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facing@uta.cl
Universidad de Tarapacá
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Torres-Silva, H.
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MAXWELL'S THEORY WITH CHIRAL CURRENTS

TEORÍA DE MAXWELL CON CORRIENTES QUIRALES

H. Torres-Silva[1]

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RESUMEN

El contenido de energía y momento de un campo electromagnético puede ser expresado enteramente, en términos de los campos a través del tensor energía momento, sin mención de las fuentes que crean los campos. Este tensor es definido introduciendo corrientes quirales. En el caso de sin fuerza se tiene $T^{00} = 0$ y $\boldsymbol{E} \times \boldsymbol{B} = 0$. Este método permite una muy simétrica derivación del contenido de energía y momento de los campos con $\boldsymbol{E} \| \boldsymbol{B}$. Esta configuración es esencial para la unificación del electromagnetismo y la gravedad, obteniendo una configuración de fuerza cero para el electrón. Para obtener esta unificación se discute la geometrización de Rainich bajo condiciones quirales.

Palabras clave: Corrientes quirales, geometrización de Rainich, tensor energía momento, unificación.

ABSTRACT

The energy and momentum content of an electromagnetic field can be expressed entirely in terms of the fields through the energy-momentum tensor with no mention of the sources creating the fields. This tensor is defined such that chiral currents are introduced. In the case of free force we have $T^{00} = 0$ *and* $\boldsymbol{E} \times \boldsymbol{B} = 0$. *This approach allows for a very symmetric derivation of the energy and momentum content of the fields with* $\boldsymbol{E} \| \boldsymbol{B}$. *This configuration is essential to the unification of electromagnetism and gravity, obtaining a force-free configuration for the electron. To obtain this unification the Rainich geometrization under chiral conditions is discussed.*

Keywords: Chiral currents, Rainich geometrization, energy-momentum tensor, unification.

INTRODUCTION

Although it's existence in this region of the universe has yet to be confirmed, magnetic charge has a strong theoretical and pedagogical history from Gilbert's initial magnetic theory to present day unified theories. Maxwell's equations for electromagnetic theory have source terms for electric charges and currents, but none for their magnetic counterparts. This, of course, reflects the experimental fact that magnetic monopoles have never been discovered [1]. Students however, should not be sheltered from the possible existence of magnetic monopoles. For example, grand unified theories, by definition, admit the existence of magnetic monopoles, and their absence represents a challenge for particle physicists and cosmologists alike [2].

Probably the most famous theoretical use of magnetic monopoles is the Dirac quantization condition [3]. The absence of magnetic source terms from Maxwell's equations allows the introduction of the electromagnetic potential, which takes on a fundamental role in the quantum theory of electrodynamics. Dirac's argument then proceeds by requiring the potential to be well defined even in a theory with magnetic monopoles, leading to the quantization of the product of the fundamental electric and magnetic charges.

The classical theory of electromagnetism can be formulated using the fields themselves as the fundamental objects and there is no need to invoke the potential formalism. This then leaves the obvious lack of symmetry between the dynamical and non dynamical Maxwell equations

[1] Instituto de Alta Investigación. Universidad de Tarapacá. Antofagasta N° 1520. Arica, Chile. E-mail: htorres@uta.cl

that can naturally be filled by postulating the existence of magnetic source terms, thus making all of Maxwell's equations dynamical [4, 5].

This short note is intended to show a symmetric derivation of the electromagnetic energy-momentum tensor from the Lorentz force law and Maxwell's equations, extended to include chiral magnetic as well as chiral electric source terms.

In section 2 we briefly review Maxwell's theory of electromagnetism with both electric and magnetic charges and currents displaying it's full theoretical symmetry. The energy momentum tensor is defined in section 3 and it's usual form is shown to follow naturally from a theory with both electric and magnetic chiral currents. In section 4, we give the Rainich geometrization under chiral conditions. We close with some discussion of our derivation in connection with unification of electromagnetism and gravity.

We will use Gaussian units and a diagonal space-time metric $g^{\mu\nu}$ with $-g^{00} = g^{11} = g^{22} = g^{33} = 1$. Greek indices will take the values 0 through 3 and Roman indices 1 through 3.

MAXWELL'S THEORY WITH CHIRAL ELECTRIC AND MAGNETIC CURRENT

The equations of electrodynamics can be extended to include chiral magnetic and electric current into Ampère's law and Faraday's law respectively. I will use the subscripts *e* and *m* to distinguish between the electric and magnetic charges and currents. In 3-vector notation Maxwell's equations for the case of chiral approach [9] without charges and monopoles (ρ_e, J_e, ρ_m, $J_m = 0$) are:

$$\nabla \cdot \boldsymbol{E} = 0 \tag{1a}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = -\frac{imc}{\hbar}\boldsymbol{E} \tag{1b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1c}$$

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = \frac{imc}{\hbar}\boldsymbol{B} \tag{1d}$$

The Lorentz invariance of the theory can be made manifest by combining the fields into the usual electromagnetic field tensor $F^{\mu\nu}$

$$F^{\mu\nu} = -F^{\nu\mu}, \quad F^{0i} = E_i, \quad F^{ij} = \varepsilon_{ijk} B_k$$

(ε_{ijk} is the totally antisymmetric Levi-Civita symbol.) The first pair of Maxwell's equations (1a) and (1b) then become

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \frac{4\pi}{c} J_e^{\mu} = -\frac{imc}{\hbar} E_e^{\mu} \tag{2}$$

Where $J_e^{\mu} = \left(c\rho_e, J_e\right)$ is the chiral electric 4-current. The electric continuity equation follows from the antisymmetry of $F^{\mu\nu}$.

The second pair of Maxwell's equations can be written in 4-vector notation by defining the pseudotensor $\tilde{F}^{\mu\nu}$, the dual of $F^{\mu\nu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \tag{3}$$

$\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric pseudotensor, = +Ĩ,1 or 0, if $\alpha\beta\mu\nu$ is an even, odd, or no, permutation of 0123. Equations (1c) and (1d) then read:

$$\frac{\partial \tilde{F}^{\mu\nu}}{\partial x^{\nu}} = \frac{4\pi}{c} J_m^{\mu} = \frac{imc}{\hbar} B_m^{\mu} \tag{4}$$

Where $J_m^{\mu} = \left(c\rho_m, J_m\right)$ is the chiral magnetic 4-current.

Thus the specification of the divergence of an antisymmetric tensor and the divergence of it's dual completely determines the tensor (and hence in this case, the fields) is a generalization of Helmholtz's theorem to four dimensional space time [8], the divergence of the dual playing the role of the curl.

The Lorentz force per unit volume on an assembly of charges is given by:

$$f^{\mu} = \frac{1}{c}\left(J_{e\nu} F^{\mu\nu} + J_{m\nu} \tilde{F}^{\mu\nu}\right) \tag{5}$$

Before we dive into the derivation of the full energy-momentum tensor, we will take a moment to derive Poynting's theorem from (5) using 3-vector notation. The 'zeroth' component of (5) is:

$$cf^0 = \boldsymbol{J}_e \cdot \boldsymbol{E} + \boldsymbol{J}_m \cdot \boldsymbol{B}$$

which is the work done by the fields on the charges per unit volume per unit time. Using (1b) and (1d) to eliminate the currents leads to:

$$cf^0 = -\nabla \cdot \left(\frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}\right) - \frac{\partial}{\partial t}\left(\frac{c}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\right)$$

which has the interpretation of: the energy per unit volume per unit time gained by the charges is equal to the energy lost by the fields through the divergence of the Poynting vector and the time rate of change of the energy density.

Note that the electric and magnetic currents were treated on equal footing, as were equations (1b) and (1d).

We now turn to the derivation of the full energy-momentum tensor. We will use 4-vector notation, which may hide some of the details. If so, the reader is encouraged to mimic the above calculation using the 3-vector part of (5) to derive the Maxwell stress tensor.

THE ENERGY-MOMENTUM TENSOR

A frequent approach to defining the energy-momentum tensor for the electromagnetic field is to generalize the definition of the Hamiltonian density to a covariant form [7]. This leads to what is called the canonical energy-momentum tensor. This has a number of draw backs for our current purpose. Firstly, the Hamiltonian approach requires the definition of the potential, which we do not wish to make and secondly, the canonical form is not symmetric (nor gauge invariant). An alternate method [8], and the approach taken here, of determining the energy-momentum tensor $T^{\mu\nu}$, is to define it such that the Lorentz force per unit volume (5) is the 4-divergence of the energy-momentum tensor

$$f^{\mu} \equiv \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} \qquad (6)$$

In this note, however, I wish to emphasize the fact that, with the introduction of magnetic source terms, each of Maxwell's equations is treated on equal footing, and the symmetric form for $T^{\mu\nu}$ follows naturally. In a theory with no magnetic charges only the first term in equation (5) exists, and Maxwell's dynamical equations (2) are used to write the source terms in terms of the derivatives of the fields. The remaining Maxwell equations (4) are then only used as no more than mathematical relations during the derivation.

If magnetic charges are admitted to the theory the electric and magnetic source terms are removed from the Lorentz force law (5) using Maxwell's equations (2) and (4), giving

$$f^{\mu} = \frac{1}{4\pi}\left(F^{\mu}{}_{\beta} \frac{\partial F^{\beta\alpha}}{\partial x^{\alpha}} + \tilde{F}^{\mu}_{\beta} \frac{\partial \tilde{F}^{\beta\alpha}}{\partial x^{\alpha}} \right)$$

Substituting in the definition of the dual field tensor (3) and using the identity

$$g_{\nu\beta}\varepsilon^{\rho\sigma\mu\nu}\varepsilon^{\gamma\delta\alpha\beta} = -\left(g^{\rho\delta}g^{\sigma\gamma}g^{\mu\alpha} + g^{\rho\gamma}g^{\sigma\alpha}g^{\mu\delta} + g^{\rho\alpha}g^{\sigma\delta}g^{\mu\gamma} - \right.$$
$$\left. - g^{\rho\gamma}g^{\sigma\delta}g^{\mu\alpha} - g^{\rho\delta}g^{\sigma\alpha}g^{\mu\gamma} - g^{\rho\alpha}g^{\sigma\gamma}g^{\mu\delta} \right)$$

gives

$$f^{\mu} = \frac{1}{4\pi}\left(F^{\mu}{}_{\beta} \frac{\partial F^{\beta\alpha}}{\partial x^{\alpha}} + \frac{\partial F^{\mu}{}_{\beta}}{\partial x^{\alpha}} F^{\beta\alpha} + \frac{1}{2} g^{\alpha\mu} F^{\gamma\delta} \frac{\partial F_{\gamma\delta}}{\partial x^{\alpha}} \right)$$

which can then be written as a total divergence

$$f^{\mu} = \frac{1}{4\pi}\frac{\partial}{\partial x^{\alpha}}\left(F^{\mu}{}_{\beta}F^{\beta\alpha} + \frac{1}{4} g^{\alpha\mu} F^{\gamma\delta} F_{\gamma\delta} \right)$$

By comparing this with (6), the symmetric energy-momentum tensor is obtained

$$T^{\alpha\mu} = \frac{1}{4\pi}\left(F^{\mu}{}_{\beta}F^{\beta\alpha} + \frac{1}{4} g^{\alpha\mu} F^{\gamma\delta} F_{\gamma\delta} \right)$$

This ends the calculation, but it is instructive to write out the components of this tensor in the more familiar 3-vector forms whose physical interpretation are given by integrating (6):

Energy density: $-T^{00} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right)$

Poynting vector: $-cT^{0i} = -cT^{i0} = \frac{c}{4\pi}\left(\boldsymbol{E} \times \boldsymbol{B} \right)_i$

Maxwell stress tensor:

$$T^{ij} = T^{ji} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right) \right)$$

In the special case where ***E=iB***, we have $-T^{00} = 0$ and $-cT^{0i} = -cT^{i0} = \frac{c}{4\pi}\left(\boldsymbol{E} \times \boldsymbol{B} \right)_i = 0$.

THE RAINICH GEOMETRIZATION UNDER CHIRAL CONDITIONS

In the literature, the algebraic Rainich conditions are obtained using special methods as spinors, duality rotations, eigenvalue problem for certain 4 x 4 matrices or artificial tensors of 4th order. Here we show an elementary procedure for to deduce an identity satisfied by determined class of second order tensors in arbitrary R_4, from which the Rainich expressions are immediate. This result is applied to chiral conditions.

Rainich [10-15] proposed a unified field theory for the geometrization of the electromagnetic field, whose basic relations can be obtained from the Einstein-Maxwell field equations under the Einstein notation:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \\ T_{\mu\nu} &= \frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} - F_{\mu\sigma} F_{\nu\tau} g^{\sigma\varsigma} \end{aligned} \qquad (7)$$

where $R_{\mu\nu}$, $R = R^{\mu}{}_{\mu}$ and $F_{\mu\sigma}$ are the Ricci tensor, scalar curvature and Faraday tensor [10], respectively.

If in (1) we contract μ with ν we find that:

$$R = 0 \qquad (8)$$

then (1) adopts the form:

$$R_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} + F_{\mu\sigma} F_{\nu\tau} g^{\sigma\varsigma} \qquad (9)$$

used by several authors [10-15] to show the identity:

$$R_{\mu\tau} R_{\nu}^{\tau} = \frac{1}{4} (R_{ab} R^{ab}) g_{\mu\nu} \qquad (10)$$

If $F_{\sigma\tau}$ is known, then (9) is an equation for $g_{\mu\nu}$ and our situation belongs to general relativity. The Rainich theory represents the inverse process: To search a solution of (8) and (10) (plus certain differential restrictions), and after with (9) to construct the corresponding electromagnetic field; from this point of view $F_{\sigma\tau}$ is a consequence of the space time geometry.

The essence of the chiral argument advanced here is that real world-space is not euclidean and that space is generally curved into the time dimension, consistent with the theory of general relativity. The curvature may not be sufficient to become obvious in a local context. However, it is sufficient to break the time-reversal symmetry that seems to characterize the laws of physics. Not only does it cause perpetual time with respect to all mass, but actually identifies a fixed direction for this It creates an arrow of time and thereby eliminates an inconsistency in the logic of physics: how reversible microscopic laws can underpin an irreversible macroscopic world. General curvature of space breaks the time-reversal symmetry and produces chiral space, manifest in the right-hand force rule of electromagnetism. The presence of matter causes space to curl up and curvature of space generates matter.

The fact that most other fundamental laws of physics do not refers the **chirality** of space, nor the arrow of time, confirms that the curvature on a local scale is barely detectable.

Now under chiral conditions $\frac{\partial F_{\nu\sigma}}{\partial x_{\sigma}} = 0$, $\frac{\partial \tilde{F}_{\nu\sigma}}{\partial x_{\sigma}} = \frac{\partial\, {}^{*}F_{\nu\sigma}}{\partial x_{\sigma}} = 0$

with, $\mu, \sigma = 4$, $\tilde{F}_{\nu\sigma} = {}^{*}F_{\nu\sigma}$ and $\partial / \partial t \to \partial / \partial t (1 + T\nabla\times)$, the Maxwell mixed tensor is

$$T^{\mu}{}_{\nu\ Maxwell} \equiv -\frac{1}{4\pi} \left[F^{u\sigma} F_{\nu\sigma} - \frac{1}{4} \delta^{\mu}{}_{\nu} F^{\tau\sigma} F_{\tau\sigma} \right] \qquad (11a)$$

$$T^{\mu}{}_{\nu\ Maxwell} \equiv -\frac{1}{8\pi} \left[F^{u\sigma} F_{\nu\sigma} + {}^{*}F^{u\sigma}\, {}^{*}F_{\nu\sigma} \right] \qquad (11b)$$

with $iF^{u\sigma} = {}^{*}F^{u\sigma}$ and $iF_{\nu\sigma} = {}^{*}F_{\nu\sigma}$. In this case we have $T^{\mu}{}_{\nu\ Maxwell} = 0$, then equation (4) is

$$R_{\mu\tau} R_{\nu}^{\tau} = \frac{1}{4} (R_{ab} R^{ab}) g_{\mu\nu} \Leftrightarrow F_{\mu\tau} F_{\nu}^{\tau} = \frac{1}{4} (F_{ab} F^{ab}) g_{\mu\nu} \qquad (12)$$

Only in this case we have a complete unification, ie, a unified field theory between the gravity and the electromagnetism

DISCUSSION

The energy and momentum content of an electromagnetic field can be expressed entirely in terms of the fields through the energy-momentum tensor with no mention of the sources creating the fields. This tensor is defined such that it's divergence gives the Lorentz force. That is, any change in the energy and momentum of a charge distribution is given by the (negative of the) change in the energy and momentum of the fields. In the case of free force we have $T^{00} = 0$ and $\boldsymbol{E} \times \boldsymbol{B} = 0$. Here there is no existence of magnetic charges, because they have never been found in nature. This approach allows for a very symmetric derivation of the energy and momentum content of the fields with $\boldsymbol{E} \| \boldsymbol{B}$. This configuration is essential to the unification of electromagnetism and gravity, obtaining a force free configuration for the electron [16]. To obtain this unification, the Rainich geometrization under chiral conditions is discussed.

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