n$, *i.e.*, the labour value of one unit of labour is less than one. When labour is indispensable to produce a basket c , we have $0 < \lambda_\ell$. (See Fujimoto and Fujita, 2008). Therefore, when the productiveness assumption A1 is given, all we have to do is to solve equation [5] first, and then calculate labour values using equations [6] and [7].

Now even in a Leontief model we are using here, there exists a set of inefficient processes hidden, typically those which waste more or less resources. With these inefficient processes included, the input coefficient matrix A is enlarged to have more columns than rows: let us suppose A is now $(n + 1) \times (m + 1)$ with $n \leq m$. We still assume there is only one homogeneous labour, and the household activity is on the last $(m + 1)$ -th column of A . Equation [3] becomes

$$(\lambda, \lambda_\ell) \leq (\lambda, 1)A$$

So, equation [4] is now to be modified to an inequality system

$$q \leq q \cdot A + (0_m, 1) \quad [8]$$

The problem is how to define the labour values and also how to compute them. The answer has already been supplied by the well-known nonsubstitution theorem.⁴ We first solve the following linear programming problem:

$$(\text{DL}) \max q_{n+1} \equiv q \cdot e_{n+1} \text{ subject to } q \leq q \cdot A + (0_m, 1) \text{ and } q \in R_+^{n+1}$$

⁴ See, for example, Fujimoto *et al.* (2003).

where $e_{n+1} \equiv (0_n, 1)' \in R_+^{n+1}$, i.e., the $(n+1)$ -column vector whose $(n+1)$ -th entry is unity with all the remaining elements being zero. Then, the labour values are obtained by applying equations [6] and [7]. Thus, our definition of labour values can be stated as:

Definition of values for a Leontief model: labour values in a Leontief model with alternative processes are nonnegative magnitudes assigned to commodities such that the labour value of labour itself be maximized under the condition that the total value of the output of each possible process should not exceed that of the input. When calculating the total value of the input of a process, unity is assigned to the direct input of the labour.⁵

And our assumption above is to be modified as:

Assumption A2. There exists a nonnegative column $(m+1)$ -vector x such that $x > Ax$.

The nonsubstitution theorem has made it clear that a sort of efficiency on the output side is involved because of the duality in linear programming. That is,

$$(PL) \min x_{m+1} \text{ subject to } x \geq A \cdot x + e_{n+1} \text{ and } x \in R_+^{m+1}$$

This programming problem dictates that while producing one unit of labour force as net product, we should minimize the gross production of labour force.⁶ Because of our assumption A2 just stated, the problem (PL) has an optimal solution, and so does the problem (DL). We can now present:

Proposition 1. Given the assumption A2, the labour value of (homogeneous) labour is less than unity.

⁵ Maximizing q_{n+1} is equivalent to maximizing $\lambda_{\mathcal{L}}$, which is obvious from equation [6].

⁶ The nonsubstitution theorem also asserts that the vector c can be any nonzero nonnegative vector, with no influence on the optimality of one and the same solution q . It is also easy to notice that the objective function can also be the total input of labour.

Proof. By the duality theorem, we know that the optimal values satisfy $q_{n+1}^* = x_{m+1}^*$. On the other hand, it is clear from the constraint in (PL) that $x_{m+1}^* \geq 1$. Therefore we get $q_{n+1}^* \geq 1$, which gives $0 \leq \lambda_\ell < 1$ because of equation [6]. QED.

It may be interesting to apply the principle of algebra in interpreting an optimal solution x_{m+1}^* to the problem (PL). Suppose we knew the labour amount which has entered directly and indirectly into one unit of labour, and write this amount as λ_ℓ , then the magnitude $x_{m+1}^*(1 - \lambda_\ell)$ shows the net residual which is not consumed in reproducing x_{m+1}^* units of gross production of labour force. Since in the problem (PL), the system produces one unit of net labour force, we should have

$$x_{m+1}^*(1 - \lambda_\ell) = 1$$

Hence

$$\lambda_\ell = \frac{x_{m+1}^* - 1}{x_{m+1}^*} = \frac{q_{n+1}^* - 1}{q_{n+1}^*}$$

which corresponds to equation [6].

LABOUR VALUES IN A GENERAL MODEL

Our model is a variant of Morishima model (Morishima, 1964) slightly generalized. That is, different from the original model in von Neumann (1945-1946), we explicitly deal with labour input coefficients, and moreover we allow for the existence of heterogeneous labour as well as proper joint production. Then, naturally we can deal with durable capital goods. Various types of labour are treated exactly like normal commodities, and so we use the symbol B and A as the output and input coefficient matrices, both of which now have n rows and m columns. There are altogether n kinds of goods, services, and various types of labour.⁷ On the

⁷ “Bads” are also allowed, though they are implicit. See Fujimoto and Opocher (2007).

other hand, there exist m production processes or household activities. This way of formulation enables us to take into consideration durable consumption commodities in household activities: a durable consumption commodity in a column of household activity of A will appear in the corresponding column of B as one period, say one year, older commodity. For each type of labour, there can be more than one household activities to reproduce that labour. Workers may save a part of their incomes, and may have properties. These complicating elements from the real world do not disturb our study while we deal with values and exploitation: this should be true in any model including Leontief models.

Now it is possible to select any commodity or a type of labour as the standard of value because goods and labour are treated in a completely symmetric way. Here, however, we choose a type of labour as the standard and let it be the i -th labour commodity. We give our definition of labour values as follows:

Definition of values for our general model: values in a general input-output model are nonnegative magnitudes assigned to commodities (including services and various types of labour) such that the value of the standard commodity be maximized under the condition that the total value of the output of each possible process should not exceed that of the input. When calculating the total value of the input of a process, unity is assigned to the direct input of the standard commodity.⁸

Our productiveness assumption here is:

Assumption A3. There exists an $x \in R_+^m$ such that

$$(B - A)x \geq e_{[i]}$$

⁸ Among a plural number of solutions, we adopt those which realize the maximum number of equalities in the constraints. And yet, a solution may not be unique.

where $e_{[i]}$ is the n -column vector whose i -th entry is unity with all the remaining elements being zero.⁹

Having defined values as above, we can now explain how to compute values in a way similar to the problem (DL) in section 3. Let us first define the following vectors:

$$\Lambda^{[i]} \equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, \lambda_i^{[i]}, \lambda_{i+1}^{[i]}, \dots, \lambda_n^{[i]}) \text{ and}$$

$$\Lambda_{[i]}^{[i]} \equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, 1, \lambda_{i+1}^{[i]}, \dots, \lambda_n^{[i]})$$

The vector $\Lambda^{[i]}$ is the vector of values with i -th labour being the standard of value, and the element $\lambda_j^{[i]}$ stands for the value of commodity j with i -commodity as the standard of value. Our definition above is rewritten like this:

Find out $\Lambda^{[i]} \geq 0$ such that $\lambda_i^{[i]}$ should be maximized

$$\text{subject to } \Lambda^{[i]} \cdot B \leq \Lambda_{[i]}^{[i]} \cdot A$$

We can proceed as we have done for a Leontief model in third section. That is, the constraint in this problem can be transformed first through adding $(1 - \lambda_i^{[i]}) \cdot b^{(i)}$ to both sides, then multiplying both sides by nonzero nonnegative v , yielding

$$v \cdot \Lambda_{[i]}^{[i]} \cdot B \leq v \cdot \Lambda_{[i]}^{[i]} \cdot A + v \cdot (1 - \lambda_i^{[i]}) \cdot b^{(i)}$$

where $b^{(i)}$ is the i -th row of B . Then, we set

$$v \cdot (1 - \lambda_i^{[i]}) = 1 \text{ or } \lambda_i^{[i]} = 1 - \frac{1}{v}$$

This normalization yields as our constraint

⁹ We can further weaken the assumption to the existence of an activity level vector which realizes a simple reproduction. See Fujimoto and Opocher (2007).

$$v \cdot \Lambda_{[i]}^{[i]} \cdot B \leq v \cdot \Lambda_{[i]}^{[i]} \cdot A + b^{(i)}$$

Since we have $\lambda_i^{[i]} = 1 - 1/v$ from our normalization, maximizing v is equivalent to maximizing $\lambda_i^{[i]}$. Writing $v \cdot \Lambda_{[i]}^{[i]}$ simply as a variable vector q , thus $q_i \equiv v$, we have the linear programming problem (DG):

$$(\text{DG}) \max q_i \text{ subject to } q' \cdot B \leq q' \cdot A + b^{(i)} \text{ and } q' \in R_+^n$$

Finally, the values can be calculated exactly as in equations [6] and [7], *i.e.*,

$$\lambda_i^{[i]} = \frac{q_i^* - 1}{q_i^*} \text{ and } \lambda_j^{[i]} = \frac{q_j^*}{q_i^*} \text{ for } j = 1, \dots, n, j \neq i \quad [9]$$

It is not difficult to establish:

Proposition 2. Given the productiveness assumption (A3), the i -th labour value of i -th labour is less than unity.

Proof. Consider the linear programming problem

$$(\text{PG}) \min b^{(i)}x \text{ subject to } Bx \geq Ax + e_{[i]} \text{ and } x \in R_+^m$$

By the duality theorem, we know that the optimal values satisfy $q_i^* = b^{(i)}x^*$. On the other hand, it is clear from the constraint in (PG) that $b^{(i)}x^* \geq 1$. Therefore we get $q_i^* \geq 1$, which gives $0 \leq \lambda_i^{[i]} < 1$ because of equation [9]. QED.

Since various types of labour and commodities are treated in a symmetric way, we can define the j -th commodity values of goods and labour exactly as in the definition above, and establish:

Proposition 3. Given the assumption (A3) with $e_{[i]}$ replaced by $e_{[j]}$, the j -th commodity value of j -th commodity is less than unity.¹⁰

¹⁰ This proposition, when labour is homogeneous, is called ‘‘commodity exploitation theorem’’. See Bowles and Gintis (1981), Roemer (1982, 1986), and Fujimoto and Fujita (2008).

At the end of this section, let us briefly discuss a model in Morishima (1964) and that in Bowles and Gintis (1977). In Morishima's model, there is only one type of homogeneous labour, and a single household activity to reproduce this homogeneous labour, with no durable consumption goods involved. Thus, B and A are of the following special form:

$$B = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix}$$

where B and A are, respectively, the $(n-1) \times (m-1)$ output and input coefficient matrices of normal production processes. In this case, the problems (DG) designating labour as the numeraire coincides with Morishima's problem (DM) in Morishima (1974):

$$(DM) \max \lambda c \text{ subject to } \lambda B \leq \lambda A + \ell \text{ and } \lambda \in R_+^{n-1}$$

Therefore our model and definition of values include Morishima's as a special case.

When B is the identity matrix, the definition of values by Bowles and Gintis (1977) is again a special case of ours. They, however, regard propositions 2 and 3 above as the existence of exploitation. Each of them is merely an alternative expression of productiveness (A3) as is explained in Fujimoto and Fujita (2008). It is not difficult to recognize that the commodity values defined by Manresa *et al.* (1998) also form a special case of ours.

EXPLOITATION

Our definition of values and our method of calculating them are useful to generalize the Fundamental Marxian Theorem as presented in Morishima (1974). Our model is surely more general than von Neumann-Morishima model used by him, allowing for heterogeneous labour, a plural number of household activities to reproduce labour power of each type, and durable

consumption goods, to say nothing of proper joint production in ordinary production processes. We do not need such concepts as “warranted rate of profit” or “capacity growth rate” defined in Morishima (1974). Moreover we need to define the concept of exploitation in a careful way because in our model there are various types of workers and they may have properties which increase their total income.

We now suppose the n -th commodity is a type of labour, n -th labour for short, and make the assumption (A3) in the preceding section for the vector $e_{[n]}$. Let us suppose the household activity vectors to reproduce the n -th labour are normalized so that they appear like this.

$$\text{Output} \begin{pmatrix} d \\ 1 \end{pmatrix} \leftarrow \text{input} \begin{pmatrix} c \\ \alpha \end{pmatrix} \quad [10]$$

That is, each household activity produces one unit of n -th labour by use of α unit of its own n -th labour and a commodity (including other types of labour) basket c . We consider only those activities whose α satisfies $0 \leq \alpha < 1$. Since some commodity in the basket c can be durable, we have on the output side a vector d , which may not be the zero vector. Both c and d are a column $(n - 1)$ -vector. One more assumption we make is:

Assumption A4. Any optimal vector x^* for the problem (PG) uses at least one of the activities that reproduce the n -th labour.

We can prove:

Proposition 4. Let commodity i such that $i \neq n$, $c_i > 0$, be arbitrarily chosen. Given the assumptions A3 and A4, the i -th commodity value of n -th labour is always greater than the i -th commodity value of the net consumption basket $(c - d)$ for any household activity used in an optimal vector of the programming problem (PG).

Proof. Take an optimal vector q^* for the problem (DG). By the assumption (A4), we can find at least one pair of vectors, which is actually used, of

the type (10) among the household activities which reproduce the n -th labour. For these vectors, we have a strict equality in the constraint in (DG) because they are used. Thus, we get

$$q^{*'} \cdot \begin{pmatrix} d \\ 1 \end{pmatrix} = q^{*'} \cdot \begin{pmatrix} c \\ \alpha \end{pmatrix} + d_i \quad [11]$$

Divide both sides of this equality by q_i^* , and we obtain, because of equation [9]

$$\Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} d \\ 0 \end{pmatrix} + \lambda_n^{[i]} = \Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} + \alpha \cdot \lambda_n^{[i]} + (1 - \lambda_i^{[i]}) \cdot d_i \quad [12]$$

This equality yields

$$(1 - \alpha) \cdot \lambda_n^{[i]} = \Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} c - d \\ 0 \end{pmatrix} + (1 - \lambda_i^{[i]}) \cdot d_i > \Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} c - d \\ 0 \end{pmatrix}$$

because $c_i > 0$ and $\lambda_i^{[i]} < 1$ by proposition 3. In sum,

$$\lambda_n^{[i]} > \Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} c - d \\ 0 \end{pmatrix} \cdot \frac{1}{1 - \alpha} \geq \Lambda_{[i]}^{[i]} \cdot \begin{pmatrix} c - d \\ 0 \end{pmatrix}$$

This ends the proof. QED.

This proposition tells us that if workers of type n exchange one unit of their labour force for the basket c and leaves d after one period, they make a loss in terms of any commodity as the standard. Equation [11], when commodity i is in fact the n -th labour, *i.e.*, $i = n$, becomes

$$\Lambda_{[n]}^{[n]} \cdot \begin{pmatrix} d \\ 0 \end{pmatrix} + 1 = \Lambda_{[n]}^{[n]} \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} + \alpha + (1 - \lambda_n^{[n]})$$

Thus, because of proposition 2, we get

$$1 > \lambda_n^{[n]} = \Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix} + \alpha \geq \Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix} \quad [13]$$

From this we get:

Proposition 5. The n -th labour value of the net consumption basket $(c - d)$ for any household activity used in an optimal vector of the programming problem (PG) is less than unity.

Homogeneous labour

First, we define exploitation for the case where there exists single homogeneous labour. Let us denote by $p \equiv (p_1, p_2, \dots, p_n)$ the row n -vector of prices and wage rate. Specifically, we write w_n for the wage rate p_n , supposing the n -th commodity is the homogeneous labour. These prices and the wage rate are what we observe for a period under consideration, and are not necessarily some magnitudes which equilibrate certain markets. Let us denote by y_n the total income of a typical worker for this particular period: $y_n \equiv w_n \cdot L + u_n$, where L is the amount of labour supplied and u_n is the other incomes accruing to his/her properties or businesses as profits, both for a given period. Denote by $\omega_n \equiv y_n/L$, the income per unit of labour supplied. Clearly when $u_n = 0$, $\omega_n = w_n$. In this subsection, the net consumption basket $(c - d)$ is the one actually used.

Definition of exploitation: a worker is exploited when his ω_n satisfies

$$\omega_n < \frac{p \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix}}{\Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix}}$$

The meaning of this definition is like this. When the above inequality were an equality, we have

$$\frac{y_n}{p \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix}} \cdot \left(\Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix} \right) = L$$

The left term on the LHS shows how many baskets $(c - d)$ could be bought by spending the total income y_n . Hence the LHS as a whole means the amount of labour value which could be obtained in exchange for the total income y_n , which is the reward for participating the production system and supplying labour force, L . Thus the above equality can be regarded as guaranteeing an equal exchange. Our definition requires that the income per unit of labour supplied be less than this level.

In the literature, the FMT is often discussed for models where there is single homogeneous labour, workers have no property to earn extra income, *i.e.*, $u_n = 0$, and the wage rate is just enough to buy the consumption basket c with $d = 0$ due to the absence of durable consumption goods. Given these assumptions, *i.e.*, $\omega_n = w_n = pc$ and $d = 0$, workers are exploited based upon our definition above, simply because

$$\Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix} < 1$$

as is clear in the expression [13]. In other cases, together with the assumption $u_n = 0$, a kind of equilibrium is stipulated in a Leontief model, *e.g.*, a uniform positive rate of profit is prevailing among industries, whence follow the inequalities $p_j/w_n > \lambda_j^{[n]}$ for all $j \neq n$, *i.e.*, each wage-unit price is larger than its corresponding labour value, which is enough to prove the existence of exploitation in our sense, provided the vector $(c - d)$ is nonnegative and nonzero. Profits are appropriated by the capitalists who are latent in our economic system.

The phenomenon of exploitation can be looked at from a different viewpoint. Let us consider the output, and assume that in a given period the economy's performance is described by

$$B \cdot x = A \cdot x + f$$

where x is an m -column activity level vector, and f an n -column *net* output vector.¹¹ Both x and f are supposed to be nonnegative nonzero vectors, and in particular labour is not involved in f , *i.e.*, $f_n = 0$. Pre-multiplying by q the both sides of the output equation above, yields

$$q' \cdot B \cdot x = q' \cdot A \cdot x + q' \cdot f$$

On the other hand, post-multiplying by x both sides of the constraint in the programming problem (DG) with $i = n$, gives

$$q' \cdot B \cdot x \leq q' \cdot A \cdot x + b^{(n)} \cdot x$$

Thus,

$$q' \cdot f = q' \cdot (B - A) \cdot x \leq b^{(n)} \cdot x$$

Divide this inequality by q_n , and we get

$$\Lambda^{[n]} \cdot f \leq (1 - \lambda_n^{[n]}) \cdot b^{(n)} \cdot x = (1 - \lambda_n^{[n]}) \cdot a^{(n)} \cdot x \quad \text{or} \quad [14]$$

$$\Lambda^{[n]} \cdot f + \lambda_n^{[n]} \cdot b^{(n)} \cdot x \leq a^{(n)} \cdot x$$

where $a^{(i)}$ is the i -th row of A . This inequality means the following almost tautological statement.

Proposition 6. The total labour value of net output plus the labour value of reproduced labour cannot exceed the total labour utilized.

This proposition is valid even in the case of heterogeneous labour provided that no type of labour is involved in the net output vector f , and

¹¹ The concept of net here is for the system of complete or extended matrices.

that values are measured by a particular type of labour. This proposition 6 combined with our definition of exploitation implies

Proposition 7. Workers as a whole are exploited unless they have the income which would enable them to buy out the whole net output.

And thus

Proposition 8 (A Fundamental Marxian Theorem). In an economy of homogeneous labour, the working class is exploited when there exist profits not all of which accrue to the working class, and some or all profits are used to buy a portion of the total product.

This is the most general FMT among those presented so far in models of homogeneous labour. Our economy may be in disequilibrium, and workers may have properties which give them extra income.

Heterogeneous labour and exploitation among workers themselves

Let us return to the case of heterogeneous labour. If we employ the definition of exploitation for the case of homogeneous labour, propositions 6 and 7 dictate that each type of workers should have the exclusive right over the whole net output; otherwise, that type is exploited. Thus each type is exploited when they share the net output without monopolizing the entire net output. Therefore, in the first instance, we have a proposition similar to proposition 8.

Proposition 9 (A Generalized Fundamental Marxian Theorem). In an economy of heterogeneous labour, the working classes as a whole are exploited when there exist profits not all of which accrue to them.

In the case of heterogeneous labour, however, the net output is a joint product by various types of labour. So, while it is natural that all the net output goes to the workers as a whole as expressed in proposition 9, it is selfish for workers of some type to claim the whole net output, insisting

that its labour value is less or equal to their contribution to its production. Thus, there can be exploitation among workers themselves. To define such a concept, we need to devise out some fair division of the output.

Let us suppose there are k types of labour, $1 \leq k < m$. We continue to consider the output equation

$$B \cdot x = A \cdot x + f \quad [15]$$

where x and f are supposed to be nonnegative nonzero vectors. As in section 5.1, we assume no type of labour is appearing in the net output vector f .¹² Here we adopt a simple concept of “fairness” in the division of the net output among workers. The total labour force rendered by i -th labour in the production expressed by equation [15] is

$$b^{(i)} \cdot x = a^{(i)} \cdot x$$

It is important to note that the labour force which has contributed to the production of net output is at the maximum¹³

$$(1 - \lambda_i^{[i]}) \cdot b^{(i)} \cdot x = (1 - \lambda_i^{[i]}) \cdot a^{(i)} \cdot x$$

And so, the contribution ratio of i -th type to the total labour force is

$$r^{[i]} \equiv \frac{(1 - \lambda_i^{[i]}) \cdot b^{(i)} \cdot x}{\sum_{j \in T} (1 - \lambda_j^{[j]}) \cdot b^{(j)} \cdot x}$$

where T is the set of indices for all the types of labour, its cardinality being k . On the other hand, the ratio of what the i -type workers could buy to the total money value of the net output is

¹² Positive elements in f are not excess supply, but are bought by some people and put to extra consumption or investment, or may be exported abroad.

¹³ More precisely, it is $\Lambda^i \cdot f$. See equation [14] with n replaced by i .

$$s^{[i]} \equiv \frac{w_i \cdot b^{(i)} \cdot x + u_i - p \cdot \begin{pmatrix} c_i - d_i \\ 0 \end{pmatrix}}{p \cdot f}$$

where w_i is the wage rate for the i -type workers and u_i the other incomes accruing to individual properties of i -type workers. The net consumption basket $(c_i - d_i)$ is the one actually used for the type i . The numerator shows what we may call the net purchasing power of i -type workers for the net output, after deducting the necessary expenses to sustain their life.

It should be remembered that if our economy is decomposable, we consider each decomposed part separately, and if that decomposed part of the economy involves only one type of labour, we merely apply the definition of exploitation for the homogeneous case.

Definition of exploitation among workers: when our economy is indecomposable, workers of type i are said to be exploited when his/her wage rate w_i and the other income u_i are such that

$$s^{[i]} < r^{[i]}$$

In the above definition, the reader may think that we have given an equal weight to each type of labour. This is not so. Various types are differentiated by their wage rates as well as consumption baskets. In other words, an equal weight is used only for calculating the shares $s^{[i]}$ and $r^{[i]}$. Rich workers may lead a lavish life with little hesitation, and they may not want to be called “workers”: they prefer “managers” or “entrepreneurs”.

NUMERICAL EXAMPLES

In this section we give three numerical examples in which there are two types of labour with no joint production. These examples are meant to tell the reader how to apply our definitions in a concrete manner, though the examples are highly abstract.

Example 1

The first economy consists of hunters and fishermen, and the technological data are

$$B \equiv \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The first row stands for beavers, the second for fish, the third for hunters' labour, and the fourth fishermen's labour. The first column means the process operated by hunters, the second by fishermen, while the third is the household activity of hunters and the fourth is that of fishermen. Assumption A3 is satisfied, *e.g.*, by $x = (1, 1, 1.1, 1.1)'$, to say nothing of assumption 4. When the standard of value is hunters' labour, the values are calculated as

$$\Lambda^{[3]} = (1/2, 0, 1/2, 0)$$

Similarly when the standard is fishermen's labour, the values become

$$\Lambda^{[4]} = (0, 1/3, 0, 1/3)$$

Since $\lambda_3^{[3]} = 1/2$ and $\lambda_4^{[4]} = 1/3$, propositions 2 and 5 are verified.

If we choose beaver and fish as the standard of value, we get, respectively,

$$\Lambda^{[1]} = (1/2, 0, 1, 0), \text{ and}$$

$$\Lambda^{[2]} = (0, 1/3, 0, 1)$$

These values shows proposition 4 is valid in this example.

Now suppose $x = (1, 1, 1, 1)'$ and the quantity equation of the period is

$$B \cdot x = A \cdot x + f$$

where $f = (1, 2, 0, 0)'$. Proposition 6 can be verified using above values. Since the economy here is (completely) decomposable, we apply the definition of exploitation for the homogeneous case. Suppose the price vector for the normal commodities $p = (\$100, \$100)$. For a hunter operating the first process, the income is \$200. This hunter needs \$100 to reproduce one unit of hunter's labour. Spending the remaining \$100, he/she can buy out the net output of one beaver. Hence no exploitation is observed. In fact we have

$$\omega_n = \frac{200}{1} = 200; \frac{p \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix}}{\Lambda^{[n]} \cdot \begin{pmatrix} c-d \\ 0 \end{pmatrix}} = \frac{100}{1/2} = 200$$

Example 2

The second example economy is indecomposable, though it consists of the same commodities and the same types of labour as in the first one. The complete input-output matrices are:

$$B \equiv \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A \equiv \begin{pmatrix} 0 & 0 & 1 & 0.2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which satisfy assumptions 3 and 4. The labour values are computed respectively as

$$\Lambda^{[3]} = (1/2, 1/20, 11/20, 3/20) \text{ and}$$

$$\Lambda^{[4]} = (1/3, 1/3, 2/3, 2/5)$$

Suppose again $p = (\$100, \$100)$, $x = (1, 1, 1, 1)'$ and the quantity equation of the period is

$$B \cdot x = A \cdot x + f$$

where $f = (0.8, 1, 0, 0)'$. All the related propositions can be verified under these values. And so, we examine who exploits who. A hunter who runs the first process contributes to the net output as much as

$$1 - \lambda_3^{[3]} = 1 - \frac{11}{20} = 1 - \left(\frac{1}{2} + \frac{1}{20}\right) = \frac{9}{20}$$

while a fisherman does

$$1 - \lambda_4^{[4]} = 1 - \frac{2}{5} = 1 - \left(\frac{1}{3} \times 0.2 + \frac{1}{3}\right) = \frac{3}{5}$$

Then let us examine how much workers can spend on the net output. A hunter gets \$200 by operating the first process, uses \$200 in his/her household activity to reproduce one unit of labour, and no money is left to buy a portion of the net output. On the other hand, a fisherman can secure the surplus money \$180, which can exactly buy the whole net output. Thus we can judge, based on our definition, fishermen exploit hunters under these prices in spite of the fact a hunter requires more to survive than a fisherman. When the price vector $p = (\$100, \$46)$, exploitation does not exist between two types of workers. This magnitude \$46 can be obtained by solving the following equation for p_2 :

$$\frac{\$100 \times 2 - (\$100 + p_2)}{p_2 \times 3 - (\$100 \times 0.2 + p_2)} = \frac{3}{4} \left(= \frac{9/20}{3/5} \right)$$

Example 3

In the third example, there are fishermen and owners of fishing-nets. Two objects are fish and services given by fishing-nets. Fishermen do not have nets, and they have to borrow nets. We assume fishing nets last for ever. The complete input-output matrices are:

$$B \equiv \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A \equiv \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The first row stands for fish, the second for the service from fishing-net, the third for fishermen's labour, and the fourth is net-owners' labour. The first column means the process ran by fishermen, the second ran by net-owners, the third is the household activity of fishermen, and the fourth is that of net-owners. The values with the fishermen labour as the standard, and those with net-owners are respectively:

$$\Lambda^{[3]} = (1/5, 1/5, 1/5, 2/5) \text{ and}$$

$$\Lambda^{[4]} = (1/10, 1/2, 1/10, 1/5)$$

Suppose again $p = (\$100, \$100)$, $x = (1, 1, 1, 1)'$ and the quantity equation of the period is

$$B \cdot x = A \cdot x + f$$

where $f = (3, 1, 0, 0)'$. A fisherman who operates the first process contributes to the net output as much as

$$1 - \lambda_3^{[3]} = 1 - \frac{1}{5} = \frac{4}{5}$$

while a net-owner's is

$$1 - \lambda_4^{[4]} = 1 - \frac{1}{5} = \frac{4}{5}$$

A fisherman gets \$500 after subtracting the rental of nets by operating the first process, uses \$100 in his/her household activity to reproduce one unit of labour, and so \$400 is left to buy all the net output. On the other hand, a net-owner has no surplus money. Hence, fishermen exploit net-owners under these prices. When the price vector $p = (\$100, \$(2800/12)) \cong (\$100, \$233)$, exploitation does not exist between two types of workers. The balancing price, p_2 , can be found by the equation:

$$\frac{\$100 \times 6 - (\$100 + p_2)}{p_2 \times 2 - \$100 \times 2} = 1 (= \frac{4/5}{4/5})$$

When the coefficient $a_{42} = 1$ reduces to 0.1, *i.e.*, net-owners can render their service with less work, and at the same time the coefficient $b_{44} = 1$ also decreases to 0.1 so that its net output remains zero at $x = (1, 1, 1, 1)'$, then the values become

$$\Lambda^{[3]} \cong (1/5, 1/5, 1/5, 4) \text{ and}$$

$$\Lambda^{[4]} \cong (1/100, 1/20, 1/100, 1/5)$$

Workers' contributions to the net output change approximately to

$$1 - \lambda_3^{[3]} = 1 - \frac{1}{5} = \frac{4}{5} = 0.8 \text{ and}$$

$$(1 - \lambda_4^{[4]}) \times 0.1 = \frac{4}{5} \times 0.1 = 0.08$$

When the price vector $p \equiv (\$100, \$(2500/21) \equiv (\$100, \$119)$, no exploitation exists, and when the price of the service of fishing-net gets higher than \$119, the net-owners exploit the fishermen. It is noted that still the smaller is the coefficient a_{42} as well as b_{44} , the smaller is the balancing price of the service from fishing-nets (nearer to the lower limit \$100), and the net-owners are likely to be exploiters.

CONCLUDING REMARKS

In modern developed economies, it seems difficult for workers to carry out a revolution to put an end to the capitalist system. Marx and Engels might think, in the 19th century, the days of proletarian dictatorship would soon be realized. While the working class has expanded since then, it has been polarized with a greater inequality of incomes among its members. As capital is accumulated more, a still higher degree of division of labour emerges, bringing about a possible cause of conflict among workers. When division of labour was in a primitive stage, *e.g.*, as in Smith (1776), Marx could devise out abstract labour to commensurate labour contents of commodities in order to help the whole working classes to unite themselves. Workers have, however, been divided and conquered, at least so far, by the capitalist system. Division of labour was an impetus for capitalist systems, and now it seems to be a support for them. Children are today taught to compete with others, to work harder to get richer, money-fetishism, and suggested that fellow workers are after all rivals, not friends. This article has tried to explain a possible conflict within the working classes using a linear model of heterogeneous labour, and to show an economic reason why workers will find it difficult to unite themselves to change their economic system. Certainly the degree of exploitation among workers matters here. Our definitions of values and exploitation are still tentative, and many variants and extensions can be made. A weak point of our method is that it is totally ahistorical, capturing a snapshot of one particular period. For example, in reality, workers migrate among various types, not simply reproducing themselves. We believe, however, this paper

can be a useful starting line for further developments, just as d'Alembert's principle in mechanics is important to understand dynamics through static snapshots. Here in this concluding section, we make some remarks on our method.

In ex-socialist countries, various services were excluded from the national account. What we have done in this article is an extreme opposite of this practice. We have subsumed every service available as rendered by workers or property-owners. Therefore there needs not be an explicit profit rate as expressed by $(1 + r)$ in mark-up ratios. There arises, however, a problem of how to determine the inputs required for those services, especially household activities to reproduce labour force. Since we allow for alternative methods, we may include every activity observed.

When we discuss labour values of commodities, it is not important whether a given economy is in equilibrium or not. In the real world, all the economies are out of equilibrium, with many sufferings from a long stagnant depression. We should be able to discuss values and surplus values also in such a disequilibrium state. Labour values depends solely on technological data. Prices as well as wage rates do play a part when we consider exploitation, unequal exchanges of labour services for what they could buy in markets. There is no need to presuppose the existence of exploiters or capitalists. This may not be accepted by Marx (1867) who supposed the grand two classes in a capitalist system, relegating landowners. A capitalist system without capitalists might be the highest stage of capitalism: capital only matters. Based on what they receive as wages, salaries or remunerations, they may be exploited by the economic system or can actively exploit the system.¹⁴ For our results to be applicable to the real world, we have to take into consideration unequal exchanges among nations. Some suggestions are made in Fujimoto and Opocher (2007) on how to define values when countries are engaged in international trade.

¹⁴ In practice, remunerations are seemingly from surplus. They can, however, be regarded as payments to services rendered, thus as costs.

The reader may have been embarrassed by the net consumption vector ($c - d$) in the definition of exploitation. When we discuss whether the exchanges of labour for a consumption basket with a given economic system is equal or unequal, we should subtract the vector d because this is not consumed after all and remain in the system, though this d is belonging to a certain worker, and this property right is protected by the system. In this respect, our treatment is again ahistorical.

The reader may also have an idea that a normal commodity can be exploited because normal commodities and various types of labour are dealt with in a symmetric way. This symmetry is only in the sphere of technological relations. Workers can refuse to participate in production when they regard the economic system as intolerably “unfair”, while normal commodities do join the production as input obediently when workers thus operate.

It may also be desirable to define “unskilled labour” or “simple labour” using only value analysis. In Fujimoto and Opocher (2007), a method is presented. Besides, a method to distinguish necessities from luxuries is also mentioned.

Finally it is necessary to introduce the government to judge whether its functions are aggravating the degree of exploitation among the workers through fiscal policies. This could be a new addition to the cursed roles of the state described by Engels (1884).

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