Least Change Secant Update Methods for Nonlinear Complementarity Problem

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Abstract

In this work, we introduce a family of Least Change Secant Update Methods for solving Nonlinear Complementarity Problems based on its reformulation as a nonsmooth system using the one-parametric class of nonlinear complementarity functions introduced by Kanzow and Kleinmichel. We prove local and superlinear convergence for the algorithms. Some numerical experiments show a good performance of this algorithm.

Key words: nonsmooth systems; nonlinear complementarity problems; generalized Jacobian; quasi-Newton methods; least change secant update methods; local convergence; superlinear convergence

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Métodos secantes de cambio mínimo para el problema de complementariedad no lineal

Resumen

En este trabajo generamos una familia de métodos secante de cambio mínimo para resolver Problemas de Complementariedad no Lineal vía su reformulación como un sistema de ecuaciones no lineales no diferenciable usando una clase de funciones de complementariedad propuesta por Kanzow and Kleinmichel. Bajo ciertas hipótesis demostramos que esta familia proporciona algoritmos local y superlinealmente convergentes. Experimentos numéricos preliminares demuestran un buen desempeño de los algoritmos propuestos.

Palabras clave: sistemas no diferenciables; complementariedad no lineal; métodos cuasi-Newton

1 Introduction

Let $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = (F_1(x), \ldots, F_n(x))$ be a continuously differentiable mapping. The Nonlinear Complementarity Problem, NCP for short, consists of finding a vector $x \in \mathbb{R}^n$ such that,

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0. \tag{1}$$

Here, $y \geq 0$ for $y \in \mathbb{R}^n$ means $y_i \geq 0$ for all $i = 1, \ldots, n$. The third condition in (1) requires that the vectors x and $F(x)$ are orthogonal; for this reason, it is called complementarity condition.

The NCP arises in many applications such as Friction Mechanical Contact problems [1], Structural Mechanics Design problems, Lubrication Elasto-hydrodynamic problems [2], Traffic Equilibrium problems [3], as well as problems related to Economic Equilibrium Models [4]. The importance of NCP in the areas of Physics, Engineering and Economics is due to the fact that the concept of complementarity is synonymous with the notion of system in equilibrium. In recent years, various techniques have been studied to solve the NCP, one of which is to reformulate it as a nonsmooth system of nonlinear equations by using special functions called complementarity functions [5]. A function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(a, b) = 0 \iff a \geq 0, \quad b \geq 0, \quad ab = 0, \tag{2}$$

is called a complementarity function.

Geometrically, the equivalence (2) means that the trace of the function φ obtained by the intersection with the xy plane is the curve formed by the positive semiaxes x and y, which is not differentiable at $(0,0)$. This lack of smoothness on the curve imply the nondifferentiability of the function φ.

In order to reformulate the NCP as a system of nonlinear equations, it is necessary to consider a complementarity function φ and to define $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi\left(x\right) = \begin{pmatrix} \varphi(x_1, F_1(x)) \\ \vdots \\ \varphi(x_n, F_n(x)) \end{pmatrix}, \tag{3}$$

then it follows from lack of smoothness of φ that the nonlinear system of equations

$$\Phi\left(x\right) = 0 \tag{4}$$

is nonsmooth. From the definition of a complementarity function (2) it follows that a vector x_* solves the system (4), if and only if, x_* solves the NCP. Different algorithms have been proposed for solving the reformulation of the NCP by a nonsmooth system of nonlinear equations (4) like nonsmooth Newton methods [6], nonsmooth quasi-Newton methods [7],[8],[9], among others [10], [11],[12],[13].

There are many complementarity functions, but the most used has been the minimum function [14] and the Fischer-Burmeister function[15], defined respectively by

$$\varphi(a,b) = \min\left\{a,b\right\}, \qquad \varphi(a,b) = \sqrt{a^2 + b^2} - a - b. \tag{5}$$

The minimum function is nonsmooth at the points of the form (a,a), while the Fischer-Burmeister function is not nonsmooth at $(0,0)$. In 1998, Kanzow and Kleinmichel [15] introduced an one-parametric class of complementarity functions φ_λ defined by

$$\varphi_\lambda(a,b) = \sqrt{(a-b)^2 + \lambda ab} - a - b, \tag{6}$$

where $\lambda \in (0, 4)$ and which we will refer to throughout this work as Kanzow function. This function is nonsmooth at $(0, 0)$. For any other vector in $\mathbb{R}^2$, the gradient vector of φ_λ is defined by

$$\nabla \varphi_\lambda(a, b) = \begin{pmatrix} \dfrac{2(a - b) + \lambda b}{2\sqrt{(a - b)^2 + \lambda ab}} - 1 \\ \dfrac{-2(a - b) + \lambda a}{2\sqrt{(a - b)^2 + \lambda ab}} - 1 \end{pmatrix} = \begin{pmatrix} \chi(a, b) - 1 \\ \psi(a, b) - 1 \end{pmatrix}. \quad (7)$$

In [16], the author makes a carefully analysis of this function and deduces some important bounds that we will use later. Moreover, In the special case $\lambda = 2$, the function φ_λ reduces to theFischer-Burmeister function, whereas in the limiting case $\lambda \to 0$, the function φ_λ becomes a multiple of the minimum function. In what follows, we denote by Φ_λ the function defined in (3) and obtained by the complementarity function φ_λ.

In this work, we propose a nonsmooth quasi Newton method for solving the NCP using the system $\Phi_\lambda(x) = 0$; for this method, we prove local convergence. Moreover, we introduce a family of least change secant update for solving the NCP based on the nonsmooth system of equations $\Phi_\lambda(x) = 0$ and, for these family, we prove local and superlinear convergence under suitable assumptions.

We organize this paper as follows. In Section 2, we reformulate the NCP as a nonsmooth system of equations using the Φ_λ function and we characterize a subset of the generalized Jacobian of Φ_λ in x. In the first part of Section 3, we propose a new algorithm quasi-Newton for solving the nonsmooth system of nonlinear equations $\Phi_\lambda(x) = 0$ and, for this method, we develop the local convergence theory. In the second part, we introduce a family of least change secant update methods following the theory developed in [17] for this type of methods. We prove, under suitable assumptions, local and superlinear convergence. In Section 4, we analyze numerically, the local performance of the algorithms introduced in the last section, for which we use 8 test problems proposed in [14],[18]. Four of this are applications problems to Economic Equilibrium and Game Theory. Finally, Section 5 contains some remarks on what we have done in this paper and present possibilities for future works.

2   Reformulation of NCP using the Kanzow function

In this section, we reformulate the NCP as a nonsmooth system of equations and from the definition of the generalized Jacobian given in [19], we construct a subset of matrices of the generalized Jacobian of Φ_λ at x. Then we show that this subset at a solution of the system $\Phi_\lambda(x) = 0$ is a compact set.

Our reformulation of NCP as a system of equations is based on the Kanzow complementarity function φ_λ defined by (6) and the Φ_λ function defined in the last section. Exploiting (2) it is readily seen that the NCP is equivalent to the following system of nonsmooth equations

$$\Phi_\lambda(x) = 0. \tag{8}$$

The most popular method for solving a differentiable system of nonlinear equations $G(x) = 0$ is Newton's method [8], which require calculating, at each iteration, the Jacobian matrix of G. There are situations where the derivatives of G are not available, or are difficult to calculate. For this cases, a less expensive alternative and widely used for solving $G(x) = 0$ are the quasi-Newton methods [8] which use, at each iteration, a matrix approximation to the Jacobian matrix. Among the latter are the so-called least change secant update methods [10], which form a family characterized by the fact that, at each iteration, the Jacobian approximation satisfies a secant equation [10] with a minimum variation property relative to some matrix norm. The price of using an approximation to the Jacobian Matrix is reflected in the decrease of the speed of convergence of the respective quasi-Newton method.

When a function is not differentiable as in the case of the function Φ_λ, the term "Jacobian matrix" does not make sense. Fortunately, Frank H. Clarke introduced the concept of Generalized Jacobian that extends the matrix Jacobian concept for some non-differentiable functions [19]. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitzian function. The Generalized Jacobian of F at x is the set given by

$$\partial F(x) = hull \left\{ \lim_{k \to \infty} F'(x_k) \in \mathbb{R}^{n \times n} : x_k \to x, \ x_k \in D_F \right\} \tag{9}$$

where D_F is the set of all points where F is differentiable and hull denotes the convex envelope of the set. The $\partial F(x)$ is a nonempty, convex and compact set [19]. In the particular case in which F is differentiable at x, $\partial F(x)$ has a single element: the Jacobian Matrix of F at x, $F'(x)$.

Since the Kanzow function φ_λ is locally Lipschitz continuous [16], so is the Φ_λ function. Thus, the Generalized Jacobian of $\Phi_\lambda(x)$ exists. In order to build matrices in this set, we consider a sequence of vectors in $\mathbb{R}^n$, $\{y_k\}$, which converges to x and such that $\Phi'_\lambda(y_k)$ exists, then we show that $\lim_{k\to\infty} \Phi'_\lambda(y_k)$ exists. To classify the indices of the components of x, we define the set

$$\beta = \beta(x) = \{i : x_i = F_i(x) = 0\}. \tag{10}$$

The sequence[1] that we will use is

$$y_k = x + \varepsilon_k z, \tag{11}$$

where $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\lim_{k\to\infty} \varepsilon_k = 0$ and the vector z is chosen such that $z_i \neq 0$ where $i \in \beta$. Obviously y_k converges to x when $k \to \infty$. To analyze the differentiability of Φ_λ in y_k, we consider two cases. If $i \notin \beta$ then $x_i \neq 0$ or $F_i(x) \neq 0$, by the continuity of F_i, we can assume ε_k so small that $y_i^k \neq 0$ or $F_i(x) \neq 0$, for which Φ_λ is differentiable at y_k. If $i \in \beta$, the $z_i \neq 0$; therefore, $y_i^k \neq 0$, which is sufficient for Φ_λ to be differentiable at y_k.

By differentiability of Φ_λ at y_k, the Jacobian matrix of Φ_λ at y_k, exist and its i th row is given by

$$\left[\Phi'_\lambda(y_k)\right]_i = \left(\chi(y_i^k, F_i(y_k)) - 1\right) e_i^T + \left(\psi(y_i^k, F_i(y_k)) - 1\right) \nabla F_i(y_k)^T$$

with χ and ψ defined by (7) and $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

For calculating the $\lim_{k\to\infty} \Phi'_\lambda(y_k)$, we consider two cases: If $i \notin \beta$, by continuity of i th row of $\Phi'_\lambda(y_k)$ we have $\lim_{k\to\infty} \nabla \varphi_\lambda(y_i^k, F_i(y_k))^T$ is $[H]_i$

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[1]We consider the same sequence of [20] which is used for the theoretical developments with the Fischer function.

where $[H]_i = (\chi(x_i, F_i(x)) - 1) e_i^T + (\psi(x_i, F_i(x)) - 1) \nabla F_i(x)^T$. If $i \in \beta$, from (11), we have

$$y_i^k = \varepsilon_k z_i, \tag{12}$$

further, by the *Taylor's theorem*,

$$F_i(y_k) = F_i(x + \varepsilon_k z) = F_i(x) + \varepsilon^k \nabla F_i \left(\zeta^k\right)^T z = \varepsilon^k \nabla F_i \left(\zeta^k\right)^T z, \tag{13}$$

where $\zeta^k \to x$ when $k \to \infty$. Substituting (12) and (13) in the i th row of $\Phi'_\lambda(y_k)$, $i \in \beta$, we obtain

$$\lim_{k \to \infty} \nabla \varphi_\lambda(y_i^k, F_i(y_k))^T = [H]_i$$

with $[H]_i = (\chi(z_i, \nabla F_i(x)^T z) - 1) e_i^T + (\psi(z_i, \nabla F_i(x)^T z) - 1) \nabla F_i(x)^T$. Then, for all $i = 1, \ldots, n$, the limit when $k \to \infty$ of each row of $\Phi'_\lambda(y_k)$ exist. Therefore,

$$\lim_{k \to \infty} \Phi'_\lambda(y_k) = H, \tag{14}$$

where

$$[H]_i = \begin{cases} (\chi(x_i, F_i(x)) - 1) e_i^T + (\psi(x_i, F_i(x)) - 1) \nabla F_i(x)^T, & i \notin \beta \\ (\chi(z_i, \nabla F_i(x)^T z) - 1) e_i^T + (\psi(z_i, \nabla F_i(x)^T z) - 1) \nabla F_i(x)^T, & i \in \beta. \end{cases}$$

Because there is an uncountable of ways of choosing the vector z, we have an uncountable set of matrices H in $\partial \Phi_\lambda(x)$ which can be calculated by the above procedure.

Now, we consider the particular case where x^* is a solution of $\Phi_\lambda(x) = 0$. If there is any index i such that $x_i = F_i(x^*) = 0$, then x^* is called a degenerate solution. We will denote the matrices (14) in x^* by $H_*(z)$. The set of these matrices we will call $\mathcal{Z}_*$. Clearly, for each $z \in \mathbb{R}^n$, there is a matrix $H_*(z)$. Thus, $\mathcal{Z}_*$ is an infinite set and further it is a compact set. To verify the compactness, it is sufficient to demonstrate that it is closed, since $\mathcal{Z}_* \subseteq \partial \Phi(x^*)$, which is compact [19], [16].

3 Algorithm and convergence theory

In the first part of this section, we propose a new quasi-Newton algorithm for solving the system $\Phi_\lambda(x) = 0$ and we develop the local convergence

theory for this method. In the second part, we develop a family of least change secant update methods, following [17]. For these family, we prove local and superlinear convergence under suitable assumptions. The following algorithm is the basic quasi-Newton algorithm applied to $\Phi_\lambda(x) = 0$.

Algorithm 1. Given x_0 an initial approximation to the solution of the problem and $\lambda \in (0, 4)$, compute $x_{k+1} = x_k - B_k^{-1}\Phi_\lambda(x_k)$, for $k = 1, 2, \ldots$, where

$$[B_k]_i = \begin{cases} (\chi(x_i, F_i(x)) - 1)\, e_i^T + (\psi(x_i, F_i(x)) - 1)\, [A_k]_i\,, & i \notin \beta \\ (\chi(z_i, [A_k]_i\, z_k) - 1)\, e_i^T + (\psi(z_i, [A_k]_i\, z_k) - 1)\, [A_k]_i\,, & i \in \beta. \end{cases} \tag{15}$$

Here $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a canonical basis of $\mathbb{R}^n$, the matrix A_k is an approximation of the Jacobian matrix of F at x_k (to see Section 6) and $z^k \in \mathbb{R}^n$ is such that $z_i^k \neq 0$, if $x_i^k = F_i(x^k) = 0$.

Under the following assumptions, we will prove that the sequence generated by the basic quasi-Newton Algorithm 1 is well define and converges linearly to a solution of $\Phi_\lambda(x) = 0$.

3.1 Local assumptions

H1. There is $x^* \in \mathbb{R}^n$ such that $\Phi_\lambda(x^*) = 0$.

H2. The jacobian matrix of F is Lipschitz continuous (with constant γ) in a neighborhood of $x^* \in \mathbb{R}^n$.

H3. The matrices of the set $\mathcal{Z}_*$ are nonsingular.

From assumption H3 and by the compactness of $\mathcal{Z}_*$, we have that there is a constant μ such that for all $H_*(z) \in \mathcal{Z}_*$,

$$\|H_*^{-1}(z)\| \leq \mu. \tag{16}$$

3.2 A local convergence theory

The following two Lemmas prepare the "Theorem of the two neighborhoods".

Lemma 3.1. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $F \in C^1$, such that its Jacobian matrix verifies H2, the matrices H and B defined by (14) and (15), respectively, and given positive constants ϵ and δ. Then, for each $x \in \mathcal{B}(x^* ; \epsilon)$ and $A \in \mathcal{B}(F'(x^*) ; \delta)$, there exists a positive constant θ such that

$$\|H - B\|_\infty \leq \delta\,\tau + \epsilon\,\omega \,=\, \theta. \tag{17}$$

where $\tau = \eta \|z\|_\infty + n\,\eta \|z\|_\infty \|\nabla F_j(x^*)\|_\infty + \sqrt{2}\,n + n$ and $\omega = n\,\gamma\,(\sqrt{2} + 1)$.

See proof in [16].

Lemma 3.2. Let $r \in (0,1)$ and B the matrix defined by (15). There exist positive constants ϵ_0 and δ_0 such that,

$$\text{if} \quad \|x - x^*\|_\infty \leq \epsilon_0 \quad \text{and} \quad \|A - F'(x^*)\|_\infty \leq \delta_0,$$

the function $\mathcal{Q}$ defined by

$$\mathcal{Q}(x, A) = x - B^{-1}\Phi_\lambda(x), \tag{18}$$

is well defined, and satisfies

$$\|\mathcal{Q}(x, A) - x^*\|_\infty \leq r \, \|x - x^*\|_\infty. \tag{19}$$

Let $r \in (0,1)$, $\hat{\epsilon} > 0$ and $\delta_0 > 0$ such as $\hat{\epsilon} < \dfrac{r}{8\,\mu\,(\omega + \sqrt{2}n\eta)}$ and $\delta_0 < \dfrac{r}{8\,\mu\,\tau}$, where ω is the constant (17), η is the Lipschitz constant of $\nabla\varphi_\lambda$ given by [16] and τ and μ are defined by Lemma 3.1. We take $x \in \mathcal{B}(x^* ; \hat{\epsilon})$, $A \in \mathcal{B}(F'(x^*) ; \delta_0)$, B the matrix associated to A by the rule (15), H_* associated to $F'(x^*)$ for the same rule and H defined by (3.8).

To prove that $\mathcal{Q}$ is well defined, we must show that B^{-1} exist. For this, we consider the inequality

$$\|B - H_*\|_\infty \;\leq\; \|B - H\|_\infty + \|H - H_*\|_\infty. \tag{20}$$

The first term on the right side of (20) is bounded by (17), thus

$$\|B - H\|_\infty \leq \delta_0\,\tau + \hat{\epsilon}\,\omega \;<\; \frac{r}{8\,\mu} + \frac{\omega r}{8\,\mu\,(\omega + \sqrt{2}n\eta)}. \tag{21}$$

We bound the second term on the right of (20) using the continuity of F, the Lipschitz continuity of $\nabla\varphi_\lambda$ and the definition of infinite matrix norm.

By the continuity of F, for all $\hat\epsilon > 0$ exist $\hat\delta > 0$ such that ,

$$\text{if} \quad \|\boldsymbol{x} - \boldsymbol{x}^*\|_\infty < \hat\delta \quad \text{then} \quad |F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*)| < \hat\epsilon.$$

Let $\tilde\epsilon = \min\left\{\hat\epsilon, \hat\delta\right\}$. If $\|\boldsymbol{x} - \boldsymbol{x}^*\|_\infty < \tilde\epsilon$ then $|F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*)| < \hat\epsilon$. On the other hand,

$$
\begin{aligned}
\|H - H_*\|_\infty &= \left\|[H]_j - [H_*]_j\right\|_1 \\
&\leq n \lim_{k\to\infty} \left\|\nabla\varphi_\lambda(y_j^k, F_j(\boldsymbol{y}^k)) - \nabla\varphi_\lambda(\overline{y_j^k}, F_j(\overline{\boldsymbol{y}^k}))\right\|_\infty \\
&\leq n \lim_{k\to\infty} \left\|\nabla\varphi_\lambda(y_j^k, F_j(\boldsymbol{y}^k)) - \nabla\varphi_\lambda(\overline{y_j^k}, F_j(\overline{\boldsymbol{y}^k}))\right\|_2
\end{aligned}
$$

Given that the gradient of φ_λ is Lipschitz continuos [16], we have

$$
\begin{aligned}
\|H - H_*\|_\infty &= n \lim_{k\to\infty} \eta \left\|\begin{pmatrix} y_j^k - \overline{y_j^k} \\ F_j(\boldsymbol{y}^k) - F_j(\overline{\boldsymbol{y}^k}) \end{pmatrix}\right\|_2 \\
&\leq \sqrt{2}\, n\, \eta \left\|\begin{pmatrix} x_j - x_j^* \\ F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*) \end{pmatrix}\right\|_\infty \\
&= \sqrt{2}\, n\, \eta \max\left\{\left|x_j - x_j^*\right|, |F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*)|\right\}.
\end{aligned}
$$

We consider the two possibilities for this maximum:

1. $\max\left\{\left|x_j - x_j^*\right|, |F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*)|\right\} = \left|x_j - x_j^*\right| \leq \|\boldsymbol{x} - \boldsymbol{x}^*\|_\infty \leq \tilde\epsilon < \hat\epsilon.$

2. $\max\left\{\left|x_j - x_j^*\right|, |F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}^*)|\right\} = \left|F_j(\boldsymbol{x}^k) - F_j(\boldsymbol{x}^*)\right| < \hat\epsilon.$

For the above,

$$\|H - H_*\|_\infty < \sqrt{2}\, n\, \eta\, \hat\epsilon < \frac{\sqrt{2}\, n\, \eta\, r}{8\,\mu\,(\omega + \sqrt{2}\, n\, \eta)}. \tag{22}$$

Substituting (21) and (22) in (20)

$$\|B - H_*\|_\infty \quad < \quad \frac{r}{8\,\mu} + \frac{\omega\,r}{8\,\mu\,(\omega + \sqrt{2}\,n\,\eta)} + \frac{\sqrt{2}\,n\,\eta\,r}{8\,\mu\,(\omega + \sqrt{2}\,n\,\eta)} \quad = \quad \frac{r}{4\,\mu}.$$

Therefore, from (16)

$$\|B - H_*\|_\infty \quad < \quad \frac{r}{4\,\mu} \quad < \quad \frac{1}{4\,\|H_*^{-1}\|_\infty}, \tag{23}$$

so,

$$\left\|H_*^{-1}B - I_n\right\|_\infty \leq \left\|H_*^{-1}\right\|_\infty \|B - H_*\|_\infty < \frac{1}{4},$$

from which $\left\|H_*^{-1}B - I_n\right\|_\infty < 1$. By Banach's Lemma[2] there exists B^{-1} and therefore the function $\mathcal{Q}$ is well defined. Moreover,

$$\|B^{-1}\|_\infty \leq \frac{\|H_*^{-1}\|_\infty}{1 - \left\|H_*^{-1}B - I_n\right\|_\infty} \leq \frac{\mu}{1 - \dfrac{1}{4}} = \frac{4}{3}\,\mu.$$

The second part of the proof is to show (19). For this, we subtract x^* in (18), we apply $\|\cdot\|_\infty$ and perform some algebraic manipulations.

$$\begin{aligned}
\|\mathcal{Q}(x, A) - x^*\|_\infty &= \left\|x - x^* - B^{-1}\Phi_\lambda(x)\right\|_\infty \\
&= \left\|(x - x^*) - B^{-1}\Phi_\lambda(x) + B^{-1}H_*\,(x - x^*) - B^{-1}H_*\,(x - x^*)\right\|_\infty \\
&= \left\|B^{-1}\,(B - H_*)\,(x - x^*) - B^{-1}\,(\Phi_\lambda(x) + H_*\,(x - x^*))\right\|_\infty \\
&= \|B^{-1}\|_\infty\,\|(B - H_*)(x - x^*) - (\Phi_\lambda(x) - \Phi_\lambda(x^*) + H_*\,(x - x^*))\|_\infty \\
&\leq \frac{4}{3}\,\mu\,\left[\|B - H_*\|_\infty\,\|x - x^*\|_\infty + \left\|\Phi_\lambda(x) - \Phi_\lambda(x^*) + H_*\,(x - x^*)\right\|_\infty\right],
\end{aligned}$$

then we obtain,

$$\left\|\mathcal{Q}(x, A) - x^*\right\|_\infty \quad \leq \quad \frac{4}{3}\,\mu\,\left[\frac{r}{4\mu} + \frac{\|\Phi_\lambda(x) - \Phi_\lambda(x^*) + H_*(x - x^*)\|_\infty}{\|x - x^*\|_\infty}\right]\|x - x^*\|_\infty. \tag{24}$$

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[2] Let $\|\cdot\|$ a matrix norm induced in $\mathbb{R}^{n \times n}$, A and $C \in \mathbb{R}^{n \times n}$. If C is non singular and $\|I_n - C^{-1}A\| < 1$ then A is non singular and $\|A^{-1}\| \leq \dfrac{\|C^{-1}\|}{1 - \|I_n - C^{-1}A\|}$.

On the other hand, for $H \in \partial \Phi_\lambda(x)$.

$$\frac{\|\Phi_\lambda(x)-\Phi_\lambda(x^*)+H_*(x-x^*)\|_\infty}{\|x-x^*\|_\infty} \leq \frac{\|\Phi_\lambda(x)-\Phi_\lambda(x^*)+H(x-x^*)\|_\infty}{\|x-x^*\|_\infty}+\|H_*-H\|_\infty, \tag{25}$$

In [15], Kanzow and Kleinmichel show that $\Phi_\lambda(x)$ is semismooth, i.e.,

$$\lim_{x \to x^*} \frac{\|\Phi_\lambda(x) - \Phi_\lambda(x^*) + H(x - x^*)\|_\infty}{\|x - x^*\|_\infty} = 0. \tag{26}$$

Thus, for any $\rho > 0$, there exists $\epsilon_2 > 0$ such that if $\|x - x^*\|_\infty < \epsilon_2$ then

$$\frac{\|\Phi_\lambda(x) - \Phi_\lambda(x^*) + H(x - x^*)\|_\infty}{\|x - x^*\|_\infty} < \rho,$$

in particular, for $\rho = \dfrac{\omega r}{8\mu(\omega + \sqrt{2}n\eta)}$ exist $\epsilon_r > 0$ such that, if $\|x - x^*\|_\infty < \epsilon_r$ then

$$\frac{\|\Phi_\lambda(x) - \Phi_\lambda(x^*) + H(x - x^*)\|_\infty}{\|x - x^*\|_\infty} < \frac{\omega\, r}{8\mu\,(\omega + \sqrt{2}\,n\,\eta)}. \tag{27}$$

Let $\epsilon_0 = \min\{\tilde{\epsilon}, \epsilon_r\}$. If $\|x - x^*\|_\infty < \epsilon_0$, and $\|A - F'(x^*)\|_\infty < \delta_0$ then, from (27), (22), (25) and (24),

$$\begin{aligned}
\|\mathcal{Q}(x, A) - x^*\|_\infty &< \frac{4}{3}\mu \left[\frac{r}{4\mu} + \frac{r}{8\mu}\right] \|x - x^*\|_\infty \\
&= \frac{r}{2} \|x - x^*\|_\infty < r \|x - x^*\|_\infty.
\end{aligned}$$

$\square$

The following theorem is analogous to the theorem of the two neighborhoods of differentiable case [10], which guarantees linear convergence of the proposed algorithm. The name of the two neighborhoods is due to, in its assumptions, it requires two neighborhoods, one for the solution which should be the starting point, and the other for the Jacobian matrix of F at the solution which should be the initial approach.

Theorem 3.1. Let H1-H3 be verified and let $r \in (0, 1)$, then there exist positive constants ϵ_1 and δ_1 such that, if $\|x_0 - x^*\|_\infty \leq \epsilon_1$ and $\|A_k - F'(x^*)\|_\infty \leq \delta_1$, for all k, then the sequence $\{x_k\}$ generated by

$x_{k+1} = x_k - B_k^{-1}\Phi_\lambda(x_k)$, with B_k the matrix whose rows are defined by (15) is well defined, converges to x^* and satisfies

$$\|x_{k+1} - x^*\|_\infty \leq r \|x_k - x^*\|_\infty, \quad \text{for all } k = 0, 1, 2, \ldots. \qquad (28)$$

We consider the function (18). Thus,

$$x_{k+1} = \mathcal{Q}(x_k, A_k) = x_k - B_k^{-1}\Phi_\lambda(x_k),$$

for all $k = 0, 1, \ldots$, with B_k defined by (15).

Given $r \in (0, 1)$, let $\epsilon_1 \in (0, \epsilon_0)$ and $\delta_1 \in (0, \delta_0)$ where ϵ_0 and δ_0 are the positive constants given by the Lema 3.2. We will use induction on k in the proof of this theorem.

- For $k = 0$. If $\|x_0 - x^*\|_\infty \leq \epsilon_1 \leq \epsilon_0$ and $\|A_0 - F'(x^*)\|_\infty \leq \delta_1 \leq \delta_0$ then $x_1 = \mathcal{Q}(x_0, A_0)$ is well defined and satisfies

$$\|x_1 - x^*\|_\infty \leq r \|x_0 - x^*\|_\infty. \qquad (29)$$

- Induction hypotheses: we assume that for $k = m - 1$,

$$\text{if} \quad \|x_{m-1} - x^*\|_\infty \leq \epsilon_1 \quad \text{and} \quad \left\|A_{m-1} - F'(x^*)\right\|_\infty \leq \delta_1,$$

then $x_m = x_{m-1} - B_{m-1}^{-1}\Phi_\lambda(x_{m-1})$, is well define, and

$$\|x_m - x^*\|_\infty = \|\mathcal{Q}(x_{m-1}, A_{m-1}) - x^*\|_\infty \leq r \|x_{m-1} - x^*\|_\infty. \qquad (30)$$

Given that $\|x_0 - x^*\|_\infty < \epsilon_1$, then

$$\|x_m - x^*\|_\infty \leq r \|x_{m-1} - x^*\|_\infty \leq r^m \|x_0 - x^*\|_\infty \leq r^m \epsilon_1 < \epsilon_0,$$

and from the assumption $\|A_m - F'(x^*)\|_\infty \leq \delta_1$, we have for the Lemma 3.2 that x_{m+1} is well defined and satisfies

$$\|x_{m+1} - x^*\|_\infty \leq r \|x_m - x^*\|_\infty. \qquad (31)$$

Therefore, we conclude that (28) is true for all $k = 0, 1, \ldots$. $\qquad \square$

We observe that in the proof of Theorem 3.1 we used the infinity norm. Therefore, if $e_k = \|x_k - x^*\|$ is the error related to any other norm, then

$e_k \le \alpha \, r^k \, e_0$, where α is a positive constant that does not depend on k, and r is as in Theorem 3.1.

Among the standard theorems of thequasi Newton theory to systems of nonlinear equations is the theorem known as Dennis-Moré condition [21] which gives a sufficient condition for superlinear convergence. The following theorem is analogous to the theorem just mentioned and it will be useful in the next section to prove superlinear convergence of Algorithm 1. In his proof, we use $\|\cdot\| = \|\cdot\|_\infty$ but, we recall that superlinear convergence results are norm-independent.

Theorem 3.2. Let H1-H3 be verified and that, for some x_0, the sequence $\{x_k\}$ generated by $x_{k+1} = x_k - B_k^{-1}\Phi_\lambda(x_k)$ converges to x^*, with B_k given by (15) and $H_* = H(x_*)$. If

$$\lim_{k \to \infty} \frac{\|(B_k - H_*)\, s_k\|}{\|s_k\|} = 0, \tag{32}$$

where $S_k = x_{k+1} - x_k$, then the sequence $\{x_k\}$ converges superlinearly to x^*.

Proof. As we mentioned above, Φ_λ is semismooth in x^*, thus

$$\lim_{x_k \to x^*} \frac{\|\Phi_\lambda(x_k) - \Phi_\lambda(x^*) - H_*(x_k - x^*)\|}{\|x_k - x^*\|} = 0,$$

where $H_* \in \partial \Phi_\lambda(x^*)$. As $\Phi_\lambda(x^*) = 0$, then

$$\lim_{x_k \to x^*} \frac{\|\Phi_\lambda(x_k) - H_*(x_k - x^*)\|}{\|x_k - x^*\|} = 0. \tag{33}$$

On the other hand,

$$\|x_k - x^*\| \le \|H_*^{-1}\| \, \|H_*(x_k - x^*)\|, \tag{34}$$

also,

$$\big|\, \|\Phi_\lambda(x_k)\| - \|H_*(x_k - x^*)\|\,\big| \le \|\Phi_\lambda(x_k) - H_*(x_k - x^*)\|. \tag{35}$$

Substituting (34) in (33) and using (35), we have

$$\lim_{x_k \to x^*} \frac{\big|\, \|\Phi_\lambda(x_k)\| - \|H_*(x_k - x^*)\|\,\big|}{\|H_*(x_k - x^*)\|} = 0. \tag{36}$$

By the limit definition, in particular for $\rho = \dfrac{1}{2}$, there exists $\epsilon > 0$, such that if $\|x_k - x^*\| < \epsilon$ then,

$$-\frac{1}{2} < \frac{\|\Phi_\lambda(x_k)\| - \|H_*(x_k - x^*)\|}{\|H_*(x_k - x^*)\|} < \frac{1}{2},$$

since

$$\frac{1}{2}\|H_*(x_k - x^*)\| < \|\Phi_\lambda(x_k)\| < \frac{3}{2}\|H_*(x_k - x^*)\|.$$

From (34),

$$\|\Phi_\lambda(x_k)\| > \frac{1}{2}\|H_*(x_k - x^*)\| \geq \frac{1}{2\left\|H_*^{-1}\right\|}\|x_k - x^*\|. \qquad (37)$$

On the other hand,

$$0 \;=\; B_k s_k + \Phi_\lambda(x_k), \qquad (38)$$

where $s_k = x_{k+1} - x_k$. We add and subtract in the equality (38) the term $H_* s_k - \Phi_\lambda(x_{k+1})$ thus

$$-\Phi_\lambda(x_{k+1}) \;=\; B_k s_k - H_* s_k + \Phi_\lambda(x_k) + H_* s_k - \Phi_\lambda(x_{k+1}).$$

Applying a norm and the triangle inequality, we obtain

$$\|\Phi_\lambda(x_{k+1})\| \;\leq\; \|(B_k - H_*)s_k\| + \|\Phi_\lambda(x_k) + H_* s_k - \Phi_\lambda(x_{k+1})\|,$$

thereby

$$\frac{\|\Phi_\lambda(x_{k+1})\|}{\|s_k\|} \;\leq\; \frac{\|(B_k - H_*)s_k\|}{\|s_k\|} + \frac{\|\Phi_\lambda(x_k) + H_* s_k - \Phi_\lambda(x_{k+1})\|}{\|s_k\|},$$

By (32), the first term of the right term converges to 0. The second term converges to 0, for the semismoothness of Φ_λ. Thus,

$$\lim_{x_k \to x^*} \frac{\|\Phi_\lambda(x_{k+1})\|}{\|s_k\|} = 0. \qquad (39)$$

From (3.2), (37) and (16)

$$
\begin{aligned}
0 = \lim_{x_k \to x^*} \frac{\|\Phi_\lambda(x_{k+1})\|}{\|s_k\|} \;\; &\geq \;\; \frac{1}{2\,\|H_*^{-1}\|} \lim_{x_k \to x^*} \frac{\|x_{k+1} - x^*\|}{\|s_k\|} \\
&\geq \;\; \frac{1}{2\,\mu} \lim_{x_k \to x^*} \frac{\|x_{k+1} - x^*\|}{\|x_{k+1} - x^*\| + \|x_k - x^*\|} \\
&= \;\; \frac{1}{2\,\mu} \lim_{x_k \to x^*} \frac{\dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|}}{\dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} + 1}.
\end{aligned}
$$

Therefore,

$$
\lim_{x_k \to x^*} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0,
$$

i.e., the sequence $\{x_k\}$ converges superlinearly to x^*. $\qquad\square$

4   Least change secant update family for solving $\Phi_\lambda(x) = 0$.

The quasi-Newton methods differ in how to update the matrix A_k at each iteration. Among the "practical" quasi-Newton algorithm are those that are called least change secant methods, in which the updating of A_k, named A_{k+1}, must satisfy the secant equation [10] given by $A_{k+1}(x_{k+1} - x_k) = F(x_{k+1}) - F(x_k)$ and its change (measured in some norm) relative to A_k must be minimum. Requiring that secant equation and a minimum change are satisfied between two consecutive updates makes the sequence of matrices $\{A_k\}$ have a property known as bounded deterioration [10] [8], which guarantees that the matrices of the sequence remain in a neighborhood of $F'(x^*)$. This is essential to demonstrate local and linear convergence. Thus, at each iteration of the least change secant algorithm, the vectors x_k and x_{k+1} defined the set V by

$$
V = V(x_k, x_{k+1}) = \left\{ A \in S \subseteq \mathbb{R}^{n \times n} : A(x_{k+1} - x_k) = F(x_{k+1}) - F(x_k) \right\}. \tag{40}
$$

Given that we need the matrix in V "nearest" A_k, it is natural to think of the orthogonal projection of this matrix on V, named $P_V(A_k) = P_{x_k,x_{k+1}}(A_k)$. Given that

$$\|P_V(A_k) - A_k\| = \inf_{A \in V} \|A - A_k\|, \tag{41}$$

and that V is a closed set, we can ensure that $P_V(A_k) \in V$. This projection is unique because V is a convex set. Therefore,

$$\|P_V(A_k) - A_k\| = \min_{A \in V} \|A - A_k\|. \tag{42}$$

Thus, $A_{k+1} = P_V(A_k)$.

Different least change secant updates are obtained by varying the matrix norm $\mathbb{R}^{n \times n}$ or the subspace S, producing the family of least change secant update methods. For example, "Good" Broyden update,"Bad" Broyden update [22], Schubert update [23] andSparse Schubert update [23].

Algorithm 2. Assume that x_0 and A_0 are arbitrary. x_{k+1} and A_{k+1} for $k = 0, 1, ...,$ are generated as follows:

$$B_k = D_a + D_b A_k \tag{43}$$
$$x_{k+1} = x_k - B_k^{-1}\Phi(x_k) \tag{44}$$
$$A_{k+1} = P_{x_k,x_{k+1}}(A_k) \tag{45}$$

where $D_a = diag(a_1, \ldots, a_n)$ and $D_b = diag(b_1, \ldots, b_n)$ with

$$a_i = \chi(x_i^k, F_i(x^k)) - 1 \qquad and \qquad b_i = \psi(x_i^k, F_i(x^k)) - 1.$$

In order to develop the theory of convergence of the least change secant update methods generated by Algorithm 2 we will assume an additional Assumption.

H4. For all x, z in a neighborhood x^*, there are $A \in V(x, z)$ and $\alpha_1 > 0$ such that

$$\|A - F'(x^*)\| \leq \alpha_1 \sigma(x, z), \tag{46}$$

where $\sigma(x, z) = \max\{\|x - x^*\|, \|z - x^*\|\}$.

5   Additional convergence results

In the next lemma, we show that a matrix generated using the rule (45) to update the matrix A_k may deteriorate, but in a controlled way.

Lemma 5.1. Let H1-H4 be verified and let A_+ be the orthogonal projection of A on the set $V(x, z)$ and $\widehat{A}$ the orthogonal projection of $F'(x)$ on $V(x, z)$ then $\|A_+ - F'(x)\| \leq \|A - F'(x)\| + \alpha_2 \sigma(x, z)$, where $\alpha_2 > 0$ and $\sigma(x, z) = \max\{\|x - x^*\|, \|z - x^*\|\}$.

Lemma 5.2. Let assumptions H1-H4 be verified. Then there exists $c > 0$ such that

$$\left\| P_{x,y}(A) - F'(x^*) \right\| \leq \left\| A - F'(x^*) \right\| + c \left\| x - x^* \right\| \qquad (47)$$

whenever the vectors x and y belong to a neighborhood of x^*, with $\|y - x^*\| \leq \|x - x^*\|$ and the matrix A in a neighborhood of $F'(x^*)$.

The two previous lemmas (see proofs in [16] and [24], respectively) and assumptions H1-H4 are central to ensuring the following result.

Theorem 5.1. Let H1 - H4 be verified and that the sequence $\{A_k\}$ is generated by (45). Given $r \in (0, 1)$, there exist positive constants $\overline{\epsilon}$ and $\overline{\delta}$ such that if $\|x_0 - x^*\| \leq \overline{\epsilon}$ and $\|A_0 - F'(x^*)\| \leq \overline{\delta}$, the sequence $\{x_k\}$ generated by $x_{k+1} = x_k - B_k^{-1}\Phi(x_k)$ is well defined, converges to x^* and, for all $k = 1, 2, \ldots,$

$$\|x_{k+1} - x^*\| \leq r \|x_k - x^*\|. \qquad (48)$$

Let $\overline{\delta} \in (0, \delta_1)$. We can choose $\overline{\epsilon} \in (0, \epsilon_1)$ with

$$\overline{\epsilon} < \frac{(\delta_1 - \overline{\delta})(1 - r)}{c}, \qquad (49)$$

where δ_1 and ϵ_1 are the positive constants of Lemma 3.2, and c is the constant defined by (52). Some considerations on how to choose the constant $\overline{\epsilon}$, are the following.

1. If $\dfrac{(\delta_1 - \overline{\delta})(1 - r)}{c} < \epsilon_1$, the constant $\overline{\delta}$ which depends on ϵ_1 because $\overline{\delta} > \delta_1 - \frac{c\,\epsilon_1}{1-r}$.

2. If $\dfrac{(\delta_1 - \overline{\delta})(1 - r)}{c} > \epsilon_1$, we can choose any value in $(0, \epsilon_1)$ as $\overline{\epsilon}$.

Thus, in either case, it is possible to choose $\overline{\epsilon}$ in $(0, \epsilon_1) \bigcap \left(0, \frac{(\delta_1 - \overline{\delta})(1-r)}{c}\right)$.
Then, from (49)

$$\overline{\delta} + c\,\frac{\overline{\epsilon}}{1 - r} \quad < \quad \overline{\delta} + c\,\frac{(\delta_1 - \overline{\delta})(1 - r)}{c\,(1 - r)} < \delta_1. \tag{50}$$

We will use induction on k in the proof of this theorem.

1. For $k = 0$, if $\|x_0 - x^*\| \le \overline{\epsilon} < \epsilon_1$ and $\|A_0 - F'(x^*)\| \le \overline{\delta} < \delta_1$ then from Lemma 3.2, $x_1 = x_0 - B_0^{-1}\Phi(x_0)$ is well defined and satisfies $\|x_1 - x^*\| \le r\,\|x_0 - x^*\|$.

2. We assume that for $k = m - 1$, if $\|x_{m-1} - x^*\| \le \overline{\epsilon}$ and $\|A_{m-1} - F'(x^*)\| \le \overline{\delta}$, $x_m = x_{m-1} - B_0^{-1}\Phi(x_{m-1})$ is well defined, and

$$\|x_m - x^*\| \le r\,\|x_{m-1} - x^*\|. \tag{51}$$

3. From (51) and (47)

$$\begin{aligned}
\|x_m - x^*\| &\le & r\,\|x_{m-1} - x^*\| \\
&\le & r^m\,\|x_0 - x^*\| < r^m\,\overline{\epsilon} < \overline{\epsilon} < \epsilon_1.
\end{aligned}$$

From (47),

$$\begin{aligned}
\|A_m - F'(x^*)\| &\le & \|A_{m-1} - F'(x^*)\| + c_{m-1}\,\|x_{m-1} - x^*\| \\
&\le & \overline{\delta} + c\overline{\epsilon}\sum_{j=0}^{m-1} r^j < \overline{\delta} + c\overline{\epsilon}\sum_{j=0}^{\infty} r^j \\
&< & \overline{\delta} + c\,\frac{\overline{\epsilon}}{1 - r} < \delta_1,
\end{aligned}$$

where

$$c = \max_{0 \le j \le m-1} c_j \tag{52}$$

and c_i is the constant of Lemma 5.2. Thus, from Lemma 3.2, x_{m+1} is well defined and satisfies

$$\|x_{m+1} - x^*\| \le r\|x_m - x^*\|. \qquad \square$$

Lemma 5.3. We assume the Assumptions H1 - H4 are verified and let the sequence $\{A_k\}$ be generated by (45). There are positive constants $\overline{\epsilon}$ y $\overline{\delta}$ such that, if $\|x_0 - x^*\| \leq \overline{\epsilon}$ and $\|A_0 - F'(x^*)\| \leq \overline{\delta}$, and the sequence $\{x_k\}$ is generated by $x_{k+1} = x_k - B_k^{-1}\Phi(x_k)$, with B_k defined by (15), then

$$\lim_{k\to\infty} \|B_{k+1} - B_k\| = 0. \tag{53}$$

With this result (See proof in [16]), we can derive sufficient condition to have super linear convergence as shown by the next Theorem.

Theorem 5.2. Let the Assumptions H1-H4 and let the sequences $\{x_k\}$ and $\{A_k\}$ be generated by the Algorithm 2 and $\lim_{k\to\infty} x_k = x^*$. If

$$\lim_{k\to\infty} \frac{\|(B_{k+1} - H_*) s_k\|}{\|s_k\|} = 0, \tag{54}$$

then the sequence $\{x_k\}$ converges superlinearly to x^*.

The proof follows in a straightforward way from Theorems 4.2 and 4.3.

$$\lim_{k\to\infty} \frac{\|(B_k - H_*)) s_k\|}{\|s_k\|} \leq \lim_{k\to\infty} \frac{\|(B_k - B_{k+1}) s_k\|}{\|s_k\|} + \lim_{k\to\infty} \frac{\|(B_{k+1} - H_*)) s_k\|}{\|s_k\|}$$

$$\leq \lim_{k\to\infty} \|B_k - B_{k+1}\| + \lim_{k\to\infty} \frac{\|(B_{k+1} - H_*) s_k\|}{\|s_k\|}.$$

From (54) and theLemma 7.3, we have that the right side expression of the last inequality is equal to zero. So

$$\lim_{k\to\infty} \frac{\|(B_k - H_*) s_k\|}{\|s_k\|} = 0,$$

This is the Dennis-Moré type condition of Theorem 3.2. Therefore, the sequence $\{x_k\}$ converges superlinearly to x^*. $\qquad\square$

6 Some numerical experiments

In this section, we analyze numerically the local behavior of the family of least change secant update methods introduced in Section 2. For this, we

compare our algorithms with the Generalized Newton method proposed in [6]. The Algorithm 3 is a Generalized Newton type method. It uses at each iteration, the matrix H_k (defined in the previous section) which uses the Jacobian Matrix of F.

Algorithm 3. Given N, x_0 and $\lambda \in (0, 4)$, for $k = 1, 2, \ldots$

    While $\|\Phi(x_k)\| \geq \sqrt{n}\, 10^{-5}$ and $k < N$

        Compute $F'(x_k)$.

        Compute H_k by (14).

        Compute $x_{k+1} = x_k - H_k^{-1}\Phi(x_k)$.

        $k \leftarrow k + 1$.

End.

The Algorithm 4, which is a least secant change type method is based on the Algorithm 2 that we proposed at Section 3. For updating the matrix A_k, in each iteration, we use the four formulas: "Good" and "bad" Broyden, Schubert and Sparse Schubert, whereby we have four versions of Algorithm 4.

Algorithm 4. Given N, x_0, A_0 and $\lambda \in (0, 4)$, for $k = 1, 2, \ldots$

    While $\|\Phi(x_k)\| \geq \sqrt{n}\, 10^{-5}$ and $k < N$

        Compute B_k by (15).

        Compute $x_{k+1} = x_k - B_k^{-1}\Phi_\lambda(x_k)$

      Update A_k.

        $k \leftarrow k + 1$.

End.

For all the tests, we use the software MATLAB®. We use 8 test problems for nonlinear complementarity, four of which we chose from a list proposed, [14] and which are considered "hard problem". These are Kojima-Shindo (application to Economic Equilibrium [25]), Kojima-Josephy, Nash-Cornout (application to the Game Theory harker) and Modified Mathiesen (application to Walrasian Economic Equilibrium [26]) problems. We generate the four remaining problems like in [27]; for this, we define $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by,

$$F_i(x) = \begin{cases} h_i(x) - h_i(x^*) & \text{if } i \text{ is odd or } i > n/2, \\ \\ h_i(x) - h_i(x^*) + 1 & \text{otherwise}. \end{cases}$$

For these functions, the vector $x^* = (1, 0, 1, 0, ...) \in \mathbb{R}^n$ is a degenerate solution and the functions h_i are given by Lukšan [18], namely, Trigonometric system, Exponential trigonometric, tridiagonal and Rosenbrock.

We use the same stopping criteria proposed in [28]. We choose the parameter $\lambda := \lambda_{\min}$ for using in the Algorithms 3 and 4 as follows

1. We vary λ in the interval $(0, 4)$ from $\lambda = 10^{-3}$ to $\lambda = 3.999$ with increments of 10^{-3}.

2. We use the generalized Newton method (Algorithm 3) with each of these values of λ.

3. We called $\lambda_{\min}$ to the value of λ for which the generalized Newton converges in fewer iterations.

For the numerical test, we vary λ in the interval $(0, 4)$ from $\lambda = 10^{-3}$ to $\lambda = 3.999$ with increments of 10^{-3}. Of all these values of λ, that for which the generalized Newton method converges in less iterations, we call it $\lambda_{\min}$, and we use it as the parameter λ in Algorithms 3 and 4. The initial approximations are the same as in [14] and [18].

Table 1 presents the results of our numerical tests. Its columns contains the following information: *Problem* means the problem name, n is the dimension problem, λ_{min} is the value of λ for which the Algorithm 3 converges in fewer iterations. We also include a column with the algorithm and the secant update used. Thus, GN means Generalized Newton; SSU, BBU, SU and GBU means Algorithm 4 with the Sparse Schubert Update, "Bad" Broyden Update, Schubert Update and the "Good" Broyden Update, respectively. A $-$ sign means divergence.

From Table 1, we observe that, for these preliminary numerical tests, the Algorithm 4 that we proposed for solving the NCP has good local behavior. In particular, we highlight the Modified Mathiesen problem, in which each method converges with the same number of iterations but to different solutions to the problem.

Problem	n	$\lambda_{\min}$	GN	SSU	BBU	SU	GBU
Kojima-Shindo	4	2.7	7	17	14	-	16
Kojima-Josephy	4	3.860	9	14	9	-	12
Nash-Cornout	5	1.540	6	8	7	135	7
Modified Mathiesen	4	0.010	3	3	3	3	3
Trigonom. System	10	1.930	5	18	24	-	-
Trigonometric exp.	100	0.010	3	6	6	8	4
Tridiagonal	20	0.010	3	5	5	8	4
Rosenbrock	20	0.010	3	3	3	3	3

Table 1: Local behavior of Algorithms 3 and 4.

7 Conclusions

In this paper, we propose a quasi-Newton method for solving the nonlinear complementarity problem when this is reformulated as a nonlinear system of equations. This method can be useful when the derivatives of the system are very expensive or difficult to obtain. Moreover, we generated a family of least change secant update methods that, under certain hypotheses, converge local and superlinearly to the solution of the problem.

Some numerical experiments shows a good local performance of this algorithm, but it is necessary more numerical tests using others well-known LCSU methods such as Column Updating method [29] Inverse Column Updating method [30]. It is necessary to incorporate a globalization strategy to the algorithm proposed and to develop theoretical and numerical analysis of the global algorithm.

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