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REVIEW PAPERS

EXPLICIT EXPRESSIONS OF THE GENERALIZED STIELTJES POLYNOMIAL

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Abstract:

The existence and uniqueness of a Kronrod type extension to the well-known Gauss-Turan quadrature formulas were proved by Li (1994, pp.71-83). For the generalized Chebyshev weight functions and for the Gori-Micchelli weight function, we found explicit formulas of the corresponding generalized Stieltjes polynomials. General real Kronrod extensions of the Gaussian quadrature formulas with multiple nodes are introduced. In some cases, the explicit expressions of the polynomials, whose zeros are the nodes of the considered quadratures, are determined.

Keywords: Stieltjes polynomials, Kronrod extension, Gori-Micchelli weight function.

Introduction

Let ω be an integrable weight function on the interval (-1,1). It is wellknown that the Gauss-Turan quadrature formula with multiple nodes

$$\int_{-1}^{1}\omega(t)f(t)dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s}A_{i,\nu}f^{(i)}(\tau_{\nu})+E_{n,s}(f)\ (n\in N;s\in N_0) \quad (1)$$

is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, and that its nodes τ_ν are the zeros of the corresponding (monic) s-ortogonal polynomial $\pi_{n,s}(t)$ of degree n which minimizes the following integral $\phi(a_0, a_1, ...a_{n-1}) = \int_{-1}^{1} \pi_n(t)^{2s+2} \omega(t)dt$, where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0$. In order to minimize ϕ, we must have

$$\int_{-1}^{1} \omega(t)\pi_n(t)^{2s+1} t^k dt = 0,\ k = 0,1,...,n-1. \tag{2}$$

which are the corresponding orthogonality relations. For s=0, we have a case of the standard orthogonal polynomials.

Following the well-known idea of Kronrod (Gautschi, Milovanović, 1988, pp.16-18), S. Li proposed to extend formula (1) to the formula (Li, 1994, pp.71-83):

$$\int_{-1}^{1} \omega(t)f(t)dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} \sigma_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1} K_\mu f(\hat{\tau}_\mu) + R_{n,s}(f), \tag{3}$$

where τ_ν are the same nodes as in (1), and the new nodes $\hat{\tau}_\nu$ and new weights $\sigma_{i,\nu}, K_\mu$ are chosen to maximize the degree of exactness of (3). It is shown in (Li, 1994, pp.71-83) that we can always obtain the maximum degree $2(s+1)n + n + 1$ by taking $\hat{\tau}_\nu$ to be the zeros of the polynomial $\hat{\pi}_{n+1}$, which we call the generalized Stieltjes polynomial, satisfying the orthogonality property $\int_{-1}^{1} \omega(t)\hat{\pi}_{n+1}(t)p(t)\pi_n(t)^{2s+1} dt = 0$, all $p \in P_n$.

At the same time, Li showed that $\hat{\pi}_{n+1}$ always exists and is unique if it is monic. In the special case when $\omega(t) = (1-t^2)^{-1/2}$, he determined $\hat{\pi}_{n+1}$ explicitly and obtained the weights in (3) for $s = 1$ and $s = 2$.

Consider the following four generalized Chebyshev weight functions $\omega(t) = \omega_i(t)$: $\omega_1(t) = (1-t^2)^{-1/2}$, $\omega_2(t) = (1-t^2)^{1/2+s}$, $\omega_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, $\omega_4(t) = (1-t)^{1/2+s}(1+t)^{1/2}$.

Bernstein (1930, pp.127-177) showed that the monic Chebyshev polynomial (orthogonal with respect to $\omega_1(t)T_n(t)/2^{n-1}$) minimized all integrals of the form $\int_{-1}^{1}\frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}dt$ $(k \geq 0)$.

This means that the Chebyshev polynomials T_n are $s-$orthogonal on (-1,1) for each $s \geq 0$. Ossicini and Rosati (1975, pp.224-237) found three other weight functions $\omega_i(t)(i=2,3,4)$ for which the $s-$orthogonal polynomials can be identified as the Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, W_n, which are defined by

$$U_n(t)=\frac{\sin(n+1)\theta}{\sin\theta},\quad V_n(t)=\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos(\theta/2)},\quad W_n(t)=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin(\theta/2)},$$

respectively, where $t=\cos\theta$. It is easy to see that $W_n(-t)=(-1)^n V_n(t)$.

Explicit expressions of the generalized Stieltjes polynomials

For an arbitrary integrable weight function $\omega(t)$ on [-1,1], Li proved that the generalized Stieltjes polynomial $\hat{\pi}_{n+1}$ exists and is unique up to a constant factor. He considered the case when $\omega(t)=\omega_1(t)$, see (Galjak, 2006). In this case, it is known that $\pi_{n,s}(t)=T_n(t)/2^{n-1}$. Li obtained that

$\hat{\pi}_2(t)=\frac{1}{2}\left(T_2(t)-\frac{s+1}{s+2}T_0(t)\right)$, and for $n \geq 2$

$$\hat{\pi}_{n+1}(t)=\frac{1}{2^n}\left(T_{n+1}(t)-T_{n-1}(t)\right)=\frac{1}{2^{n-1}}\left(t^2-1\right)U_{n-1}(t).$$

Let first $\omega(t)$ be $\omega_2(t)$. In this case, it is known that $\pi_{n,s}(t)=U_n(t)/2^n$. We have just proved the previous statement (Milovanović, Spalević, 2006, pp.171-195), (Milovanović et al, 2006b, pp.22-28) and (Milovanović et al, 2009, pp. 246-250).

Theorem 1. Let $\hat{\pi}_{n+1}$ be the monic polynomial of degree $n+1$ satisfying the orthogonality relation

$$\int_{-1}^{1}\left(1-t^2\right)^{1/2+s}\hat{\pi}_{n+1}(t)p(t)\pi_n(t)^{2s+1}dt=0, \text{ all } p\in P_n. \tag{4}$$

Then

$$\hat{\pi}_{n+1}(t)=\frac{1}{2^n}T_{n+1}(t). \tag{5}$$

Proof: In this case, orthogonality conditions (4) have the form

$$\int_{-1}^{1}\left(1-t^2\right)^{1/2+s}\hat{\pi}_{n+1}(t)t^k\left[U_n(t)\right]^{2s+1}dt=0,\ k=0,1,\ldots,n. \tag{6}$$

We have

$$\left(1-t^2\right)^s\left[U_n(t)\right]^{2s+1}=\sum_{j=0}^{s}\beta_jU_{n(2j+1)+2j}(t), \tag{7}$$

Where $\beta_j=2^{-2s}(-1)^j\binom{2s+1}{s-j}$.

Conditions (4) can be written in the form

$$\int_{-1}^{1}\sqrt{1-t^2}\,\hat{\pi}_{n+1}(t)t^k\left(1-t^2\right)^s\left[U_n(t)\right]^{2s+1}dt=0,\ k=0,1,\ldots,n. \tag{8}$$

By using (7), the last conditions (8) obtain the form

$$\sum_{j=0}^{s}\beta_j\int_{-1}^{1}\sqrt{1-t^2}\,\hat{\pi}_{n+1}(t)t^kU_{n(2j+1)+2j}(t)dt=0,\qquad k=0,1,\ldots,n. \tag{9}$$

Let $\hat{\pi}_{n+1}(t)$ be $T_{n+1}(t)/2^n$. By using (Monegato, 1982, pp.137-158) $2T_n(t)U_{n-1}(t)=U_{2n-1}(t)$ the integral under the sum in (9) for $j=0$ has the form $\int_{-1}^{1}\sqrt{1-t^2}T_{n+1}(t)t^kU_n(t)dt=\frac{1}{2}\int_{-1}^{1}\sqrt{1-t^2}t^kU_{2n+1}(t)dt$, and it is equal to 0 for $k=0,1,\ldots,2n.$

For $j=1$, the integral under the sum in (9) has the form $\int_{-1}^{1}\sqrt{1-t^2}T_{n+1}(t)t^kU_{3n+2}(t)dt$, and it is equal to 0 for $k=0,1,\ldots,n$, if $3n+2>2n+1$, which is always fulfilled. As for $j=1$, the same conclusions for $j\geq 2$ are obtained. Therefore, conditions (9) are fulfilled. Finally, (5) holds because of the uniqueness of the generalized Stieltjes polynomial.

Let now $\omega(t)=\omega_3(t)$ and we have just proved the previous statement (Galjak, 2006), (Milovanović, Spalević, 2006, pp.171-195), (Milovanović, Spalević, 2003, pp.1855-1873).

Theorem 2. Let $\hat{\pi}_{n+1}(t)$ be the monic polynomial of degree $n+1$ satisfying the orthogonality relation

$$\int_{-1}^{1}(1-t)^{-1/2}(1+t)^{1/2+s}\hat{\pi}_{n+1}(t)p(t)\pi_n(t)^{2s+1}dt=0,\text{ all }p\in P_n. \quad (10)$$

Then

$$\hat{\pi}_{n+1}(t)=\frac{2^n(n!)^2}{(2n)!}(t-1)P_n^{(1/2,-1/2)}(t), \quad (11)$$

where $P_n^{(1/2,-1/2)}(t)$ is the orthogonal polynomial with respect to the weight function $\omega(t)=\sqrt{\dfrac{1-t}{1+t}}$.

Proof: In this case, it is known that the corresponding monic $s-$orthogonal polynomial of degree n is

$\pi_{n,s}(t)=\dfrac{1}{2^n}V_n(t)=\dfrac{2^n(n!)^2}{(2n)!}P_n^{(-1/2,1/2)}(t)$, where $P_n^{(-1/2,1/2)}(t)$ is the ordinary orthogonal polynomial with respect to the weight function $\omega(t)=\sqrt{\dfrac{1+t}{1-t}}$. In this case, orthogonality conditions (10) have the form

$$\int_{-1}^{1}(1-t)^{-1/2}(1+t)^{1/2+s}\hat{\pi}_{n+1}(t)t^k\left[P_n^{(-1/2,1/2)}(t)\right]^{2s+1}dt=0,\quad\text{for } k=0,1,\ldots,n. \quad (12)$$

We have (Ossicini, Rosati, 1975, pp.224-237)

$$(1+t)^s\left[P_n^{(-1/2,1/2)}(t)\right]^{2s+1}=\sum_{j=0}^{s}\gamma_j P_{n(2j+1)+j}^{(-1/2,1/2)}(t), \quad (13)$$

where $\gamma_j=2^{-s}\dfrac{c_n^{2s+1}}{c_{n(2j+1)+j}}\dbinom{2s+1}{s-j}$ and $c_n=(2n)!/\left(2^{2n}(n!)^2\right)$.

The above conditions (12) can be written in the form

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\hat{\pi}_{n+1}(t)t^k(1+t)^s\left[P_n^{(-1/2,1/2)}(t)\right]^{2s+1}dt=0,\text{ for } k=0,1,\ldots,n. \quad (14)$$

By using (13), the last conditions (14) obtain the form

$$\sum_{j=0}^{s}\gamma_j\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\hat{\pi}_{n+1}(t)t^k P_{n(2j+1)+j}^{(-1/2,1/2)}(t)dt=0,\ k=0,1,...,n. \tag{15}$$

Let $\hat{\pi}_{n+1}(t)$ be $\frac{2^n(n!)^2}{(2n)!}(t-1)P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t)$. By using $P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t)P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t)=k_n^2U_{2n}(x)$, where $k_n=((2n-1)!!/(2n)!!)$ (Monegato, 1982, pp.137-158), the integral under the sum in (15) for $j=0$ has the form

$$\int_{-1}^{1}\sqrt{1-t^2}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t)P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t)t^k dt=k_n^2\int_{-1}^{1}\sqrt{1-t^2}U_{2n}(t)t^k dt,$$

and it is equal to 0 for $k=0,1,...,2n-1$. For $j=1,2,...,s,$ the integrals under the sum in (15) have the form

$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}(1-t)t^k P_n^{(1/2,-1/2)}(t)P_{n(2j+1)+j}^{(-1/2,1/2)}(t)t^k dt,$ and they are equal to 0 for $k=0,1,...,n,$ if $2n+1<3n+1\le n(2j+1)+j$, which is always fulfilled. Using these conclusions, we have that conditions (15) are fulfilled. Finally, (11) holds because of the uniqueness of the generalized Stieltjes polynomial.

Let ω be an integrable weight function on the interval (a,b). Take now a sequence of nonnegative integers $\sigma=(s_1,s_2,...)$. For any $n\in N$, we denote the corresponding finite sequence $(s_1,s_2,...,s_n)$ by σ_n and consider a generalization of the Gaurss-Turan quadrature formula (1) to rules having nodes with arbitrary multiplicities

$$\int_a^b\omega(t)f(t)dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}A_{i,\nu}f^{(i)}(\tau_\nu)+\overline{R}(f), \tag{16}$$

Where $A_{i,\nu}=A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu=\tau_\nu^{(n,\sigma)}$ $(i=0,1,...,2s_\nu;\nu=1,...,n)$. Such formulas were derived independently by Chakalov and Popoviciu. A significant theoretical progress in this subject was made by Stancu see (Milovanović, 2001, pp.267-286).

In this case, it is important to assume that the nodes $\tau_\nu=\tau_\nu^{(n,\sigma)}$ are ordered, say

$$\tau_1<\tau_2<...<\tau_n,\ \tau_\nu\in[a,b], \tag{17}$$

with odd multiplicities $2s_1+1, 2s_2+1, ..., 2s_n+1,$ respectively, in order to have the uniqueness of the Chakalov-Popoviciu quadrature formula (16) (Karlin, Pinkus, 1976, pp.113-141). Then, this quadrature formula has the maximum degree of exactness $d_{\max}=2\sum_{\nu=1}^{n}s_\nu+2n-1$ if and only if

$$\int_a^b\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}t^k\omega(t)dt=0,\qquad k=0,1,...,n-1.$$

The existence of such quadrature rules was proved by Chakalov, Popoviciu, Morelli and Verna, and the existence and uniqueness subject to (17) by Ghizzetti and Ossicini (Ghizzetti, Ossicini, 1995), (Milovanović, 2001, pp.267-286), and also by (Milovanović, Spalević, 2002, pp.619-637). Conditions (17) define a sequence of polynomials

$$\{\pi_{n,\sigma}\}_{n\in N_0}\ \pi_{n,\sigma}(t)=\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right),\ \tau_1^{(n,\sigma)}<\tau_2^{(n,\sigma)}<...<\tau_n^{(n,\sigma)},\ \tau_\nu^{(n,\sigma)}\in[a,b],$$

such that $\int_a^b\pi_{k,\sigma}(t)\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right)^{2s_\nu+1}\omega(t)dt=0,\ k=0,1,...,n-1.$

These polynomials are called σ- orthogonal polynomials and they correspond to the sequence $\sigma=(s_1,s_2,...)$. We will often write simple τ_ν instead of $\tau_\nu^{(n,\sigma)}$. If we have $\sigma=(s,s,...)$, the above polynomials reduce to the s - orthogonal polynomials.

General Kronrod extensions of the Chakalov-Popoviciu quadratures

Let $\sigma_m^*=(s_1^*,s_2^*,...,s_m^*)\ (s_\mu^*\in N_0,\mu=1,2,...,m)$. Following the well-known idea of Kronrod (Gautschi, Li, 1990, pp.315-329), we extend formula (16) to the interpolatory quadrature formula

$$\int_a^b f(t)\omega(t)dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}B_{i,\nu}f^{(i)}(\tau_\nu)+\sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*}C_{j,\mu}^*f^{(j)}(\tau_\mu^*)+R_{n,m}(f),\tag{18}$$

where τ_ν are the same nodes as in (16), and the new nodes τ_μ^* and new weights $B_{i,\nu},C_{j,\mu}^*$ are chosen to maximize the degree of exactness of (18) which is greater than or equal to

$$\sum_{\nu=1}^{n}(2s_\nu+1)+\sum_{\mu=1}^{m}(2s_\mu^*+1)+m-1=2\left(\sum_{\nu=1}^{n}s_\nu+\sum_{\mu=1}^{m}s_\mu^*\right)+n+2m-1.$$

We called quadrature formula (18) the Chakalov-Popoviciu-Kronrod quadrature formula. The particular case is the Gauss-Turan-Kronrod quadrature formula, if $s_1=s_2=...=s_n=s$. In the theory of the Gauss-Kronrod quadrature formulas, the important role is played by the Stieltjes polynomials $E_{n+1}(x)$ whose zeros are the nodes τ_μ^*, namely $E_{n+1}(x)=\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)$.

Gori and Micchelli (Li, 1994, pp.71-83) have introduced for each n an interesting class of weight functions. Consider a subclass of the Gori-Micchelli weight functions,

$$\omega_{n,l}(t)=\left[\frac{U_{n-1}(t)}{n}\right]^{2l}(1-t^2)^{l-1/2},\ l\in\{0,1,...,s\}. \tag{19}$$

In the particular case $l=0$, (19) reduced to the Chebyshev weight function of the first kind $\omega_{n,0}=(1-t^2)^{-1/2}$. In this case, $(n\geq 2,\sigma_{n+1}^*=((s-l)/2,s-l,...,s-l,(s-l)/2))$ when quadrature formula (18) has the form $(\tau_1^*=-1,\tau_{n+1}^*=1)$

$$\int_{-1}^{1}f(t)\omega_{n,l}(t)dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s}B_{i,\nu}f^{(i)}(\tau_\nu)+\sum_{\mu=2}^{n}\sum_{j=0}^{2(s-l)}C_{j,\mu}^*f^{(j)}(\tau_\mu^*)+\sum_{j=0}^{s-l}\left(C_{j,1}^*f^{(j)}(-1)+C_{j,n+1}^*f^{(j)}(1)\right)+R_n(f) \tag{20}$$

Theorem 3. In the Kronrod extension (20) of the Gauss-Turan quadrature formula (16) with the weight function (19), and for $n\geq 2$, the corresponding generalized Stieltjes polynomial

$E_{n+1}^{(\sigma^*)}(t)$ $(\sigma_{n+1}^*=((s-l)/2,s-l,...,s-l,(s-l)/2))$ is given by

$E_{n+1}^{(\sigma^*)}(t)=(t^2-1)U_{n-1}(t)$, i.e., the nodes τ_μ^* $(\mu=2,...,n)$ are the zeros of

$U_{n-1}(t)$ (the Chebyshev polynomial of the second kind of the degree $n-1$), and $\tau_1^*=-1,\tau_{n+1}^*=1$.

When quadrature formula (18) has the form

$$\int_{-1}^{1} f(t)\left(1-t^2\right)^{1/2+s} dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s} C^*_{j,\mu} f^{(j)}(\tau^*_\mu) + R_n(f), \qquad (21)$$

we have just proved the previous statement (Galjak, 2006), (Milovanović et al, 2006a, pp.291-305), (Milovanović et al, 2006b, pp. 22-28).

Theorem 4. In the Kronrod extension (21) of the Gauss-Turan quadrature formula (16) with the weight function $\omega_2(t) = \left(1-t^2\right)^{1/2+s}$ the corresponding generalized Stieltjes polynomial $E^{(\sigma^*)}_{n+1}(t)$ $\left(\sigma^*_{n+1} = (s, s, ..., s)\right)$ is given by $E^{(\sigma^*)}_{n+1}(t) = T_{n+1}(t)$, i.e., the nodes τ^*_μ $(\mu = 1, ..., n+1)$ are the zeros of $T_{n+1}(t)$ (the Chebyshev polynomial of the first kind of the degree $n+1$).

When quadrature formula (18) has the form $\left(\tau^*_1 = -1\right)$

$$\int_{-1}^{1} f(t)(1-t)^{1/2+s}(1+t)^{-1/2} dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=2}^{n+1}\sum_{j=0}^{2s} C^*_{j,\mu} f^{(j)}(\tau^*_\mu) + \sum_{j=0}^{s} C^*_{j,1} f^{(j)}(-1) + R_n(f), \qquad (22)$$

we have just proved the previous statement.

Theorem 5. In the Kronrod extension (22) of the Gauss-Turan quadrature formula (16) with the weight function $\omega_3(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$ the corresponding generalized Stieltjes polynomial $E^{(\sigma^*)}_{n+1}(t)$ $\left(\sigma^*_{n+1} = (s/2, s, ..., s)\right)$ is given by $E^{(\sigma^*)}_{n+1}(t) = (t+1) P^{(-1/2,1/2)}_n(t)$, i.e., the nodes τ^*_μ $(\mu = 2, ..., n+1)$ are the zeros of $P^{(-1/2,1/2)}_n(t)$ (the Chebyshev polynomial of the fourth kind of the degree n), and $\tau^*_1 = -1$.

When quadrature formula (18) has the form ($\tau^*_{n+1} = 1$).

$$\int_{-1}^{1} f(t)\omega_4(t) dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n}\sum_{j=0}^{2s} C^*_{j,\mu} f^{(j)}(\tau^*_\mu) + \sum_{j=0}^{s} C^*_{j,n+1} f^{(j)}(1) + R_n(f), \qquad (23)$$

where $\omega_4(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$ is the Chebyshev weight of the fourth kind, in a similar way as in the previous case, the previous statement can be proved.

Theorem 6. In the Kronrod extension (23) of the Gauss-Turan quadrature formula (16) with the weight function $\omega_4(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$ the corresponding generalized Stieltjes polynomial

$E_{n+1}^{(\sigma^*)}(t)$ $(\sigma_{n+1}^* = (s,s,...,s/2))$ is given by $E_{n+1}^{(\sigma^*)}(t) = (t-1)P_n^{(1/2,-1/2)}(t)$, i.e., the nodes τ_μ^* $(\mu = 1,...,n)$ are the zeros of $P_n^{(1/2,-1/2)}(t)$ (the Chebyshev polynomial of the third kind of the degree n), and $\tau_{n+1}^* = 1$.

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ЭКСПЛИЦИТНЫЕ ВЫРАЖЕНИЯ ДЛЯ ОБОБЩЕННЫХ ПОДМНОЖЕСТВ СТИЛТЬЕСА

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ЯЗЫК СТАТЬИ: английский

Резюме:

Уникальное применение теории расширения функций Конрода относительно квадратурной формулы Гаусс-Тюрана доказал Ли (Li, 1994, стр.71-83). Определены эксплицитные выражения для обобщенных подмножеств Стилтьесапо отношению квесовым функциям Чебышева и весовым функциям Гори-Мишелли. Определено расширение функций Конрода по квадратурной формуле Гаусса с узлами многочлена.

В отдельных случаях определены эксплицитные выражения множеств, нули которых являются узлами исследуемых квадратур.

Ключевые слова: подмножества Стилтьеса, расширение функций Кронрода, Гори-Мишелливесовая функция.

ЕКСПЛИЦИТНИ ИЗРАЗИ УОПШТЕНИХ СТИЛТЈЕСОВИХ ПОЛИНОМА

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: прегледни чланак
ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

Егзистенцију и јединственост Кронродових екстензија за добро познате Гаус-Туранове квадратурне формуле доказао је Ли (Li, 1994, стр.71-83). Одређени су експлицитни изрази за уопштене Стилтјесове полиноме у односу на Чебишевљеве тежинске функције, као и у односу на тежинску функцију Гори-Мичели. Дефинисана је Кронродова екстензија за Гаусове квадратурне формуле са вишеструким чворовима. У неким случајевима одређени су експлицитни изрази полинома, чије су нуле чворови посматраних квадратура.

Кључне речи: Стилтјесови полиноми, Кронродова екстензија, тежинска функција Гори-Мичели.

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