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EXISTENCE AND UNIQUENESS OF THE SOLUTIONS OF SOME CLASSES OF INTEGRAL EQUATIONS C*-ALGEBRA-VALUED *b*-METRIC SPACES

Esad Jakupović[a], *Hashem* P. Masiha[b], *Zoran* D. Mitrović[c], *Seyede* S. Razavi[d], *Reza* Saadati[e]

[a]Academy of Sciences and Arts of the Republic of Srpska, Banja Luka, Republic of Srpska, Bosnia and Herzegovina,
e-mail: esadjakupovic50@gmail.com,
ORCID ID: https://orcid.org/0000-0003-2354-5532

[b]K. N. Toosi University of Technology, Faculty of Mathematics, Tehran, Islamic Republic of Iran,
e-mail: masihahp@kntu.ac.ir,
ORCID ID: https://orcid.org/0000-0001-9751-6828

[c]University of Banja Luka, Faculty of Electrical Engineering, Banja Luka, Republic of Srpska, Bosnia and Herzegovina,
e-mail: zoran.mitrovic@etf.unibl.org, **corresponding author,**
ORCID ID: https://orcid.org/0000-0001-9993-9082

[d]K. N. Toosi University of Technology, Faculty of Mathematics, Tehran, Islamic Republic of Iran,
e-mail: ssrazavi@kntu.ac.ir,
ORCID ID: https://orcid.org/0000-0002-9772-1140

[e]Iran University of Science and Technology, School of Mathematics, Narmak, Tehran, Islamic Republic of Iran,
e-mail: rsaadati@iust.ac.ir,
ORCID ID: https://orcid.org/0000-0002-6770-6951

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Abstract:

Introduction/purpose: The aim of the paper is to establish some coupled fixed point results in C-algebra-valued b-metric spaces. Moreover, the obtained results are used to define the sufficient conditions for the existence of the solutions of some classes of integral equations.*

Methods: The method of coupled fixed points gives the sufficient conditions for the existence of the solution of some classes of integral equations.

*Results: New results were obtained on coupled fixed points in C * -algebra-valued b-metric space.*

Conclusion: The obtained results represent a contribution in the fixed point theory and open new possibilities of application in the theory of differential and integral equations.

Key words: Coupled fixed point, C-algebra, integral equation.*

Basic definitions

In this section, we review some facts of the C*-algebras which are needed in this paper. The references (Ali Abou Bakr, 2019), (Bai, 2016), (Bonsal, 1962), (Hussain et al, 2018), (Huang et al, 2018), (Hussain&Mitrović, 2017), (Kadelburg et al, 2016), (Kadelburg&Radenović, 2016), (Kongban&Kumam, 2018), (Ma et al, 2014), (Ma&Jiang, 2015), (Mitrović et al, 2019), (Radenović et al, 2017), (Radenović et al, 2019), (Vujaković et al, 2019), (Zoto et al, 2019), (Wu&Zhao, 2018), (Cao&Xin, 2016) and (Todorčević, 2019) are useful.

We denote $\mathcal{A}$ as a unital C*-algebra with the unit $1_{\mathcal{A}}$.

Let

$$\mathcal{A}_h = \{t \in \mathcal{A} : t = t^*\}.$$

We say $t \in \mathcal{A}$ a positive element, showed it by $t \succeq 0_{\mathcal{A}}$ if $t \in \mathcal{A}_h$ and $\sigma(t) \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is the zero element in $\mathcal{A}$ and $\sigma(t)$ is the spectrum of t.

On the set $\mathcal{A}_h$ we have a partial ordering given by $v \succeq u$ if and only if $v - u \succeq 0_{\mathcal{A}}$. Also, we will denote

$$\mathcal{A}_+ = \{t \in \mathcal{A} : t \succeq 0_{\mathcal{A}}\}$$

and

$$\mathcal{A}' = \{t \in \mathcal{A} : tk = kt \text{ for all } k \in \mathcal{A}\}.$$

Definition 1. *Assume that $\mathcal{X} \neq \emptyset$. As usual suppose that $\delta : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is a function fulfilling:*

(1) $\delta(u, v) \succeq 0_{\mathcal{A}}$ for each u and v in $\mathcal{X}$;

(2) $\delta(v, u) = 0_{\mathcal{A}}$ if $v = u$;

(3) $\delta(u, v) = \delta(v, u)$ for each u and v in $\mathcal{X}$;

(4) $\delta(u, v) \preceq \delta(u, t) + \delta(t, v)$ for each u, v and $t \in \mathcal{X}$.

Then δ is a C-algebra-valued metric (shortly, C*-AV-M).*

Definition 2. *Assume that* $\mathcal{X} \neq \emptyset$. *Suppose that* $b \in \mathcal{A}'$ *such that* $||b|| \geq 1$. *A function* $\delta_b : \mathcal{X}^2 \to \mathcal{A}$ *is said to be a C*-algebra-valued b-metric (in short C*-AV-BM) on* $\mathcal{X}$ *if for every* $u, v, t \in \mathcal{A}$*:*

(1) $\delta_b(u, v) \succeq 0_{\mathcal{A}}$ *for each* u *and* v *in* $\mathcal{X}$ *and* $\delta_b(u, v) = 0$ *if* $u = v$*;*

(2) $\delta_b(u, v) = \delta_b(v, u)$*;*

(3) $\delta_b(u, v) \preceq b[\delta_b(u, t) + \delta_b(t, v)]$.

Then $(\mathcal{X}, \mathcal{A}, \delta_b)$ *is a C*-AV-BM space with the coefficient* b.

Example 1 (Ma&Jiang (2015)). Assume that $\mathcal{X} = \mathbb{R}$ and $\mathcal{A} = M_n(\mathbb{R})$. Now, we define

$$\delta(u, v) = diag(c_1|u - v|^p, c_2|u - v|^p, \ldots, c_n|u - v|^p),$$

where $diag$ denotes a diagonal matrix, and where $u, v \in \mathbb{R}$, $c_1, \ldots, c_n$ positive constants and $p \in (1, +\infty)$. It can be shown that $(\mathcal{X}, \mathcal{A}, \delta)$ is a complete C*-AV-BM. We only prove the third statement of Definition 2. For this we have:

$$|u - v|^p \leq 2^p(|u - t|^p + |t - v|^p),$$

then $\delta(u, v) \preceq A[\delta(u, t) + \delta(t, v)]$ for every $u, v, t \in \mathcal{X}$, where $A = 2^p I \in \mathcal{A}'$ and $A > I$ by $2^p > 1$. Since $|u - v|^p \leq |u - t|^p + |t - v|^p$ is impossible for all $u \succ t \succ v$, $(\mathcal{X}, M_n(\mathbb{R}), \delta)$ is not a C*-AV-M space.

Definition 3. *Assume that* $(\mathcal{X}, \mathcal{A}, \delta_b)$ *is a C*-AV-BM space.* $(u, v) \in \mathcal{X} \times \mathcal{X}$ *is called a coupled fixed point (shortly FP) of the function* $\psi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *if* $\psi(u, v) = u$ *and* $\psi(v, u) = v$.

For some useful applications, see (Kongban&Kumam, 2018) and (Ali Abou Bakr, 2019).

Main results

The following theorem is one of our main results.

Theorem 1. *Assume that* $(\mathcal{X}, \mathcal{A}, \delta_b)$ *is a C*-AV-BM space and suppose that* $\psi : \mathcal{X}^2 \to \mathcal{X}$ *is a function satisfying*

$$\delta_b(\psi(u, v), \psi(t, s)) \preceq a^*\delta_b(u, t)a + a^*\delta_b(v, s)a, \tag{1}$$

for every $u, v, t, s \in \mathcal{X}$*, in which* $a \in \mathcal{A}$ *with* $||a|| < \frac{1}{\sqrt{2}}$. *Then* ψ *has a unique coupled FP. Moreover,* ψ *has a unique FP in* $\mathcal{X}$.

Proof. Let $u_0, v_0 \in \mathcal{X}$. Set $u_1 = \psi(u_0, v_0)$ and $v_1 = \psi(v_0, u_0)$. We obtain two sequences $\{u_n\}$, $\{v_n\}$ in $\mathcal{X}$ such that $u_{n+1} = \psi(u_n, v_n)$ and $v_{n+1} = \psi(v_n, u_n)$ if we continue the above process. From (1) we have

$$\begin{aligned}\delta_b(u_n, u_{n+1}) &= \delta_b(\psi(u_{n-1}, v_{n-1}), \psi(u_n, v_n)) \\ &\preceq a^*\delta_b(u_{n-1}, u_n)a + a^*\delta_b(v_{n-1}, v_n)a \\ &\preceq a^*(\delta_b(u_{n-1}, u_n) + \delta_b(v_{n-1}, v_n))a.\end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned}\delta_b(v_n, v_{n+1}) &= \delta_b(\psi(v_{n-1}, u_{n-1}), \psi(v_n, u_n)) \\ &\preceq a^*\delta_b(v_{n-1}, v_n)a + a^*\delta_b(u_{n-1}, u_n)a \\ &\preceq a^*(\delta_b(v_{n-1}, v_n) + \delta_b(u_{n-1}, u_n))a.\end{aligned} \tag{3}$$

Let

$$\delta_n = \delta_b(u_n, u_{n+1}) + \delta_b(v_n, v_{n+1}).$$

From (2) and (3), we have

$$\begin{aligned}\delta_n &= \delta_b(u_n, u_{n+1}) + \delta_b(v_n, v_{n+1}) \\ &\preceq a^*(\delta_b(u_{n-1}, u_n) + \delta_b(v_{n-1}, v_n))a + a^*(\delta_b(v_{n-1}, v_n) + \delta_b(u_{n-1}, u_n))a \\ &= 2(a^*(\delta_b(u_{n-1}, u_n) + \delta_b(v_{n-1}, v_n))a) \\ &\preceq (\sqrt{2}a)^*(\delta_b(u_{n-1}, u_n) + \delta_b(v_{n-1}, v_n))(\sqrt{2}a) \\ &\preceq (\sqrt{2}a)^*\delta_{n-1}(\sqrt{2}a).\end{aligned}$$

Due to the following property: (if $t, k \in \mathcal{A}_h$, then $t \preceq k$ implies $u^*tu \preceq u^*ku$), we can obtain for any $n \in \mathbb{N}$,

$$0_{\mathcal{A}} \preceq \delta_n \preceq (\sqrt{2}a)^*\delta_{n-1}(\sqrt{2}a) \preceq \ldots \preceq ((\sqrt{2}a)^*)^n\delta_0(\sqrt{2}a)^n.$$

If $\delta_0 = 0_{\mathcal{A}}$, then from (2) of Definition 2 we know (u_0, v_0) is a coupled FP of F.

Now, by letting $\delta_0 \preceq 0_{\mathcal{A}}$, we can obtain for $m, n \in \mathbb{N}$, $n > m$

$$\begin{aligned}\delta_b(u_n, u_m) &\preceq b(\delta_b(u_n, u_{n-1}) + \delta_b(u_{n-1}, u_m)) \\ &\preceq b\delta_b(u_n, u_{n-1}) + b^2\delta_b(u_{n-1}, u_{n-2}) + b^2\delta_b(u_{n-2}, u_m) \\ &\preceq b\delta_b(u_n, u_{n-1}) + b^2\delta_b(u_{n-1}, u_{n-2})\end{aligned}$$

$$
\begin{aligned}
&+ b^3\delta_b(u_{n-2}, u_{n-3}) + \ldots + b^{n-m}\delta_b(u_{m+1}, u_m) \\
&\preceq b\delta_b(u_n, u_{n-1}) + b^2\delta_b(u_{n-1}, u_{n-2}) + \ldots \\
&+ b^{n-m-1}\delta_b\left(u_{m+2}, u_{m+1}\right) + b^{n-m-1}\delta_b\left(u_{m+1}, u_m\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\delta_b(v_n, v_m) \quad \preceq \quad & b\delta_b(v_n, v_{n-1}) + b^2\delta_b(v_{n-1}, v_{n-2}) + \ldots + \\
& b^{n-m-1}\delta_b\left(v_{m+2}, v_{m+1}\right) + b^{n-m-1}\delta_b\left(v_{m+1}, v_m\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\delta_b(u_n, u_m) + \delta_b(v_n, v_m) &\preceq b(\delta_b(u_n, u_{n-1}) + \delta_b(v_n, v_{n-1})) \\
&+ b^2(\delta_b(u_{n-1}, u_{n-2}) + \delta_b(v_{n-1}, v_{n-2})) \\
&+ \ldots + b^{n-m-1}\left(\delta_b\left(u_{m+2}, u_{m+1}\right) + \delta_b\left(v_{m+2}, v_{m+1}\right)\right) \\
&+ b^{n-m}\left(\delta_b\left(u_{m+1}, u_m\right) + \delta_b\left(v_{m+1}, v_m\right)\right) \\
&\preceq b\delta_{n-1} + b^2\delta_{n-2} + \ldots + b^{n-m}\delta_m \\
&\preceq b((2^{\frac{1}{2}}a)^*)^{n-1}\delta_0(2^{\frac{1}{2}}a)^{n-1} + b^2((2^{\frac{1}{2}}a)^*)^{n-2}\delta_0(2^{\frac{1}{2}}a)^{n-2} \\
&+ \ldots + b^{n-m}((2^{\frac{1}{2}}a)^*)^m\delta_0(2^{\frac{1}{2}}a)^m \\
&\preceq \sum_{k=n-1}^{m} b^{n-k}((2^{\frac{1}{2}}a)^*)^k\delta_0(2^{\frac{1}{2}}a)^k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||\delta_b(u_n, u_m) + \delta_b(v_n, v_m)|| &\leq \sum_{k=n-1}^{m} ||b||^{n-k}||\sqrt{2}a||^{2k}\delta_0 \\
&\leq \sum_{k=n-1}^{\infty} ||b||^{n-k}||\sqrt{2}a||^{2k}\delta_0 \\
&\leq \frac{||b||||\sqrt{2}a||^{2(n-1)}}{1 - ||b||^{-1}||\sqrt{2}a||^2}\delta_0 \\
&= \frac{||b||}{1 - ||b||^{-1}||\sqrt{2}a||^2}||\sqrt{2}a||^{2(n-1)}\delta_0.
\end{aligned}
$$

Since $||a|| < \frac{1}{\sqrt{2}}$, we have

$$||\delta_b(u_n, u_m) + \delta_b(v_n, v_m)|| \leq \frac{||b||}{1 - ||b||^{-1}||\sqrt{2}a||^2}||\sqrt{2}a||^{2(n-1)}\delta_0 \longrightarrow 0,$$

$$\delta_b(u_n, u_m) \preceq \delta_b(u_n, u_m) + \delta_b(v_n, v_m),$$

and

$$\delta_b(v_n, v_m) \preceq \delta_b(v_n, v_m) + \delta_b(u_n, u_m),$$

yields that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $\mathcal{X}$, so we can find u and v in $\mathcal{X}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Now, we prove that $\psi(u, v) = u$ and $\psi(v, u) = v$.

From the triangle inequality and (1), we obtain

$$\begin{aligned}\delta_b(\psi(u, v), u) &\preceq b[\delta_b(\psi(u, v), u_{n+1}) + \delta_b(u_{n+1}, u)]\\ &\preceq b[\delta_b(\psi(u, v), \psi(u_n, v_n)) + \delta_b(u_{n+1}, u)]\\ &\preceq b[a^*\delta_b(u, u_n)a + a^*\delta_b(v, v_n)a + \delta_b(u_{n+1}, u)],\end{aligned}$$

taking $n \to \infty$, we have that $\delta_b(\psi(u, v), u) = 0_{\mathcal{A}}$, and consequently $\psi(u, v) = u$. In the same way $\psi(v, u) = v$. Therefore, a coupled FP of ψ is (u, v).

Assume that another coupled FP of ψ is (u', v'), so

$$\delta_b(u, u') = \delta_b(\psi(u, v), \psi(u', v')) \preceq a^*\delta_b(u, u')a + a^*\delta_b(v, v')a,$$

$$\delta_b(v, v') = \delta_b(\psi(v, u), \psi(v', u')) \preceq a^*\delta_b(v, v')a + a^*\delta_b(u, u')a,$$

and hence

$$\delta_b(u, u') + \delta_b(v, v') \preceq (\sqrt{2}a)^*(\delta_b(u, u') + \delta_b(v, v'))(\sqrt{2}a).$$

The above inequality with $||\sqrt{2}a|| < 1$ yields that

$$||\delta_b(u, u') + \delta_b(v, v')|| \leq ||\sqrt{2}a||^2||\delta_b(u, u') + \delta_b(v, v')||.$$

The above inequality holds only when $||\delta_b(u, u')+\delta_b(v, v')|| = 0$, which gives $(u', v') = (u, v)$. So the coupled FP is unique.

Now, we prove that $u = v$ to show that ψ has a unique FP. Note that

$$\delta_b(u, v) = \delta_b(\psi(u, v) + \psi(v, u)) \preceq a^*\delta_b(u, v)a + a^*\delta_b(v, u)a,$$

and hence

$$||\delta_b(u, v)|| \leq ||a||^2||\delta_b(u, v)|| + ||a||^2||\delta_b(v, u)||$$

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$$\leq 2||a||^2||\delta_b(u,v)||.$$

In fact, from $||a|| < \frac{1}{\sqrt{2}}$ we get $||\delta_b(u,v)|| = 0$, thus $u = v$. □

Lemma 1. *(Ma et al, 2014)*

1) *If $u \in \mathcal{A}_+$ with $||u|| < \frac{1}{2}$, then $1_{\mathcal{A}} - u$ is invertible.*
2) *If $u, v \in \mathcal{A}_+$ with $uv = vu$, then $uv \succeq 0_{\mathcal{A}}$.*
3) *If $u, v \in \mathcal{A}_h$ and $t \in \mathcal{A}'_+$, then $u \preceq v$ deduces $tu \preceq tv$, where $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.*

Theorem 2. *Assume that $(X, \mathcal{A}, \delta_b)$ is a complete C*-AV-BM space and suppose that the function $\psi : \mathcal{X}^2 \to \mathcal{X}$ satisfies*

$$\delta_b(\psi(u,v),\psi(t,s)) \preceq a\delta_b(\psi(u,v),u) + b\delta_b(\psi(t,s),t),\ \forall u,v,t,s \in \mathcal{X} \quad (4)$$

in which $a, b, c, \in \mathcal{A}'_+$ with $||a|| + ||b|| < 1$, $||ac|| + ||bc|| < 1$, $||c|| > 1$. Then ψ has a unique coupled FP. Also, ψ has a unique FP in $\mathcal{X}$.

Proof. As in Theorem 1, choose $\{u_n\}$ and $\{v_n\}$ in $\mathcal{X}$ and set $u_{n+1} = \psi(u_n, v_n)$ and $v_{n+1} = \psi(v_n, u_n)$. Then from (4),

$$\begin{aligned}\delta_b(u_n,u_{n+1}) &= \delta_b(\psi(u_{n-1},v_{n-1}),\psi(u_n,v_n))\\ &\preceq a\delta_b(\psi(u_{n-1},v_{n-1}),u_{n-1}) + b\delta_b(\psi(u_n,v_n),u_n)\\ &= a\delta_b(u_n,u_{n-1}) + b\delta_b(u_{n+1},u_n),\end{aligned}$$

Thus,

$$(1_{\mathcal{A}} - b)\delta_b(u_n,u_{n+1}) \preceq a\delta_b(u_n,u_{n-1}),$$

Similarly,

$$(1_{\mathcal{A}} - b)\delta_b(v_n,v_{n+1}) \preceq a\delta_b(v_n,v_{n-1}),$$

Since $a, b \in \mathcal{A}'_+$ with $||a|| + ||b|| < 1$, then $1_{\mathcal{A}} - b$ is invertible and $(1_{\mathcal{A}} - b)^{-1}a \in \mathcal{A}'_+$. Therefore

$$\delta_b(u_n,u_{n+1}) \preceq (1_{\mathcal{A}} - b)^{-1}a\delta_b(u_n,u_{n-1}),$$

$$\delta_b(v_n,v_{n+1}) \preceq (1_{\mathcal{A}} - b)^{-1}a\delta_b(v_n,v_{n-1}),$$

Then

$$||\delta_b(u_n,u_{n+1})|| \leq ||(1_{\mathcal{A}} - b)^{-1}a||||\delta_b(u_n,u_{n-1})||,$$

$$||\delta_b(v_n, v_{n+1})|| \le ||(1_{\mathcal{A}} - b)^{-1}a||||\delta_b(v_n, v_{n-1})||,$$

It follows from the fact

$$||(1_{\mathcal{A}} - b)^{-1}a|| \le ||(1_{\mathcal{A}} - b)^{-1}||||a|| \le \sum_{k=0}^{\infty} ||b||^k ||a|| = \frac{||a||}{1 - ||b||} < 1.$$

Hence $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $\mathcal{X}$. By the completeness of $\mathcal{X}$, there are $u, v \in \mathcal{X}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. As

$$\begin{aligned}
\delta_b(\psi(u,v), u) &\preceq c[\delta_b(u_{n+1}, \psi(u,v)) + \delta_b(u_{n+1}, u)] \\
&= c\delta_b(\psi(u_n, v_n), \psi(u,v)) + c\delta_b(u_{n+1}, u) \\
&\preceq ca\delta_b(\psi(u_n, v_n), u_n) + cb\delta_b(\psi(u,v), u) + c\delta_b(u_{n+1}, u) \\
&\preceq ca\delta_b(u_{n+1}, u_n) + cb\delta_b(\psi(u,v), u) + c\delta_b(u_{n+1}, u),
\end{aligned}$$

which deduces that

$$\delta_b(\psi(u,v), u) \preceq (1_{\mathcal{A}} - cb)^{-1} ca\delta_b(u_{n+1}, u_n) + (1_{\mathcal{A}} - cb)^{-1} c\delta_b(u_{n+1}, u).$$

like above

$$||(1_{\mathcal{A}} - cb)^{-1}ca|| \le ||(1_{\mathcal{A}} - cb)^{-1}||||ca|| \le \sum_{k=0}^{\infty} ||cb||^k ||ca|| = \frac{||ca||}{1 - ||cb||} < 1.$$

$$||(1_{\mathcal{A}} - cb)^{-1}c|| \le ||(1_{\mathcal{A}} - cb)^{-1}||||c|| \le \sum_{k=0}^{\infty} ||cb||^k ||c|| = \frac{||c||}{1 - ||cb||} < 1.$$

Then $\delta_b(\psi(u,v), u) = 0_{\mathcal{A}}$ or equivalently $\psi(u,v) = u$. In the same way, $\psi(v,u) = v$. Now if (u', v') is another coupled FP of ψ, then from (4), we get

$$\begin{aligned}
\delta_b(u, u') &= \delta_b(\psi(u,v), \psi(u', v')) \\
&\preceq a\delta_b(u', \psi(u', v')) + b\delta_b(u, \psi(u,v)) = 0_{\mathcal{A}},
\end{aligned}$$

Hence $\delta_b(u', u) = 0_{\mathcal{A}}$, and then$u' = u$. In the same way, we can obtain that $v' = v$. That is, (u,v) is the unique coupled FP of ψ. Now, we prove the uniqueness of FPs of ψ. Using (4), we obtain

$$\begin{aligned}
\delta_b(u, v) &= \delta_b(\psi(u,v), \psi(v,u)) \\
&\preceq a\delta_b(\psi(u,v), u) + b\delta_b(\psi(v,u), v) \\
&= a\delta_b(u,u) + b\delta_b(v,v) = 0_{\mathcal{A}}.
\end{aligned}$$

This yields that $u = v$. □

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Theorem 3. *Assume that* $(\mathcal{X}, \mathcal{A}, \delta_b)$ *is a complete C*-AV-BM space and let the function* $F : \mathcal{X}^2 \to \mathcal{X}$ *hold*

$$\delta_b(\psi(u,v), \psi(t,s)) \preceq a\delta_b(\psi(u,v), t) + b\delta_b(\psi(t,s), u), \ \forall u, v, t, s \in \mathcal{X} \qquad (5)$$

where $a, b, c \in \mathcal{A}'_+$ *with* $||a|| + ||b|| < 1$, $||ac|| + ||bc|| < 1$, $||c|| > 1$. *then* ψ *has a unique coupled FP. Also,* ψ *has a unique FP in* $\mathcal{X}$.

Proof. Similar to Theorem 2, choose two sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{X}$ and set $u_{n+1} = \psi(u_n, v_n)$ and $v_{n+1} = \psi(v_n, u_n)$. Then from (5) we obtain

$$\begin{aligned}
\delta_b(u_n, u_{n+1}) &= \delta_b(\psi(u_{n-1}v_{n-1}), \psi(u_n, v_n)) \\
&\preceq a\delta_b(\psi(u_{n-1}, v_{n-1}), u_n) + b\delta_b(\psi(u_n, v_n), u_{n-1}) \\
&= a\delta_b(u_n, u_n) + b\delta_b(u_{n+1}, u_{n-1}) \\
&= b\delta_b(u_{n+1}, u_{n-1}) \\
&\preceq cb(\delta_b(u_{n+1}, u_n) + \delta_b(u_n, u_{n-1})) \\
&= cb\delta_b(u_{n+1}, u_n) + cb\delta_b(u_n, u_{n-1}),
\end{aligned}$$

Thus,

$$(1_{\mathcal{A}} - cb)\delta_b(u_n, u_{n+1}) \preceq cb\delta_b(u_n, u_{n-1}). \qquad (6)$$

By the symmetry in (5),

$$\begin{aligned}
\delta_b(u_{n+1}, u_n) &= \delta_b(\psi(u_n v_n), \psi(u_{n-1}, v_{n-1})) \\
&\preceq a\delta_b(\psi(u_n, v_n), u_{n-1}) + b\delta_b(\psi(u_{n-1}, v_{n-1}), u_n) \\
&= a\delta_b(u_{n+1}, u_{n-1}) + b\delta_b(u_n, u_n) \\
&= a\delta_b(u_{n+1}, u_{n-1}) \\
&\preceq ca(\delta_b(u_{n+1}, u_n) + \delta_b(u_n, u_{n-1})) \\
&= ca\delta_b(u_{n+1}, u_n) + ca\delta_b(u_n, u_{n-1}),
\end{aligned}$$

that is,

$$(1_{\mathcal{A}} - ca)\delta_b(u_n, u_{n+1}) \preceq ca\delta_b(u_n, u_{n-1}). \qquad (7)$$

Now, from (6) and (7) we obtain

$$(1_{\mathcal{A}} - \frac{ca + cb}{2})\delta_b(u_n, u_{n+1}) \preceq \frac{ca + cb}{2}\delta_b(u_n, u_{n-1}).$$

Since $a, b, c \in \mathcal{A}'_+$ with $||ca+cb|| \le ||ca||+||cb|| < 1$, then $(1_\mathcal{A} - \frac{ca+cb}{2})^{-1} \in \mathcal{A}'_+$, which together with [3, Lemma 1] yields that

$$\delta_b(u_n, u_{n+1}) \preceq (1_\mathcal{A} - \frac{ca+cb}{2})^{-1} \frac{ca+cb}{2} \delta_b(u_n, u_{n-1}).$$

Let $t = (1_\mathcal{A} - \frac{ca+cb}{2})^{-1} \frac{ca+cb}{2}$, then $||t|| = ||(1_\mathcal{A} - \frac{ca+cb}{2})^{-1} \frac{ca+cb}{2}|| < 1$.

By using the same argument of Theorem 2, we obtain $\{u_n\}$ which is a Cauchy sequence in X. Also, one can show that $\{v_n\}$ is a Cauchy sequence in $\mathcal{X}$. Therefore by the completeness of $\mathcal{X}$, there are $u, v \in \mathcal{X}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Now, we obtain that $\psi(u, v) = u$ and $\psi(v, u) = v$. To do this, we have

$$\begin{aligned}
\delta_b(\psi(u,v),u) &\preceq c[\delta_b(u_{n+1},\psi(u,v)) + \delta_b(u_{n+1},u)] \\
&= c\delta_b(\psi(u_n,v_n),\psi(u,v)) + c\delta_b(u_{n+1},u) \\
&\preceq ca\delta_b(\psi(u_n,v_n),u) + cb\delta_b(\psi(u,v),u_n) + c\delta_b(u_{n+1},u) \\
&\preceq ca\delta_b(u_{n+1},u) + cb\delta_b(\psi(u,v),u_n) + c\delta_b(u_{n+1},u),
\end{aligned}$$

and then

$$||\delta_b(\psi(u,v),u)|| \le ||ca||||\delta_b(u_{n+1},u)|| + ||cb||||\delta_b(\psi(u,v),u_n)|| + ||c||||\delta_b(u_{n+1},u)||,$$

by the continuity of the metric and the norm, we get

$$||\delta_b(\psi(u,v),u)|| \le ||cb||||\delta_b(\psi(u,v),u)||.$$

Since $||cb|| < 1$, it implies that $||\delta_b(\psi(u,v),u)|| = 0$, and then $\psi(u,v) = u$. In the same way, $\psi(v,u) = v$, which implies that (u,v) is a coupled FP of ψ. By the same reasoning in Theorem 2, we obtain $u = v$, which means that ψ has a unique FP in $\mathcal{X}$. □

Existence and uniqueness

Consider the next equation:

$$x(m) = \int_\mathcal{E} (T_1(m,n) + T_2(m,n))(\alpha(n,x(n)) + \beta(n,x(n))dn + J(m), \ m \in \mathcal{E} \quad (8)$$

for the Lebesgue measurable set of $\mathcal{E}$ in which $m(\mathcal{E}) < \infty$.

In what follows, we always let $\mathcal{X} = L^{\infty}(\mathcal{E})$ denote a class of essentially bounded measurable functions on $\mathcal{E}$, where $\mathcal{E}$ is a Lebesgue measurable set such that $m(\mathcal{E}) < \infty$.

Now, we consider the functions T_1, T_2, α, β fulfill the following assumptions:

(i) T_1 from $\mathcal{E} \times \mathcal{E}$ to $\mathbb{R}^{\geq 0}$, T_2 from $\mathcal{E} \times \mathcal{E}$ to $\mathbb{R}^{\leq 0}$. Also, two integrable functions α and β are from $\mathcal{E} \times \mathbb{R}$ to $\mathbb{R}$, and $J \in L^{\infty}(\mathcal{E})$;

(ii) there exists $\ell \in (0, \frac{1}{2})$ such that

$$0 \leq \alpha(m, x) - \alpha(m, y) \leq \ell(x - y)$$

and

$$-\ell(x - y) \leq \beta(m, x) - \beta(m, y) \leq 0$$

for $m \in \mathcal{E}$ and $x, y \in \mathbb{R}$;

(iii) $\sup_{m \in \mathcal{E}} \int_{\mathcal{E}} (T_1(m, n) - T_2(m, n)) dn \leq 1$.

Theorem 4. *Suppose that assumptions (i)-(iii) hold. Then the integral equation (8) has a unique solution in $L^{\infty}(\mathcal{E})$.*

Proof. Let $\mathcal{X} = L^{\infty}(\mathcal{E})$ and $B(L^2(\mathcal{E}))$ be the set of bounded linear operators on a Hilbert space $L^2(\mathcal{E})$. We endow $\mathcal{X}$ with the b-metric $\delta_b : \mathcal{X} \times \mathcal{X} \to B(L^2(\mathcal{E}))$ defined by

$$\delta_b(\alpha, \beta) = M_{|\alpha - \beta|^p}$$

where $M_{|\alpha-\beta|^p}$ is the multiplication operator on $L^2(\mathcal{E})$. Hence $(\mathcal{X}, B(L^2(\mathcal{E})), \delta_b)$ is a complete C*-AV-BM space. Define the self-mapping $\Psi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$\Psi(x, y)(m) = \int_{\mathcal{E}} T_1(m, n)(\alpha(n, x(n)) + \beta(n, y(n)))dn$$
$$+ T_2(m, n)(\alpha(n, y(n)) + \beta(n, x(n)))dn + J(m),$$

for all $m \in \mathcal{E}$. Now, we have

$$\delta_b(\Psi(x, y), \Psi(u, v)) = M_{|\Psi(x,y) - \Psi(u,v)|^p}.$$

We obtain,

$$|(\Psi(x, y) - \Psi(u, v))(m)|^p = |\int_{\mathcal{E}} T_1(m, n)(\alpha(n, x(n)) + \beta(n, y(n)))dn$$

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$$
\begin{aligned}
+ & \int_{\mathcal{E}} T_2(m,n)(\alpha(n,y(n))+\beta(n,x(n)))dn - \int_{\mathcal{E}} T_1(m,n)(\alpha(n,u(n)) \\
+ & \beta(n,v(n)))dn - \int_{\mathcal{E}} T_2(m,n)(\alpha(n,v(n))+\beta(n,u(n)))dn|^p \\
= & (|\int_{\mathcal{E}} T_1(m,n)(\alpha(n,x(n))-\alpha(n,u(n))+\beta(n,y(n))-\beta(n,v(n)))dn| \\
+ & |\int_{\mathcal{E}} T_2(m,n)(\alpha(n,y(n))-\alpha(n,v(n))+\beta(n,x(n))-\beta(n,u(n)))dn|)^p \\
\leq & (\int_{\mathcal{E}} T_1(m,n)|\alpha(n,x(n))-\alpha(n,u(n))+\beta(n,y(n))-\beta(n,v(n))|dn \\
- & \int_{\mathcal{E}} T_2(m,n)|\alpha(n,y(n))-\alpha(n,v(n))+\beta(n,x(n))-\beta(n,u(n))|dn)^p \\
\leq & (\sup_{n\in\mathcal{E}}[\ell|x(n)-u(n)|+\ell|y(n)-v(n)|]\times\int_{\mathcal{E}}(T_1(m,n)-T_2(m,n))dn)^p \\
\leq & ([\ell||x-u||_\infty+\ell||y-v||_\infty]\sup_{m\in\mathcal{E}}\int_{\mathcal{E}}(T_1(m,n)-T_2(m,n))dn)^p \\
\leq & (\ell||x-u||_\infty+\ell||y-v||_\infty)^p \\
\leq & \ell(||x-u||_\infty+||y-v||_\infty)^p.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||\delta_b(\Psi(x,y),\Psi(u,v))|| &= ||M_{|\Psi(x,y)-\Psi(u,v)|^p}|| \\
&= \sup_{||\varphi||=1}(M_{|\Psi(x,y)-\Psi(u,v)|^p}\varphi,\varphi) \\
&= \sup_{||\varphi||=1}\int_{\mathcal{E}}|(\Psi(x,y)-\Psi(u,v))(m)|^p\varphi(m)\bar{\varphi(m)}dm \\
&\leq \sup_{||\varphi||=1}\int_{\mathcal{E}}|\varphi(m)|^2dt(\ell||x-u||_\infty+\ell||y-v||_\infty)^p \\
&\leq (\ell||x-u||_\infty+\ell||y-v||_\infty)^p \\
&\leq \ell(||x-u||_\infty+||y-v||_\infty)^p.
\end{aligned}
$$

Set $a=\sqrt{\ell}1_{B(L^2(\mathcal{E}))}$, then $a\in B(L^2(\mathcal{E}))$ and $||a||=|\sqrt{\ell}|<\frac{1}{\sqrt{2}}$. Hence by applying Theorem 1, we get the desired result. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed to the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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СУЩЕСТВОВАНИЕ И ЕДИНСТВЕННОСТЬ РЕШЕНИЙ НЕКОТОРЫХ КЛАССОВ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ C*-АЛГЕБРЫ В *b*-МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

Есад Якупович[а], *Хашем* П. Масиха[б], *Зоран* Д. Митрович[в], **корессподент**, *Сеиде* С. Разави[б], *Реза* Саадати[г]

[а]Академия наук и художеств Республики Сербской, г. Баня-Лука, Республика Сербская, Босния и Герцеговина

[б]Технологический университет «K. H. Тооси», математический факультет, г. Тегеран, Исламская Республика Иран

[в]Университет в г. Баня-Лука, электротехнический факультет, г. Баня-Лука, Республика Сербская, Босния и Герцеговина

[г]Иранский научно-технологический университет, математический колледж, Нармак, г. Тегеран, Исламская Республика Иран

РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА:
27.25.17 Метрическая теория функций,
27.33.00 Интегральные уравнения,
27.39.29 Приближенные методы функционального анализа

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

*Введение/цель: Цель данной статьи заключается в выявлении сопряженных неподвижных точек C *-алгебры в b-метрических пространствах. На основании полученных результатов обсуждается существование решений интегральных уравнений.*

Методы: С помощью методов сопряжения неподвижных точек представлены необходимые условия для существования решений некоторых классов интегральных уравнений.

*Результаты: Получены новые результаты о сопряженных неподвижных точках C *-алгебры в b-метрическом пространстве.*

Выводы: Полученные результаты вносят вклад в теорию неподвижных точек и открывают новые возможности применения в теории дифференциальных и интегральных уравнений.

*Ключевые слова: сопряженная неподвижная точка, C *-алгебра, интегральное уравнение.*

ЕГЗИСТЕНЦИЈА И ЈЕДИНСТВЕНОСТ РЈЕШЕЊА НЕКИХ КЛАСА ИНТЕГРАЛНИХ ЈЕДНАЧИНА У *b*-МЕТРИЧКИМ ПРОСТОРИМА НАД C*-АЛГЕБРАМА

Есад Јакуповић[а], *Hashem* P. Masiha[б], *Зоран* Д. Митровић[в], **аутор за преписку**, *Seyede* S. Razavi[б], *Reza* Saadati[г]

[а]Академија наука и умјетности Републике Српске, Бања Лука, Република Српска, Босна и Херцеговина

[б]Технолошки универзитет „K. N. Toosi", Математички факултет, Техеран, Исламска Република Иран

[в]Универзитет у Бањој Луци, Електротехнички факултет, Бања Лука, Република Српска, Босна и Херцеговина

[г]Ирански научно-технолошки универзитет, Колеџ математике, Нармак, Техеран, Исламска Република Иран

ОБЛАСТ: математика
ВРСТА ЧЛАНКА: оригинални научни рад

Сажетак:

*Увод/циљ: Циљ овог рада јесте да се добију одређени резултати о придруженим непокретним тачкама у C *-алгебра b-метричким просторима. Користећи ове резултате дати су довољни услови за егзистенцију рјешења неких класа интегралних једначина.*

Методе: Коришћењем методе придружених фиксних тачака дати су довољни услови за егзистенцију рјешења неких класа интегралних једначина.

Резултати: Добијени су нови резултати о придруженим фиксним тачкама у b-метричком простору над C-алгебром.*

Закључак: Добијени резултати представљају допринос теорији фиксних тачака и отварају нове могућности за примене у теорији диференцијалних и интегралних једначина.

*Кључне речи: придружена фиксна тачка, C *-алгебра, интегрална једначина.*

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