Vojnotehnicki glasnik/Military Technical Courier
ISSN: 0042-8469
vojnotehnicki.glasnik@mod.gov.rs
University of Defence
Serbia

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Vojnotehnicki glasnik/Military Technical Courier, vol. 65, núm. 3, 2017, pp. 617-640
University of Defence

# CONSTRUCTIONS OF SQUARING THE CIRCLE, DOUBLING THE CUBE AND ANGLE TRISECTION

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http://dx.doi.org/10.5937/vojtehg65-13404

FIELD: Mathematics
ARTICLE TYPE: Original Scientific Paper
ARTICLE LANGUAGE: English

*Abstract:*

*The constructions of three classical Greek problems (squaring the circle, doubling the cube and angle trisection) using only a ruler and a compass are considered unsolvable. The aim of this article is to explain the original methods of construction of the above-mentioned problems, which is something new in geometry. For the construction of squaring the circle and doubling the cube the Thales' theorem of proportional lengths has been used, whereas the angle trisection relies on a rotation of the unit circle in the Cartesian coordinate system and the axioms of angle measurement. The constructions are not related to the precise drawing figures in practice, but the intention is to find a theoretical solution, by using a ruler and a compass, under the assumption that the above-mentioned instruments are perfectly precise.*

*Keywords: construction, squaring the circle, doubling the cube, angle trisection, coordinate system, unit circle, rotation, proportion.*

## Introduction

Three problems were proposed in the time of the ancient Greeks, between 600 and 450 BC. Even though the problems of squaring the circle, doubling the cube and angle trisection date back to Thales's times, it is not known who proposed them. Many Greeks from that period until 500 AD attempted to solve the problems using only Euclidean constructions, but without success. However, they did find a series of solutions using tools other than a straight edge and a compass which made a significant contribution to mathematics at the time.

No progress on the unsolved problems was made until 19th century when abstract algebra was developed and concluded that the three Greek problems cannot be solved. The arguments put forward to prove the unsolvability of squaring the circle, doubling the cube and angle trisection were the impossibility of constructing the square root of $\pi$, the cube root of 2 and the angle trisection of 60º, respectively (Courant, Robbins, 1973, pp.108-113). These individual cases prejudiced mathematicians against the unsolvability of the three Greek problems.

Squaring the circle is related to constructing a square with the same area as a given circle. Doubling the cube is the problem of determining the length of the sides of a cube whose volume is double that of a given cube. Angle trisecection concerns the construction of an angle equal to one third of a given arbitrary angle. The above-mentioned problems are allowed to be constructed using only a straightedge and a compass, i.e. using elementary Euclidean construction.

Through an original method based on pure geometry, the three problems have been solved. The work methodology is based on the problem-solving process, i.e. constructive task-solving, consisting of four parts: analysis (description of the construction), construction, proof and discussion.

A reader should use a straightedge (ruler), a compass and a sheet of paper to follow the procedure for solving the problems.

## The construction of squaring the circle

In everyday speech we may hear the expression “squaring the circle” used as a metaphor for trying to solve something impossible. The origin of the phrase is not familiar to many of those who resort to its usage in conversation, but it is widely known among mathematicians that it refers to a problem proposed by ancient Greeks, related to constructing a square with the same area as a given circle by using only a compass and a straightedge (ruler).

When the length $X = \sqrt{2}$ is constructed, one can notice that the construction of a side of a square, which meets the requirements of the problem, is similar to the construction of the length $X = \sqrt{2}$ .

### *Description of the length construction* $X = \sqrt{2}$

We will consider a line *p* to contain an arbitrary length AB (Fig. 1).

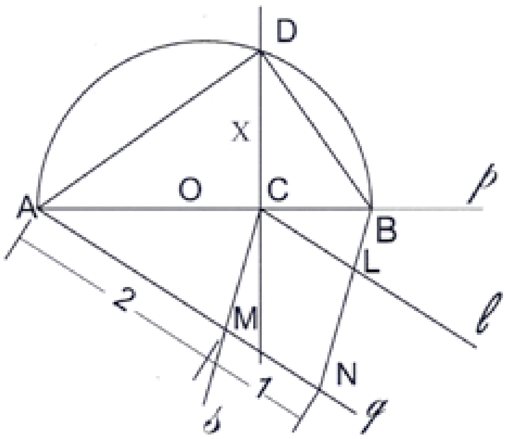


*Figure 1 – Division of the length AB in the ratio 1:2*
*Рис. 1 – Длина отрезка AB, в отношении 1:2*
*Слика 1 – Подјела дужи AB у односу 1:2*

The point C divides the given length in the ratio of integers 2:1, i.e. AC:CB = 2:1, in the following way: We construct an arbitrary ray A*q* and by a compass determine the points M and N so that the length AM=2, and the length MN=1. Then we construct the length NB. The line *s* passing through the point M is parallel to the length NB. We denote the intersection of the lines *s* and *p* by C. Then we construct the line *l* passing through the point C parallel to the ray A*q* and denote its intersection with the length NB by L. Let us prove that the length AB is divided by the point C in the ratio 2:1.

In Fig. 1, the triangles ACM and CBL are similar because the corresponding angles at the vertices A and C, i.e. M and L are as equal as angles with the parallel arms in the same direction.

The following proportion is true:

$$AC : AM = CB : CL \tag{1}$$

by replacing: AM = 2 and MN = CL = 1 in

we obtain: $AC : 2 = CB : 1 \Rightarrow AC : CB = 2 : 1$,

Q.E.D. (Quod erat demonstrandum).

Further, let us construct a semicircle on the length AB (Fig. 1).

With a compass and a straightedge, we construct the line *n* perpendicular to the length AB through the point C and denote its intersection with the semicircle by the point D. We construct the lengths AD and BD. Let us prove that the length CD equals the real number X= $\sqrt{2}$ .

The right-angled triangles ACD and DCB are similar, because the angles at the vertices A and D are as equal as angles with the perpendicular arms. The vertex angle D of the triangle ADB is right-angled because it is peripheral, whereas the straight angle BOA is 180° as the central angle, which is two times as great as the peripheral angle.

The following proportion is true:

AC : CD = CD : CB (2)

$CD^2 = AC \cdot CB$

if we replace

AC = 2 and CB = 1 in (2)

we obtain

$CD^2 = 2 \cdot 1$

$CD = \sqrt{2}$,

Q.E.D. (Quod erat demonstrandum).

In the 19th and 20th century, many mathematicians were trying to prove the unsolvability of squaring the circle using an algebraic method relying on the fact that $\sqrt{2}$ cannot be written as a fraction; that is why it is considered an approximate number. However, in geometry, as we have shown, $\sqrt{2}$ is the length, because there are *no approximate lengths.* Therefore, the value of $\sqrt{2}$ corresponds to the real number between 1 and 2, i.e. the relation is the following: $1<\sqrt{2}<2$. The relation can be proven in a classical, well-known way described below.

The Cartesian coordinate system is given (Fig. 2). In the first quadrant, we construct the square OABC whose side OA equals 1.

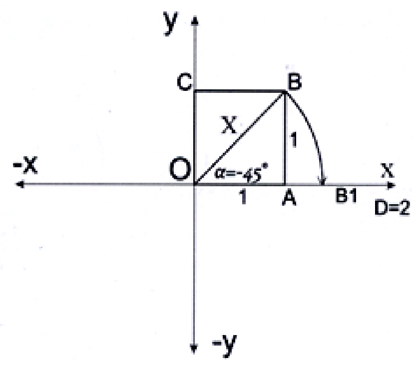


*Figure 2 – Value of the length X between real numbers 1 and 2*
*Рис. 2 – Значение длины X между действительными числами 1 и 2*
*Слика 2 – Вриједност дужи X између природних вриједности 1 и 2*

We denote the diagonal OB by X. According to the Pythagorean Theorem, the equation of the right-angled triangle OAB is the following:

$X^2 = 1^2 + 1^2 \Rightarrow X^2 = 2 \Rightarrow X = \sqrt{2}$

If we rotate the length OB around the point O as a centre of rotation by a negative angle $\alpha = -45^{\circ}$, the point B will be mapped onto the point $B_1$ which is situated between the points A and D on the axis O*x*. (Fig. 2) The length $OB_1$ corresponds to the real number and it is bigger than the length OA and smaller than the length OD.

"On the basis of Cantor's axiom which states that there is a one-to-one correspondence between real numbers and points on a line, every point on the real number line corresponds to a real number" (Dolićanin, 1984, p.62). It can be concluded that the real point $B_1$ is situated between integers 1 and 2.

*Squaring the circle using only a straightedge and a compass is possible*

Description of the construction:

A given circle[1] with the central point O and the radius *r* are denoted by *k*(O, *r*). The length AB is the diameter of an arbitrary circle *k* (Fig. 3). As shown by the previous method, when constructing the length $X=\sqrt{2}$, we divide the diameter AB by the point C in the ratio of integers 11000000 and 3005681, i.e. AC : CB = 11000000 : 3005681, in the following way:

On the arbitrary ray A*q*, we determine the point M by "transferring" 11000000 arbitrary unit lengths. Then we determine the point N so that the length MN equals 3005681 arbitrary unit lengths.

Then we construct the length NB. Through the point M, we draw a line *s* parallel to the length NB. The intersection of the line *s* and the length AB is denoted by C. Through the point C we construct the line *l* so that it is parallel to the ray A*q* and its intersection with the length NB we denote by the point L (Fig. 3). The length AB is divided in the above-mentioned ratio by the point C.

[1] Instead of a circle, in Figure 3 a semicircle is constructed for the sake of clarity.

Proof:

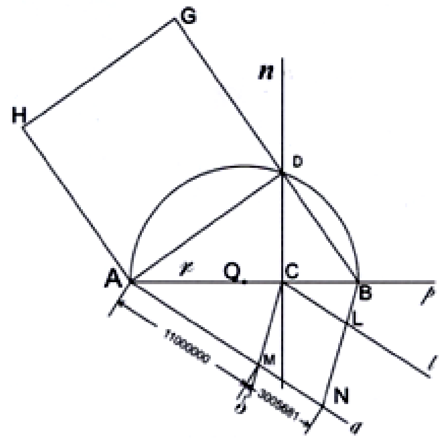


*Figure 3 – Proportion of the lengths AC and CB in the ratio 11000000 : 3005681*
*Рис. 3 – Пропорциональность длины отрезка AC и CB в отношении 11000000 : 3005681*
*Слика 3 – Пропорционалност дужи AC и CB у односу 11000000 : 3005681*

The triangles AMC and CLB are similar, so we can form the proportion:

AC : AM = CB : CL (3)

Based on relation (3), we replace:

$AM = 11 \cdot 10^6$ and

$MN = CL = 3005681 = 3.005681 \cdot 10^6$

It follows that

$AC : CB = 11 \cdot 10^6 : 3.005681 \cdot 10^6$

After having it shortened with $10^6$, we get:

$AC : CB = 11 : 3.005681 = t$ (4)

Based on relation (4)

$AC : 11 = t \Rightarrow AC = 11t$ and $CB : 3.005681 = t \Rightarrow CB = 3.005681t$, (5)

where $t$ is a non-negative real number, i.e. $t > 0$ and $t$ € R.

Let us construct a line *n* through the point C to be perpendicular to the diameter AB, and denote its (one) intersection with the periphery of the circle by D. Then we draw the lengths AD and BD. *AD represents the side of the square whose area is equal to the area of the given circle. Then we construct the square ADGH* (Fig. 3).

*Discussion:* The problem of squaring the circle always has two solutions, because the line *n* with the circle *k*(O,r), apart from the point D, has one more intersection point $D_1$ and thus the lengths AD and $AD_1$ are equal. We can construct one more square of the same area as the given circle. Thus the problem of squaring the circle has been proven solvable.

*Proof of squaring the circle by calculation*

By calculation, we shall now prove that the area of the given circle $k$(O, r) equals the area of the square ADGH (Fig. 3).

Radius $r(t) = (11t + 3.005681t) : 2$

$r(t) = 7.0028405 \cdot t$ (6)

r($t$) is a linear function whose graph (the part of line) belongs to the first quadrant and is defined for every $t > 0$.[2]

By using the formula to calculate the area of circle

$P = r^2\pi$ , $r = 7.0028405 \cdot t$,

we obtain the equation:

$P(t) = (7.0028405t)^2\pi \Rightarrow P(t) = 49.039775t^2 \cdot \pi$

$\Rightarrow P(t) = 154.062t^2$ (7)

$\Rightarrow P_0 = 154.06\, t^2$

P($t$) is a square function whose graph (part of the parabola) belongs to the first quadrant, and is defined for every $t > 0$.[3]

Now we shall calculate the area of the square ADGH.

The area of the square ADGH equals $AD^2$ (Fig. 3).

If we apply the Pythagorean Theorem on the right-angled triangle ACD in Fig. 3, we obtain the relation:

$AD^2 = AC^2 + CD^2$ (8)

$AC = 11t \Rightarrow AC^2 = 121t^2$ (9)

Based on the similarity of the triangles ACD and DCB in Fig. 3, the proportion is true:

$AC : CD = CD : CB$

$\Rightarrow AC^2 = AC \cdot CB$ (10)

if $AC = 11t \Rightarrow AC = 121t^2$

By replacing $AC = 11t$ and $CB = 3.005681t$ in (9), we obtain

$CD^2 = 11t \cdot 3.005681t$

$CD^2 = 33.062t^2$ (11)

---

[2] Drawing the function graph r($t$) is left to the reader.

[3] Drawing the function graph P($t$) is left to the reader.

If we replace relation (10) and (11) with relation (8), we get

$$AD^2 = 121t^2 + 33{,}062t^2$$
$$\Rightarrow P_\square = 154.06t^2 \qquad (12)$$

By comparing relations (7) and (12), one may notice that the area of the circle is equal to the area of the square, i.e.

$P_0 = P_\square$.

Thus the Greek problem of squaring the circle has been proven solvable by calculation.

## Constuction of doubling the cube (hexahedron)

The problem of doubling the cube relates to the construction of the edge of a second cube whose volume is double that of the first, using only a compass and a straightedge.

*Doubling of the cube using only a straightedge and a compass is possible*

*Description of the construction to determine the edges of the cube:*

On the arbitrary line *p,* we determine the points A and B so that the length AB is equal to the edge of the given cube, i.e. AB = a in Fig. 4.

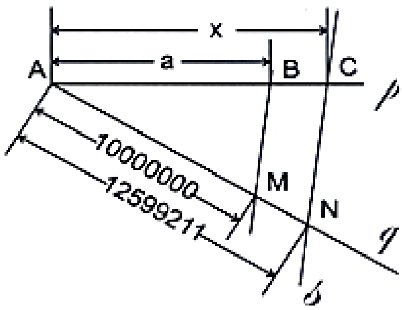


*Figure 4 – Proportion of the lengths AB and AC in the ratio 10000000 : 12599211*
*Рис. 4 – Пропорциональность длины отрезка AC и CB в отношении 10000000 : 12599211*
*Слика 4 – Пропорционалност дужи AB и AC у односу 10000000 : 12599211*

We will construct an arbitrary ray A*q* and determine the points M and N on the ray so that the length AM = $10^7$ arbitrary unit lengths, and the length AN=12599211 arbitrary unit lengths, i.e.
AM : AN = $10^7$ : 12599211.

Let us construct the length BM. Then we construct a line *s* through the point N parallel to the length BM. The intersection of the line *s* and the line *p* is denoted by C. We will prove that the length AC is the edge of the cube, whose volume is double that of the given cube.

Let AC equals x. The triangles ABM and ACN are similar because the angle at the vertex A is common, and the angles at the vertices B and C are as equal as angles with parallel arms in the same direction.
Based on the similarity of the triangles, the following proportion is true:

$$AB : AM = AC : AN \tag{13}$$

by replacing AB = a and AC = x, AM = $10^7$ and AN = $1.2599211 \cdot 10^7$ in (13) we obtain:

$$a : 10^7 = x : 1.2599211 \cdot 10^7. \tag{14}$$

After having it shortened with 107 in relation (14), we get

$$a : 1 = x : 1.2599211$$

it follows that

$$x = 1.1599211 \cdot a. \tag{15}$$

The cubed equation (15)

$$x^3 = (1.2599211 \cdot a)^3$$

$$x^3 = 1.2599211^3 \cdot a^3 \tag{16}$$

$$x^3 = 2 \cdot a^3$$

$$\text{If } a^3 = V_1 \text{ and } x^3 = V_2 \tag{17}$$

$$V_2 = 2 \cdot V_1$$

Q.E.D. (Quod erat demonstrandum)

### *Constructions of doubling the cube in oblique projection using a straightedge and a compass*

Description of the construction:
Let us construct the right-angled trihedral O*xyz*. (Fig. 5)

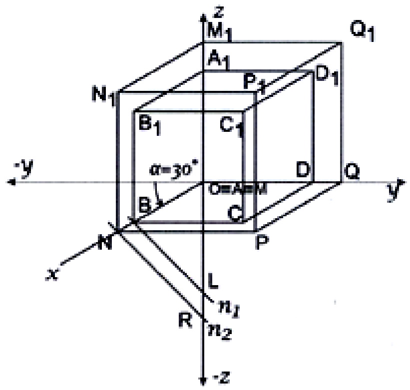


*Figure 5 – Oblique projection of a cube*
*Рис. 5 – Наклонная проекция куба*
*Слика 5 – Коса пројекција коцке*

For the oblique picture to be more transparent (clearer), we will take the angle α = 30º, and $q$ = 1 : 2 (α = 30º, angle –$y$O$x$).

The ratio $q$ = 1 : 2 represents the ratio of the length on the O$x$-axis and O-$z$-axes, i.e. if an arbitrary length on the O$x$-axis equals 1, then the length on the O$z$-axis is twice longer ($\alpha$ is called a reduction angle, $q$ is a reduction ratio).

On the negative part of the $z$-axis we determine a point L so that the length OL is equal to the edge of the cube in its true size (the length AB = a in Fig. 4). Through the point L we construct the line $n1$ perpendicular to O$x$ and we denote their intersection by B (Fig. 5).

The point D is determined on the ray Oy so that the length OD is equal to the edge of the given cube in Fig. 4 in its true size. The point A coincides with the vertex of the trihedral. Then, we determine the point C by constructing a parallelogram ABCD (it is the oblique picture of the lower base of the given cube).

Through the points B, C and D we construct lines parallel to the $z$-axis. On the $z$-axis and the parallel lines we determine the points $A_1B_1C_1D_1$ so that $AA_1 = BB_1 = CC_1 = DD_1 = a$. Let us construct other lengths where $ABCDA_1B_1C_1D_1$ presents an oblique picture of the given cube.

In a similar way, we construct a cube so that its volume is double that of the given cube, i.e. whose edge is $x$ (it is the length AC in Fig. 4). On the negative part of the $z$-axis in Fig. 5 we determine the point R so that the length OR equals the edge of the cube x = AC in Fig. 4. Through the point R we construct the line $n2$ perpendicular to the axis O$x$ and denote the intersection with the O$x$-axis by N. The point M coincides with the trihedral vertex. The point Q is determined on the O$y$ axis so that MQ = $x$ (in its true size). Let us construct a parallelogram MNPQ (it is the oblique picture of the lower base of the new cube). Through the points N, P, Q in Fig. 5 we construct lines parallel to the *z*-axis.

On the *z*-axis and all the parallel lines we construct the length *x* in its true size by a compass and we determine the points $M_1$, $N_1$, $P_1$ and $Q_1$ so that $MM_1=NN_1=PP_1=QQ_1=x$. (Fig. 4). The parallelogram $M_1N_1P_1Q_1$ *represents the oblique picture of the upper base of the new cube*, whereas $MNPQM_1N_1P_1Q_1$ *is the oblique picture of the new cube.*

In this way, we have constructed the cube $MNPQM_1N_1P_1Q_1$ whose volume is double that of the given cube $ABCDA_1B_1C_1D_1$.

*Discussion:* The above method has proven the solvability of doubling the cube using only a straightedge and a compass. The problem always has a solution, i.e. every cube can be doubled.

## The construction of angle trisection

Angle trisection is related to dividing an arbitrary angle into three equal parts in a constructive way using only a straightedge and a compass.

Dividing an angle into 3 equal parts does not seem to be a particular problem. For instance, it is easy to construct one-third of the angles of 45º, 67º 30', 90º, 135º, 180º, 202º 30', 270º, 360º, etc. using a straightedge and a compass. However, the general problem arises when an *arbitrary* angle should be divided into three equal parts.

In order to solve this problem, we present some wellknown geometry properties (axioms, theorems, definitions) which will be used here.

(1) Axiom on the measurement of angles: The degree measure of an angle equals the sum of degree measures of the angle divided by an arbitrary ray which passes through its arms.

(2) By convention, the rotation of an angle arm counter-clockwise is called positive rotation, whereas negative rotation goes clockwise. Positive rotation is denoted by $R(O, \alpha)$, negative by $R(O, -\alpha)$, where the point O is the centre of rotation and the oriented angle α is the angle of rotation.

(3) The base angles of an isosceles triangle are always equal.

(4) The exterior angle of a triangle equals the sum of two interior opposite non-adjacent angles.

(5) Vertically opposite angles are equal.

(6) Corresponding angles are equal in measure if and only if two parallel lines are cut by a transversal.

(7) Alternate angles are equal in measure if and only if two parallel lines are cut by a transversal.

(8) A circle is a centrally symmetric figure.

(9) A circle is an axially symmetric figure.

(10) The central angle (in a circle) is twice the size of the periphery angle which lies over the same arc of the circle.

(11) Given the circle with the centre at the origin of Cartesian coordinate system. Each chord constructed to be parallel to the coordinate axis cuts equal circular arcs on the given circle.

(12) In mathematics, a unit circle is a circle with a radius of one.

(13) An angle bisector is a ray that divides an angle into two equal angles.

(14) Each length can be divided into any (arbitrary) number of equal parts.

### *Angle trisection using a straightedge and a compass is possible*

Proof:

Given an acute angle with the vertex at point O and the arms *p* and *q* (Fig. 6).

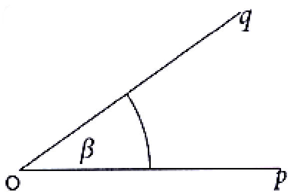


*Figure 6 – Model of the angle pOq*
*Рис. 6 – Модель угла pOq*
*Слика 6 – Модел угла pOq*

Let us construct the Cartesian coordinate system *x*O*y* so that the positive part of the *x*-axis corresponds to the arm *p* of the given angle (marked as $x \equiv p$). (Fig. 7)

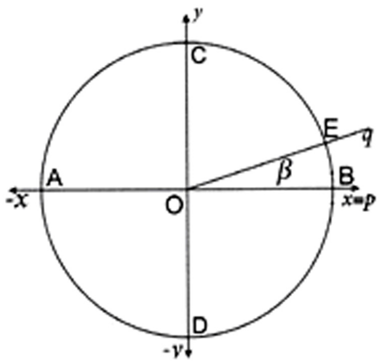


*Figure 7 – Unit circle with with the centre at the origin of coordinate system*
*Рис. 7 – Единичная окружность с центром в начале системы координат*
*Слика 7 – Јединична кружница са центром у координатном почетку*

Then we construct the circle $k$ with the centre O and a radius that equals one (in Figure 7 marked as $k$(O, 1)).

On the basis of property (12), this will hereinafter be referred to as the *unit* circle.

The intersections of the unit circle $k$(O,1) with the $x$-axis and the $y$-axis are denoted by A and B, and C and D, respectively.

The intersection of the unit circle $k$(O,1) and the arm $q$ is denoted by E (Fig. 7).

The problem arises when we want to divide the angle BOE into three equal parts in a constructive way using only a straightedge and a compass.

The first step is to divide the angle $\beta$ = 22º 30' into three equal parts. On the unit circle, we construct the angle β by a 45-degree bisector (Fig. 8).

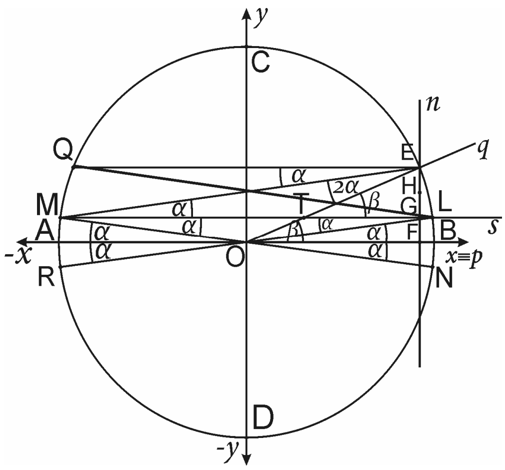


*Figure 8 – Angle trisection β = 22º 30'*
*Рис. 8 – Трисекция угла β = 22º 30'*
*Слика 8 – Трисекција угла β = 22º 30'*

Description of the construction:

Through the point E (Fig. 8) we draw a perpendicular $n$ to the $x$-axis and its intersection with the $x$-axis is denoted by F. Then, we divide the length FE into three equal parts and denote the points by G and H (Fig. 9)[4]

[4] The division of the length FE has been constructed separately for the sake of clarity in Figure 8.

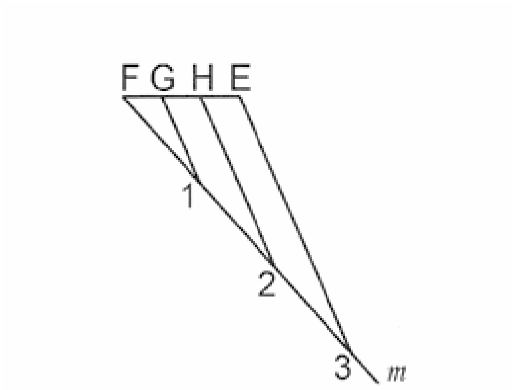


*Figure 9 – Division of the length FE for angle trisection of β = 22° 30'*
*Рис. 9 – Длина деления отрезка FE по трисекции угла β = 22° 30'*
*Слика 9 – Подјела дужи FE за трисекцију угла β = 22° 30'*

From the *x*-axis, the set of points on the length FE is F-G-H-E (Ostojić, 1980, p.164). Let us construct a line *s* passing through the point G parallel to the *x*-axis and denote its intersections with the unit circle *k* in the first quadrant by the point L and in the second quadrant by the point M. (Fig.8) According to (11), the circular arcs AM and BL are equal on the unit circle *k*(O, 1). Given ∢BOE = β, and BOL = α, we will prove that

BOL= $\frac{1}{3}$ ∢BOE, ie.

$$\alpha = \frac{\beta}{3}.$$

Based on (9), the point L is the symmetric point of M with respect to the *y*-axis. Let the point M be the symmetric point of N with respect to the origin of the Coordinate system based on (8) and the point L is the symmetric point of N with respect to the *x*-axis based on property (9). Further, if the point L is the symmetric point of R with respect to the origin, then the point M is the symmetric point of R with respect to the *x*-axis (Fig. 8).

Based on the properties of the circle as an axially symmetric and centrally symmetric figure, it follows that the circular arcs are equal, i.e. BL = BN = AM = AR. Therefore, on the unit circle, the central angles which lie over the equal circular arcs are equal. Based on (5), it follows that ∢BON = ∢AOM = α and ∢BOL = ∢AOR = α.

Applying property (10), the central angle NOL is twice the size of the periphery angle NML which lies over the arc of the circle NL, i.e. ∢NML = (2α) : 2 = α. Since the triangle OLM is an isosceles triangle (the base is the chord ML), then according to (3) ∢OLM = ∢OML = α.

Let the symmetric point E with respect to the *y*-axis be Q (Fig. 8). The circular arcs AQ and BE are equal in accordance with property (11). It was pointed out that circular arcs AM and BL are equal. Accordingly, the chords EQ and LM are parallel (marked as EQ || LM). Based on (7) $\sphericalangle$MEQ = $\sphericalangle$LME as alternate angles are equal and the line determined by M and E is called transversal and it follows that $\sphericalangle$MEQ = $\sphericalangle$LME = α.

Let us construct the length LQ. The chords ME and QL intersect at the point S. OMSL quadrilateral is a rhombus. The diagonal ML of the rhombus divides the angle OME into two equal parts. It follows that $\sphericalangle$OME = 2α. OME is an isosceles triangle and the angle OEM = 2α.

The angle NOE is central, and $\sphericalangle$NME peripheral over the same arc NE. Based on (10) it follows that:

$$\alpha + \beta = 2 \cdot 2\alpha \Rightarrow \alpha + \beta = 4\alpha$$

$$\Rightarrow \beta = 3\alpha$$

$$\Rightarrow \alpha = \frac{\beta}{3}$$

*Alternative proof:*

The intersection of the arm *q* and the line *s* is determined by the point T. According to (3), the triangle OEM is an isosceles triangle (the chord ME is the base of the triangle), so that $\sphericalangle$OME = 2α and $\sphericalangle$OEM = $\sphericalangle$TEM = 2α. The angle LME = $\sphericalangle$TME =α (because the points L and T belong to the line *s*).

The angle BOE = $\sphericalangle$LTE = *β*, as they are corresponding angles in accordance with property (6). Further, based on (4), the external angle of the triangle TME is equal to the sum of the two internal non-adjacent angles, i.e.

$$\sphericalangle LTE = \sphericalangle TEM + \sphericalangle TME$$

If we replace the angles with Greek letters ($\sphericalangle$ LTE = *β*, $\sphericalangle$TEM = 2α, $\sphericalangle$TME = α), we obtain

$$\beta = 2\alpha + \alpha,$$

$$\beta = 3\alpha,$$

$$\alpha = \frac{\beta}{3}$$

Q.E.D

The angle trisection of 22º 30’ is α = 7º 30’, and thus the angle trisection has been proven.

### *Angle trisection of β < 22º 30’*

Proof: Given the acute angle *β* less than 22º 30’ in Fig.10.

Description of the construction:

The procedure, denoting (marking) and proof of the angle trisection less than 22º 30’ are completely the same as the procedure, denoting (marking) and proof we explained when we constructed the angle trisection of $\beta$ = 22º 30’ (shown in Figs. 8 and 9). The construction of $\beta$ < 22º 30’ is shown in Figs. 10 and 11.

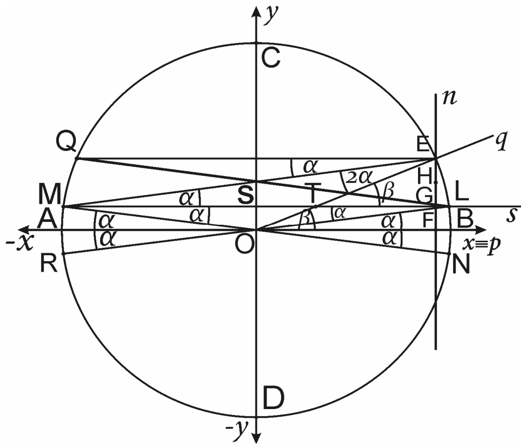


*Figure 10 – Angle trisection β < 22º 30’*
*Рис. 10 – Трисекция угла β < 22º 30’*
*Слика 10 –Трисекција угла β < 22º 30’*

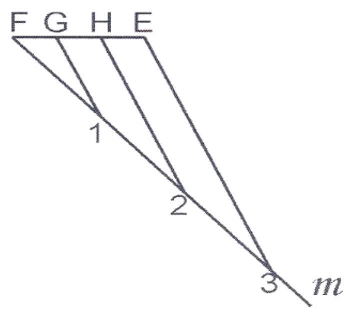


*Figure 11 – Division of the length FE for the angle trisection of β < 22º 30’*
*Рис. 11 – Длина деления отрезка FE по трисекции угла β < 22º 30’*
*Слика 11 – Подјела дужи FE за трисекцију угла β < 22º 30’*

Accordingly, if the acute angle *p*O*q* is given and if it belongs to the first quadrant, then we compare it to the angle of 22º 30’. Every acute angle may be: less than 22º 30’, less than 45º, less than 67º 30’ and less than 90º.

We shall explain the angle trisection for all the abovementioned cases:

a) The trisection of an angle less than 22º 30’ has been described in Figs. 10 and 11.
b) If the acute angle is less than 45º, then two angle trisections are performed – of 22º 30’ and of less than 22º 30’, because the given angle was divided into two parts by the ray. The angle of 22º 30’ is constructed using the angle bisector of 45º. In accordance with property 1, thirds of the circular arcs (which have been described in Figs. 8, 9, 10 and 11) are summed up and as a whole they represent the third of the angle.
c) If the acute angle is less than 67º 30’, then the angle is divided into two angles: one angle of 45º and the other of less than 22º 30’ by the ray. The angle trisection of 45º is performed first. The angle of 15º is obtained by constructing the angle bisector of 30º. The angle of less than 22º30’ is divided (as described in Figs. 10 and 11). The third of the circular arc of the angle of 45º and the third of the circular arc less than 22º 30’ are added up and as a whole represent the third of the given angle.
d) If the acute angle is less than 90º, then the angle is divided into one angle of 67º 30’ and the other angle less than 22º 30’ by the ray. Trisections of both angles are performed separately. The angle trisection of 67º 30’ is simple as it is the angle of 22º 30’ and it is obtained by the angle bisector of 45º. The third of the angle of less than 22º 30’has been described in Figs. 10 and 11. The sum of thirds of circular arcs, as a whole, is transferred to the circular arc of the given angle (three times) and thus the division of the unit circle into three equal parts has been performed, i.e. the trisection of an arbitrary acute angle.

*Arbitrary angle trisection*

1. The acute angle trisection has been described.

2. Given the obtuse angle β`= 90º + β (the arm p coincides with the positive part of the x-axis, and the arm q belongs to the second quadrant), Fig. 12. Then,

$$\frac{\beta'}{3} = 30^{\circ} + \frac{\beta}{3}.$$

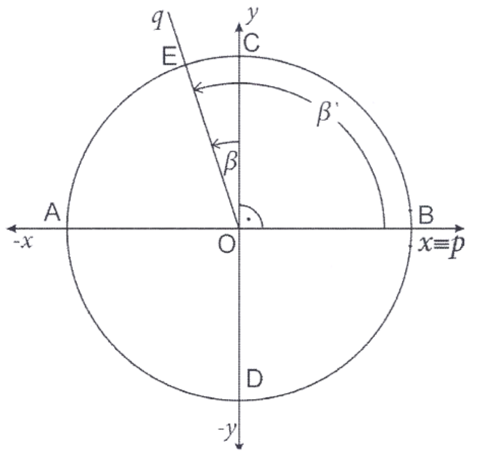


*Figure 12 – Arm q of β' angle belongs to the second quadrant*
*Рис. 12 – Луч q угла β' принадлежит второму квадранту*
*Слика 12 – Крак q угла β' припада другом квадранту*

Applying property (2) we perform a rotation of *β'* for the angle (-90º), or R(O, -90º). Then the arm *p* ≡ coincides with the negative part of the *y*-axis and the arm *q* belongs to the first quadrant. A rotation of the points A, B, C and D has also been performed. (Fig. 13)

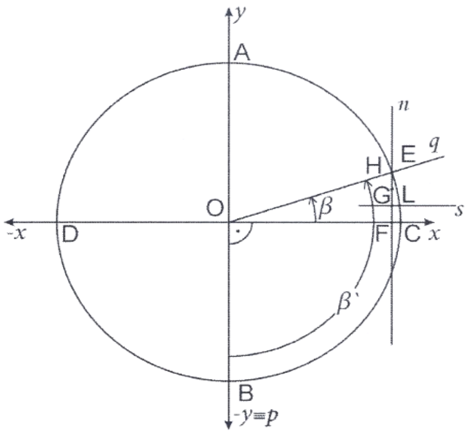


*Figure 13 – Potation of β' for the angle (-90º)*
*Рис. 13 – Поворот угла β' на (-90º)*
*Слика 13 – Ротација угла β' за (-90º)*

After the rotation of $\beta'$, the angle $\beta$ becomes acute and its trisection explained in the abovementioned cases for the acute angle can be applied (see a, b, c, d in 3.2.). The third of circular arc $\beta$ is added to the circular arc of 30º on the unit circle (Fig.14).

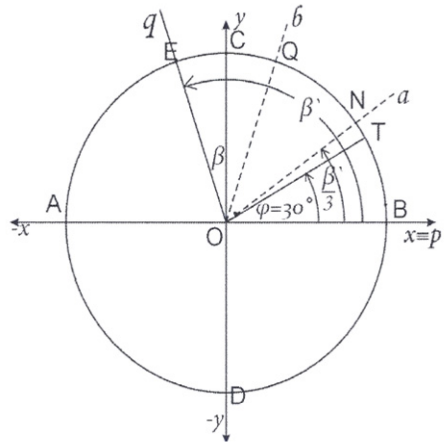


*Figure 14 – Angle trisection of β' when the arm q belongs to the second quadrant*
*Рис. 14 – Трисекция угла β' в случае, когда луч q угла β' принадлежит второму квадранту*
*Слика 14 – Трисекција угла β' када крак q угла β' припада другом квадранту*

The angle BOE is divided into three equal parts by the dashed lines *a* and *b*, and it follows ∢BON = ∢NOQ = ∢QOE = $\frac{\beta'}{3}$. The angle BOT equals 30º. (Fig. 14)

The circular arc TN in Fig. 14 equals the circular arc CL in Fig. 13. The sum of the circular arcs BT and TN is equal to the circular arc BN.

It follows that ∢BON = $\frac{1}{3}$∢ BOE. (The construction has been performed on the unit circle in Fig. 14).

3. If the angle *p*O*q* is less than 270º, i.e. if the arm q belongs to the third quadrant, then we write down $\beta$` =180º + $\beta$ (Fig. 15). The rotation of the angle $\beta'$ for the angle (-180º) is performed, i.e. R(O,-180º). Then the *p* is congruent to the negative part of the *x*-axis, while the *q* belongs to the first quadrant. The angle $\beta$ as the acute angle belongs to the first quadrant. (Fig. 16) The angle $\varphi$ is straight. (Fig. 15)
Then,

$\frac{\beta'}{3}$ = 60º + $\frac{\beta}{3}$.

It follows that the circular arc of 60º is added to the third of circular arc of the acute angle *β*.

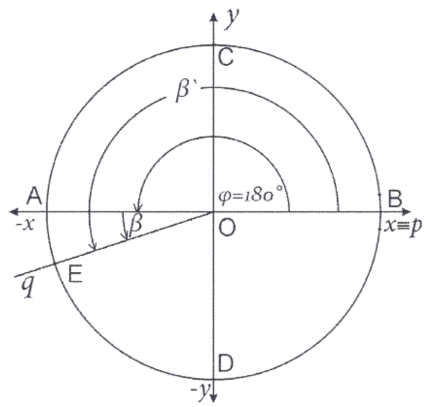


*Figure 15 – Arm q  of the β' angle belongs to the third quadrant*
*Рис. 15 – Луч q  угла β' принадлежит третьему квадранту*
*Слика 15 – Крак q  угла β' припада трећем квадранту*

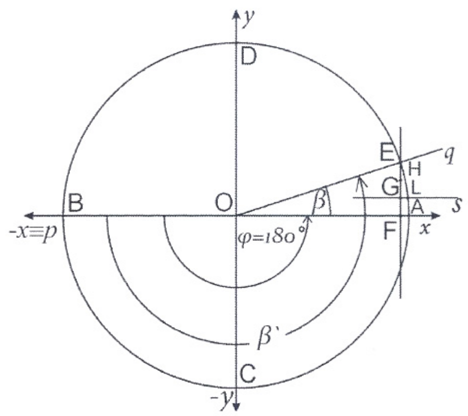


*Figure 16 – Potation of β' for the angle (-180º)*
*Рис. 16 – Поворот угла β' на (-180º)*
*Слика 16 – Ротација угла β' за (-180º)*

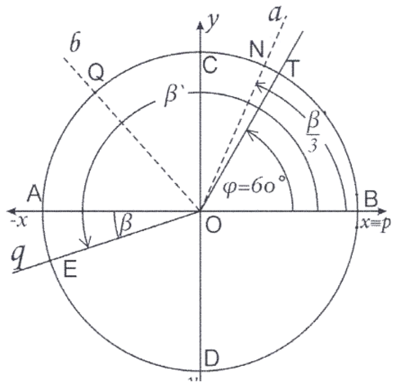


*Figure 17 – Angle trisection of β' when the arm q belongs to the third quadrant*
*Рис. 17 – Трисекция угла β' в случае, когда луч q угла β' принадлежит третьему квадранту*
*Слика 17 – Трисекција угла β' када крак q угла β' припада трећем квадранту*

The angle BOE is divided into three equal parts by the dashed lines *a* and *b* (in Fig. 17). The circular arc TN in Fig. 17 equals the circular arc AL in Fig.16.

The sum of the circular arc BT and the circular arc TN equals the circular arc BN (on the unit circle), or $\measuredangle$ BON = $\frac{1}{3}\measuredangle$ BOE in Fig. 17 where the angle BOT equals 60º.

4. Finally, if the angle *p*O*q* is less than 360º, i.e. if the arm *q* belongs to the fourth quadrant, $\beta$`= 270º + $\beta$ (Fig. 18), then we perform a rotation of the angle $\beta'$ for the angle of -270º, marked as R(O,-270º). The arm *p* coincides with the positive part of the *y*-axis and the arm *q* belongs to the first quadrant (in Fig. 19).

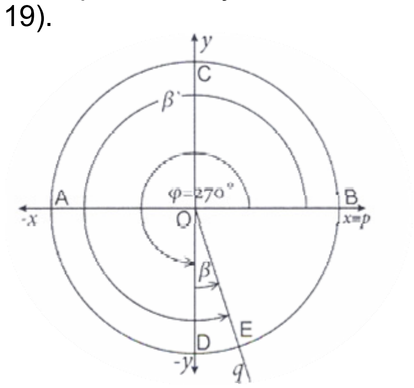


*Figure 18 – Arm q of the β' angle belongs to the fourth quadrant*
*Рис. 18 – Луч q угла β' принадлежит четвертому квадранту*
*Слика 18 – Крак q угла β' припада четвртом квадранту*

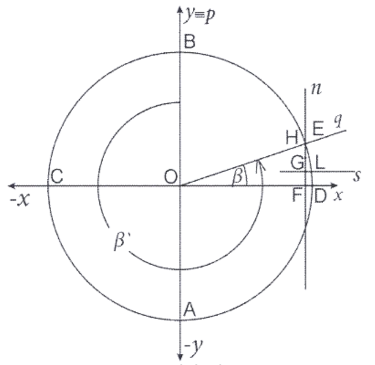


*Figure 19 – Potation of β' for the angle (-270º)*
*Рис. 19 – Поворот угла β'на (-270º)*
*Слика 19 – Ротација угла β' за (-270º)*

Since in this case $\frac{\beta'}{3}$ = 90º + $\frac{\beta}{3}$ the trisection of the acute angle $\beta$ (which is situated in the first quadrant after the rotation of the angle $\beta'$) is constructed in the same way as that described of the trisection of acute angles, then the third of the circular arc $\beta$ is added to the circular arc corresponding to the angle of 90º which belongs to the unit circle (Fig. 20).

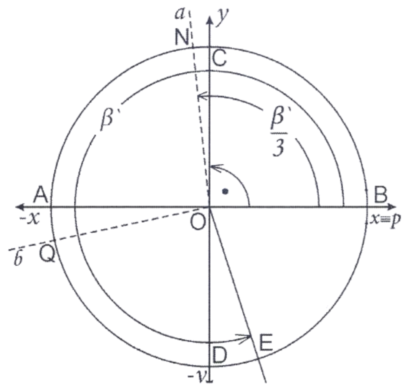


*Figure 17 – Angle trisection of β' when the arm q belongs to the fourth quadrant*
*Рис. 17– Трисекция угла β' в случае, когда луч q угла β' принадлежит четвертому квадранту*
*Слика 17 – Трисекција угла β' када крак q угла β' припада четвртом квадранту*

The angle BOE is divided into three equal parts by the dashed lines *a* and *b*. (Fig. 20) The angle BOC equals 90º. The circular arc CN in Fig. 20 equals the circular arc DL in Fig. 19.

It follows that $\measuredangle$ BON = $\frac{1}{3}$ $\measuredangle$ BOE.

*Discussion:* Applying the described method, the angle trisection can be performed for every angle. The diameter of the unit circle has been taken arbitrarily and cannot be changed while constructing without changing the lengths of circle arcs. Depending on the size of the angle, thirds of a circle arc less than 22º 30' on the unit circle can be added to a circle arc of 7º 30', 15º, 22º 30', 30º, 60º and 90º.

## *References*


Courant, R., Robbins, H., 1973. What is Mathematics? An Elementary Approach to Ideas and Methods. Oxford University Press, London.

Dolićanin, Ć., 1984. *Mathematics*. Institute for Textbooks and Educational Materials, Priština.

Ostojić, O., 1980. *Theoretical Basis of Initial Mathematics Teaching*, Handbook for Math Teachers, Republic Institute for the Improvement of Education, Titograd.


КОНСТРУКЦИИ КВАДРАТУРА КРУГА,
УДВОЕНИЕ КУБА И ТРИСЕКЦИЯ УГЛА


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ЯЗЫК СТАТЬИ: английский



*Резюме:*

*Построение трех классических античных задач (квадратура круга, удвоение куба и трисекция угла), с помощью линейки и циркуля до сих пор считается неразрешимым. В данной статье представлены оригинальные методы построения вышеупомянутых задач, что представляет собой инновационный подход в геометрии. Для построения квадратуры круга и удвоения куба была использована теорема Фалеса о пропорциональных отрезках, а трисекция угла основанная на повороте единичного круга в декартовой системе координат и аксиомах измерения угла.*

*Данные построения не претендуют на создание точных чертежей на практике, так как главная наша цель заключается в нахождении теоретического решения, основанного на построении данных фигур с помощью линейки и циркуля, с учетом предположения, что вышеупомянутые инструменты являются совершенно точными.*



*Ключевые слова: построение, квадратура круга, удвоение куба, трисекция угла, система координат, единичный круг, вращение, пропорция.*

# КОНСТРУКЦИЈЕ КВАДРАТУРЕ КРУГА, УДВАЈАЊА КОЦКЕ И ТРИСЕКЦИЈЕ УГЛА

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ЈЕЗИК ЧЛАНКА: енглески

*Сажетак:*

*Конструкције три класична грчка проблема (квадратура круга, удавајање коцке и трисекција угла), уз употребу само лењира и шестара, до данас се сматрају нерешивим. Циљ овог чланка јесте да се оригиналним методама објасне конструкције поменутих проблема, што представља новину у геометрији. За конструкцију квадратуре круга и удвајање коцке коришћена је Талесова теорема о пропорционалним дужима, а за трисекцију угла ротација јединичне кружнице у правоуглом координатном систему и аксиоме о мјерењу угла. Констукције се не односе на прецизно цртање фигура у пракси, већ је намјера да се употребом лењира и шестара нађе теоријско рјешење под претпоставком да су поменути инструменти савршено прецизни.*

*Кључне ријечи: конструкција, квадратура круга, удвајање коцке, трисекција угла, координатни систем, јединична кружница, ротација, пропорција.*

Paper received on / Дата получения работы / Датум пријема чланка: 08.03.2017.
Manuscript corrections submitted on / Дата получения исправленной версии работы / Датум достављања исправки рукописа: 24.03.2017.
Paper accepted for publishing on / Дата окончательного согласования работы / Датум коначног прихватања чланка за објављивање: 26.03.2017.