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ПРЕГЛЕДНИ РАДОВИ

ОБЗОРНЫЕ СТАТЬИ

REVIEW PAPERS

ZETA FUNCTION AND SOME OF ITS PROPERTIES

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Abstract:

Introduction/purpose: Some properties of the zeta function will be shown as well as its applications in calculus, in particular the "golden nugget formula" for the value of the infinite sum $1 + 2 + 3 + \cdots$ *. Some applications in physics will also be mentioned.*

Methods: Complex plane integrations and properties of the Gamma function will be used from the definition of the function to its analytic extension.

Results: From the original definition of the $\zeta(s)$ *function valid for* $s > 1$ *a meromorphic function is obtained on the whole complex plane with a simple pole in* $s = 1$*.*

Conclusion: The relevance of the zeta function cannot be overstated, ranging from the infinite series to the number theory, regularization in theoretical physics, the Casimir force, and many other fields.

Key words: Zeta function, analytic continuation, complex plane integration.

Definition of the Zeta Function and its generalization

Consider the complex variable $s = \sigma + it$ with σ and t being real. For $\sigma > 1$ the series

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

is convergent. This defines the zeta function, already known to Euler (Euler, 1738), (Euler, 1740), the properties of which were discovered by Riemann (Riemann, 1859) more than 100 years after Euler's works.

One generalization of this function is the following function

$$\zeta(s,a)=\sum_{n=0}^{+\infty}\frac{1}{(n+a)^s} \tag{2}$$

with $a>0$ due to Hurwitz (Hurwitz, 1932). For $a=1$ one recovers the zeta function. Working with the Hurwitz zeta allows us to obtain more general results that can immediately be translated for the zeta function in the case $\zeta(s,1)$.

By considering the definition of the Γ function

$$\Gamma(s)=\int_0^{+\infty} x^{s-1}e^{-x}dx \tag{3}$$

and by applying the substitution $x\to(n+a)x$, where n is an integer, we get

$$\Gamma(s)=\int_0^{+\infty} x^{s-1}(n+a)^s e^{-x(n+a)}dx\ , \tag{4}$$

that is

$$\Gamma(s)\frac{1}{(n+a)^s}=\int_0^{+\infty} x^{s-1}e^{-x(n+a)}dx\ . \tag{5}$$

Summing of n on both sides of (5) and using (2) leads to

$$\Gamma(s)\zeta(s,a)=\int_0^{+\infty} x^{s-1}\sum_{n=0}^{+\infty}e^{-x(n+a)}dx=\int_0^{+\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx\ , \tag{6}$$

the integral converges for $\sigma>1$. This formula furnishes us with an integral expression for $\zeta(s,a)$ that will be useful to analytically extend the zeta function even for $\sigma<1$. (2) could be rewritten as

$$\zeta(s,a)=\frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx\ , \tag{7}$$

which bears resemblance to the Γ function itself (3).

The integral on the complex plane

From formula (7) we will consider the integral

$$I_a(s) = \int_{\mathcal{C}} \frac{(-z)^{s-1} e^{-az}}{1 - e^{-z}} dz \tag{8}$$

extended over the complex plane, $z \in \mathbb{C}$, on the contour $\mathcal{C}$. When observing the integrand we see that there is a branch point at $z = 0$ and that there exist simple poles for $z = \pm 2n\pi i$, where $n = 1, 2, 3, \ldots$. Assuming there is a cut on the real positive axis, we will make use of a Hankel's type of contour for the integral (Hankel, 1869). It comes from $+\infty$ just above the real positive axis, goes around the origin and returns back to $+\infty$ this time below the real positive axis, so it does not contain any of the above mentioned poles, and does not pass through the branch point $z = 0$. We conclude that such an integral (8) provides us with an analytical function for *all values of* s. We can write the relation

$$\int_{\mathcal{C}} \frac{(-z)^{s-1} e^{-az}}{1 - e^{-z}} dz = [e^{\pi i(s-1)} - e^{-\pi i(s-1)}] \int_0^{+\infty} \frac{(-x)^{s-1} e^{-ax}}{1 - e^{-x}} dx =$$
$$2i \sin[\pi(s-1)] \int_0^{+\infty} \frac{(-x)^{s-1} e^{-ax}}{1 - e^{-x}} dx = -2i \sin(\pi s) \int_0^{+\infty} \frac{(-x)^{s-1} e^{-ax}}{1 - e^{-x}} dx \; . \tag{9}$$

Remembering the reflection property of the Γ function, for which

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \tag{10}$$

combining with the result of (9) and plugging all back in (7), we end up with an expression for Hurwitz's ζ from an integral on the complex plane

$$\zeta(s, a) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{(-z)^{s-1} e^{-az}}{1 - e^{-z}} dz = -\frac{\Gamma(1-s)}{2\pi i} I_a(s) \; . \tag{11}$$

Now, we have already noticed that the last integral $I_a(s)$ gives an analytical function for *all* values of s on the complex plane. Therefore the only possible poles of $\zeta(s, a)$ could be given by the $\Gamma(1-s)$ function, that is at the points $1, 2, 3, \ldots$.

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On the other hand, from definition (2), we already know that $\zeta(s,a)$ converges for $\sigma > 1$. We, therefore, conclude that the only possible pole for $\zeta(s,a)$ is to be found at the point $s = 1$, and it is a simple pole, because the Γ function has only simple poles for negative integers, of the form

$$\Gamma(x) = \frac{(-1)^n}{n!(x+n)} + \frac{(-1)^n\psi(n+1)}{n!} + \mathcal{O}(x+n) \; , \tag{12}$$

where $\psi(x) = d/dx[\ln(\Gamma(x))]$. For $s = 1$, $I_a(1)$ is written as

$$I_a(1) = \int\limits_{\mathcal{C}} \frac{e^{-az}}{1-e^{-z}} dz = 2\pi i \operatorname*{Res}_{z=0} \left(\frac{e^{-az}}{1-e^{-z}} \right) = 2\pi i \; , \tag{13}$$

because for $a > 0$ the integrand is zero at infinity, and its residue is +1. Hence, from (11) we obtain that

$$\lim_{s\to 1} \frac{\zeta(s,a)}{\Gamma(1-s)} = -1 \; . \tag{14}$$

Equation (12) tells us that $\Gamma(1-s)$ has a single pole for $s = 1$ with the residue of -1. It, therefore, follows that $\zeta(s,a)$ has $s = 1$ as the only singularity, which is a single pole with the residue of +1.

Functional equation

Formula (11) provides an expression for the Hurwitz zeta function $\zeta(s,a)$ which is valid for all values of $s \in \mathbb{C}\backslash\{1\}$ by means of an expression containing a complex integral (8) $I_a(s)$ (Hurwitz, 1932).

Consider now the real positive point $(2N+1)\pi$, N being an integer, and define the Hankel's contour $\mathcal{C}_N$ in analogy to the previous one encountered. The path runs from the point $(2N+1)\pi$ towards 0 just above the positive real axis, goes around the origin $z = 0$ in a counterclockwise direction without intersecting it, and returns to the original point just below the real positive axis, and does not contain any of the simple points $\pm 2n\pi i$ of the integrand. Consider then a circle $\mathcal{C}'_N$ centered in the origin with the radius $(2N+1)\pi$, ending at the beginning of $\mathcal{C}_N$. The full path $\mathcal{C}_N + \mathcal{C}'_N$ is closed, and the origin lies outside of it, therefore we obtain

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$$-\frac{1}{2\pi i}\int\limits_{\mathcal{C}_N+\mathcal{C}'_N}\frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz=\sum_{n=1}^{N}(R_n^+ + R_n^-)\ , \tag{15}$$

where R_n^+ and R_n^- are the residues of the integrand at the points $+2n\pi i$ and $-2n\pi i$ respectively, for $n = 1, \ldots, N$. This is true because all the poles are inside the above defined contour, and the minus sign is necessary as this contour runs clockwise. The residues at points $\pm 2\pi in$ are given by

$$R_n^{\pm}=(2\pi n)^{s-1}e^{\mp\frac{i}{2}\pi(s-1)}e^{\mp 2\pi ina}\ , \tag{16}$$

hence

$$R_n^+ + R_n^- = (2\pi n)^{s-1}\,2\sin\left(\frac{\pi}{2}s+2\pi na\right)=\frac{2\sin\left(\frac{\pi}{2}s\right)}{(2\pi)^{1-s}}\frac{\cos(2\pi an)}{n^{1-s}}+ \frac{2\cos\left(\frac{\pi}{2}s\right)}{(2\pi)^{1-s}}\frac{\sin(2\pi an)}{n^{1-s}}\ . \tag{17}$$

For a large N, the first part of the contour, $\mathcal{C}_{\mathcal{N}}$, becomes the Hankel's contour already encountered in (8), $\lim_{N\to+\infty}\mathcal{C}_{\mathcal{N}}=\mathcal{C}$. The integral on the circle $\mathcal{C}'_N$ does not contribute in the limit $N\to+\infty$. In fact, parametrizing the variable on the circle path $z=(2N+1)\pi e^{i\theta}$, $-\pi\le\theta\le+\pi$ and for $a>0$ one has

$$\left|\int\limits_{\mathcal{C}'_N}\frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz\right|\le\int_{-\pi}^{+\pi}[(2N+1)\pi]^s e^{-(2N+1)\pi}d\theta = 2\pi[(2N+1)\pi]^s e^{-(2N+1)\pi}, \tag{18}$$

and this expression clearly goes to zero for $N\to+\infty$.

In the limit $N\to+\infty$ applied to (15), we therefore obtain

$$\lim_{N\to+\infty}-\frac{1}{2\pi i}\int\limits_{\mathcal{C}_N}\frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz=-\frac{1}{2\pi i}\int\limits_{\mathcal{C}}\frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz= \frac{2\sin\left(\frac{\pi}{2}s\right)}{(2\pi)^{1-s}}\sum_{n=1}^{+\infty}\frac{\cos(2\pi an)}{n^{1-s}}+\frac{2\cos\left(\frac{\pi}{2}s\right)}{(2\pi)^{1-s}}\sum_{n=1}^{+\infty}\frac{\sin(2\pi an)}{n^{1-s}}\ , \tag{19}$$

and with the aid of (11) we can eventually write the converging series

$$\zeta(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left[2\sin\left(\frac{\pi}{2}s\right) \sum_{n=1}^{+\infty} \frac{\cos(2\pi a n)}{n^{1-s}} + 2\cos\left(\frac{\pi}{2}s\right) \sum_{n=1}^{+\infty} \frac{\sin(2\pi a n)}{n^{1-s}} \right] . \tag{20}$$

By writing $a = 1$ in (20) we obtain the functional equation linking the values of $\zeta(s)$ and $\zeta(1-s)$ discovered by Riemann

$$\zeta(s) = (2\pi)^s \frac{\Gamma(1-s)}{\pi} \sin\left(\frac{\pi}{2}s\right) \zeta(1-s) \ , \tag{21}$$

and by making use of the Γ function reflection property (10), we obtain the alternative expression for the ζ function reflection property

$$2^{1-s}\Gamma(s)\zeta(s)\cos\left(\frac{\pi}{2}s\right) = \pi^s \zeta(1-s) \ . \tag{22}$$

Some special values of ζ

In the previous section, we have seen that $\zeta(s)$ is a meromorphic function on the complex plane, with a simple pole for $s = 1$. We will now discuss some notable values of this function.

$\zeta(2)$: the Basel problem

Historically, the first who posed the problem of the values of $\sum_{n=1}^{+\infty} 1/n^2$ was Mengoli in 1650 (Mengoli, 1650). Euler solved this problem in the years after 1730, and he named it after his hometown (Euler, 1740).

Consider the series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{23}$$

and divide this expression by x, obtaining the infinite polynomial

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \ . \tag{24}$$

This function is well defined for all $x \in \mathbb{R}$ and has its zeroes at the points $\pm n\pi$, for $n = 1, 2, 3, \ldots$. Hence, this polynomial can be written as an infinite product of the factors

$$\frac{\sin(x)}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$
$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots = \prod_{n=1}^{+\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) . \quad (25)$$

That has to be compared with $\zeta(2)$, namely

$$\zeta(2) = \sum_{n=1}^{+\infty}\frac{1}{n^2} = \left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right) . \quad (26)$$

Consider the coefficient of the polynomial (25) in x^2. It is obtained by picking exactly once each x^2 term in the factors, multiplyng it with all the constant terms and then adding together all those terms. An analogy that helps would be with the characteristic polynomial of a matrix, where the coefficient of the lowest term of the polynomial (in our case, x^2) is given by the trace of the matrix that is the sum of all eigenvalues. This gives for the x^2 term in (25)

$$-x^2\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right) = -\frac{x^2}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right) = -\frac{x^2}{\pi^2}\zeta(2) . \quad (27)$$

Comparing this result with the x^2 term of (24), one obtains

$$-\frac{x^2}{3!} = -\frac{x^2}{\pi^2}\zeta(2) \quad (28)$$

giving eventually the required value for ζ

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.6449 . \quad (29)$$

Notice that, even though this brilliant procedure leads to the correct value, it is wrong. In fact, one could multiply (25) by an arbitrary positive function, say $\exp(x)$, and retain the same result, because the roots of the polynomial will not change. Yet the series expansion of this new function would be quite different from (24). The culprit is that it is not valid to treat an infinite product or sum expecting it to behave like a finite one.

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$\zeta(-1)$

The infinite sum

$$1+2+3+4+5+\cdots=\sum_{n=1}^{+\infty}\frac{1}{n^{-1}} \tag{30}$$

is equivalent to $\zeta(-1)$. By writing $s=-1$ in (21), we get

$$\zeta(-1)=\frac{1}{2\pi}\frac{\Gamma(2)}{\pi}\sin\left(-\frac{\pi}{2}\right)\zeta(2) \tag{31}$$

remembering the properties of the Γ function and eq. (29) for the value of $\zeta(2)$ we have

$$\zeta(-1)=-\frac{1}{12}=1+2+3+4+5+\cdots\ , \tag{32}$$

so we end up with an astonishing result: the sum of all positive integers is not only finite, but also negative! Of course this does not mean that the usual addition rules turned out to be magically spoiled. In fact, consider the finite sum

$$1+2+3+\cdots+N=\sum_{n=1}^{N}n=\frac{N(N+1)}{2}\ , \tag{33}$$

and this expression becomes infinite when $N\to+\infty$.

What it really means is that the definition of ζ given in (1) is no longer valid whenever $\sigma\leq 1$, like in our case of $s=-1$. The $\zeta(s)$ that obeys to (21) is the analytical continuation on the whole complex plane of the one defined in (1), and they do coincide only when $\sigma>1$.

One could also think of equation (30) as an equivalence to a so–called regularized ζ function defined in (1). This technique is used very often in theoretical physics and lays on a rigorous basis, too vast to be described here. Loosely speaking, it is like considering the behavior of a function near a pole, then discarding the divergent part while retaining the finite part. The latter assigns the value of the function on the pole.

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For instance, by considering the behavior of the Γ function close to negative integers described in (12), near the origin we have

$$\lim_{x\to 0}\Gamma(x)=\frac{1}{x}-\gamma+\frac{6\gamma^2+\pi^2}{12}x+\mathcal{O}(x^2) \tag{34}$$

so that a regularized Γ function should assume the value

$$\mathrm{Reg}[\Gamma(0)]=-\gamma\approx -0.5772\ . \tag{35}$$

It is worth noticing that the value of $\zeta(-1)$ appears in physics when computing the Casimir force in one dimension, that arises as a fluctuation of the vacuum energy when quantizing the electromagnetic field, or when computing the ground state energy of the bosonic string theory, a model which was an attempt to unify gravitational with other fundamental forces.

$\zeta(0)$

The infinite sum

$$1+1+1+1+1+\cdots=\sum_{n=1}^{+\infty}\frac{1}{n^0} \tag{36}$$

is equivalent to $\zeta(0)$. By writing $s=0$ in (21), we get

$$\zeta(0)=\lim_{s\to 1}\frac{1}{\pi}\Gamma(1)\sin\left(\frac{\pi}{2}s\right)\zeta(1-s)\ , \tag{37}$$

and, remembering that ζ has a simple pole with residue +1 in $s=1$, we have

$$\lim_{s\to 1}\sin\left(\frac{\pi}{2}s\right)\zeta(1-s)=-\frac{\pi}{2} \tag{38}$$

Thus, by putting all the values in (37) we obtain

$$\zeta(0)=-\frac{1}{2}=1+1+1+1+1+\cdots\ , \tag{39}$$

obtaining once again a negative value for an infinite sum of positive terms.

The same considerations given for the case of $\zeta(-1)$ apply here.

Other useful information on complex functions could be found for instance in (Denjoy, 1926), (Wolff, 1926), (Došenović, 2018), (Todorčević, 2019).

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ДЗЕТА-ФУНКЦИЯ И ЕЕ ОСОБЕННОСТИ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА:
27.25.17 Метрическая теория функций,
27.33.00 Интегральные уравнения,
27.39.29 Приближенные методы функционального анализа
ВИД СТАТЬИ: обзорная статья

Резюме:

Введение/цель: В данной статье представлены некоторые особенности дзета-функции, а также ее применение в математическом анализе, особое внимание уделено формуле «golden nugget» при вычислении бесконечной суммы 1 + 2 + 3 + • • •. В статье также приводятся примеры ее применения в физике.

Методы: Интеграция в комплексной плоскости и свойства гамма-функции используются на всех этапах: от определения функции до ее аналитического расширения.

Результаты: Из первоначального определения функции ζ (s), относящейся к s> 1, выводится мероморфная функция на всей комплексной плоскости с простым полюсом в s = 1.

Выводы: Дзета-функция безусловно играет важнейшую роль во многих областях, начиная от бесконечных рядов и заканчивая теорией чисел, регуляризации в теоретической физике, силе Казимира и пр.

Ключевые слова: дзета-функция, аналитическое продолжение, интеграция в комплексной плоскости.

ЗЕТА-ФУНКЦИЈА И НЕКЕ ЊЕНЕ ОСОБИНЕ

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: прегледни рад

Сажетак:

Увод/циљ: У раду су приказане неке особине зета-функције, као и њена примена у математичкој анализи, нарочито формула „golden nugget” за вредност бесконачног збира 1+2+3+…. Такође, поменуте су и неке њене примене у физици.

Методе: Интеграције комплексне равни и особине гама-функције биће искоришћене од дефиниције функције до њене аналитичке екстензије.

Резултати: Од оригиналне дефиниције функције ζ(s) валидне за s >1, добија се мероморфна функција на целој комплексној равни са простим полом на s = 1.

Закључак: Изузетан значај зета-функције је несумњив, од бесконачних низова до теорије бројева, регуларизације у теоријској физици, Казимирове силе и многих других области.

Кључне речи: зета-функција, аналитичка континуација, интеграција комплексне равни.

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