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Discrete Likelihood Ratio Order for Power Series Distribution

Orden de la razón de verosimilitud discreta para la distribución de series de potencias

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Abstract

It is well-known that some discrete distributions belong to the power series distribution (PSD) family, so it seems useful to study conditions to establish the discrete likelihood ratio order for this family. In this paper, conditions to some cases of PSD family under which the discrete likelihood ratio order we have looked at the holds. Also, we study the discrete version of the proportional likelihood ratio as an extension of the likelihood ratio order. Then we compare some members of the PSD family by discrete proportional likelihood ratio order.

***Key words*:** Binomial distribution, Geometric distribution, Logarithmic series distribution, Negative binomial distribution, Poisson distribution, Proportional likelihood ratio order.

Resumen

Es bien conocido en la literatura que algunas distribuciones discretas pertenecen a la familia de distribuciones de series de potencias (PSD, power series distributions por sus siglas en inglés). Por lo tanto, es útil estudiar algunas condiciones para establecer el orden de la razón de verosimilitud para esta familia. En este artículo, se estudian las condiciones para algunos casos de la familia PSD bajo las cuales se mantiene el orden de la razón de verosimilitud. Otros autores han introducido y estudiado el orden de la razón de verosimilitud proporcional como una extensión del orden de razón de verosimilitud para variables aleatorias continuas. Aquí, se presenta el

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orden de razón de verosimilitud proporcional para variables aleatorias discretas y se estudian para la familia PSD.

***Palabras clave*:** distribución binomial, distribución binomial negativa, distribución de series logarítmicas, distribución geométrica, distribución Poisson, orden de la razón de verosimilitud proporcional.

1. Introduction

Recently, many papers have been devoted to compare random variables according to stochastic orderings in particular likelihood ratio order. Most of the contributions are for the continuous random variables. We refer to Shanthikumar & Yao (1986), Lillo, Nanda & Shaked (2001), Hu, Nanda, Xie & Zhu (2003), Shaked & Shanthikumar (2007), Misra, Gupta & Dhariyal (2008), Blazej (2008), Navarro (2008) and Bartoszewicz (2009) for more details.

Ramos-Romero & Sordo-Diaz (2001) introduced a new stochastic order between two continuous and non-negative random variables and called it proportional likelihood ratio (PLR) order, which is closely related to the usual likelihood ratio order. Belzunce, Ruiz & Ruiz (2002), extended hazard rate and reversed hazard rate orders to proportional state in the same manner and called them proportional (reversed) hazard rate orders. So, they studied their properties, preservations and relations with other orders. In general, the proportional versions are stronger orderings and easy to verify in many situations, so they are helpful to check what components are more reliable, and consequently systems formed from them.

In the next section, we recall the discrete likelihood ratio order and then compare some members of PSD family. Then we present discrete proportional likelihood ratio order and study it for PSD family at the last section of this paper.

2. Discrete Likelihood Ratio Order for Power Series Distribution Family

We obtain the conditions under which the discrete likelihood ratio order is established for some cases of the power series distribution family.

Definition 1. Let X and Y be discrete non-negative random variables with probability functions $P_X(x)$ and $P_Y(x)$ respectively. X is said to be smaller than Y in the discrete likelihood ratio order (denoted by $X \leq_{lr} Y$), if

$$\frac{P_Y(x)}{P_X(x)} \text{ is increasing in } x \in N. \tag{1}$$

Noack (1950) defined a random variable X taking non-negative integer values with probabilities

$$P(X = x) = \frac{a_x \theta^x}{b(\theta)},\ a_x \geq 0,\ x = 0, 1, 2, \dots \tag{2}$$

He called the discrete probability distribution given by (2) a power series distribution and derived some of its properties relating its moments, cumulants, etc. Patil (1961, 1962) studied the generalized power series distribution (GPSD) family with probability function like (2), whose support is any non-empty and enumerable set of non-negative integers.

Note that the Poisson, negative binomial and geometric distributions belong to PSD family and binomial and logarithmic distributions are in the GPSD family.

Suppose that X and Y have probability functions $P(X = x) = \frac{\alpha_x \theta_1^x}{b(\theta_1)}$ and $P(Y = x) = \frac{\beta_x \theta_2^x}{b(\theta_2)}$ respectively. So, using Definition 1, $X \leq_{lr} Y$ if $\frac{P_Y(x)}{P_X(x)} \leq \frac{P_Y(x+1)}{P_X(x+1)}$ for all x, or equivalently

$$(\frac{\alpha_{x+1}}{\alpha_x})(\frac{\beta_x}{\beta_{x+1}}) \leq \frac{\theta_2}{\theta_1}. \tag{3}$$

Now, we check equation (3) for some members of the PSD family:

Poisson Distribution: In equation (2), $a_x = \frac{1}{x!}$ and $b(\lambda) = e^{\lambda}$, leads to the Poisson distribution with parameter λ. Also, we get

$$\frac{P_X(x+1)}{P_X(x)} = \frac{\lambda}{1+x}.$$

Now, if X and Y possess Poisson distribution with parameters λ_1 and λ_2 respectively, then, using (3), $X \leq_{lr} Y$ if and only if $\lambda_1 \leq \lambda_2$.

Binomial Distribution: Suppose that X has binomial distribution with parameters n_1 and p_1 and Y has binomial distribution with parameters n_2 and p_2, for all $n_1 < n_2$. Using (3) and after simplification,

$$\left(\frac{n_1 - x}{n_2 - x}\right)\left(\frac{p_1}{1 - p_1}\right)\left(\frac{1 - p_2}{p_2}\right) \leq 1,\ x = 0, 1, \ldots, n_1 - 1$$

the left side of the above inequality gets its maximum at $x = 0$, so, if $n_1 < n_2$ and $\frac{n_1 p_1}{1-p_1} \leq \frac{n_2 p_2}{1-p_2}$ then $X \leq_{lr} Y$.

Negative Binomial Distribution: Suppose that X has negative binomial distribution with parameters r_1 and p_1 and Y has negative binomial distribution with parameters r_2 and p_2. Using (3)

$$\left(\frac{r_1 + x}{r_2 + x}\right)\left(\frac{1 - p_1}{1 - p_2}\right) \leq 1,\ x = 0, 1, \ldots$$

if $r_2 \leq r_1$ then, $\frac{r_1+x}{r_2+x} \leq 1$ is decreasing in $x \in N$, so gets maximum at $x = 0$. Therefore, $r_2 \leq r_1$ and $r_1(1 - p_1) \leq r_2(1 - p_2)$ imply that $X \leq_{lr} Y$.

Geometric Distribution: If X and Y are random variables of geometric distribution with parameters p_1 and p_2 respectively, then $p_2 \leq p_1$ implies that $X \leq_{lr} Y$ (it is evident that the geometric distribution is obtained from the negative binomial distribution where $r = 1$).

Logarithmic Series Distribution: For random variables X and Y with logarithmic series distribution with parameters θ_1 and θ_2 respectively, if $\theta_1 \leq \theta_2$ then $X \leq_{lr} Y$.

Binomial Distribution versus Poisson Distribution: If X is binomial distribution with parameters n and p and Y is Poisson distribution with parameter λ, then $X \leq_{lr} Y$ if

$$\left(\frac{p}{1-p}\right)\left(\frac{n-x}{\lambda}\right) \leq 1,\ x = 0, 1, 2, \ldots, n$$

Also, maximum of the left side expression of the above inequality are given at $x = 0$, so, if $np \leq \lambda(1-p)$ then $X \leq_{lr} Y$.

Poisson Distribution versus Negative Binomial distribution: Consider random variable X having Poisson distribution with parameter λ and Y having negative binomial distribution with parameters r and p. Since $\frac{1}{r+x}$ is decreasing in x, then $\lambda \leq r(1-p)$ leads to $X \leq_{lr} Y$.

Poisson Distribution versus Geometric distribution: If X is Poisson distribution with parameter λ and Y is geometric distribution with parameter p, then, $X \leq_{lr} Y \Longleftrightarrow \lambda \leq 1-p$.

Poisson Distribution versus Logarithmic Series Distribution: Let X and Y be random variables of Poisson and logarithmic series distributions with parameters θ_1 and θ_2 respectively. So, $X \leq_{lr} Y \Longleftrightarrow \theta_1 \leq \theta_2$.

Negative Binomial versus Logarithmic Series Distribution: The random variable X of negative binomial with parameters r and p is smaller in sense of likelihood ratio order than Y of logarithmic series distribution with parameter θ in the likelihood ratio order if $\theta \geq (1-p)(r+1)$.

Table 1: Necessary conditions for establishment discrete likelihood ratio order.

$X \leq_{lr} Y$	Conditions
$X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$	$\lambda_1 \leq \lambda_2$
$X \sim Bin(n_1, p_1)$ and $Y \sim Bin(n_2, p_2)$	$n_1 \leq n_2$ and $\frac{n_1 p_1}{1-p_1} \leq \frac{n_2 p_2}{1-p_2}$
$X \sim Nb(r_1, p_1)$ and $Y \sim Nb(r_2, p_2)$	$r_2 \leq r_1$ and $r_2(1-p_2) \geq r_1(1-p_1)$
$X \sim Ge(p_1)$ and $Y \sim Ge(p_2)$	$p_1 \geq p_2$
$X \sim Ls(\theta_1)$ and $Y \sim Ls(\theta_2)$	$\theta_1 \leq \theta_2$
$X \sim Bin(n, p)$ and $Y \sim Poi(\lambda)$	$np \leq \lambda(1-p)$
$X \sim Poi(\lambda)$ and $Y \sim Nb(r, p)$	$\lambda \leq r(1-p)$
$X \sim Poi(\lambda)$ and $Y \sim Ge(p)$	$\lambda \leq (1-p)$
$X \sim Poi(\lambda)$ and $Y \sim Ls(\theta)$	$\lambda \leq \theta$
$X \sim Nb(r, p)$ and $Y \sim Ls(\theta)$	$\theta \geq (r+1)(1-p)$

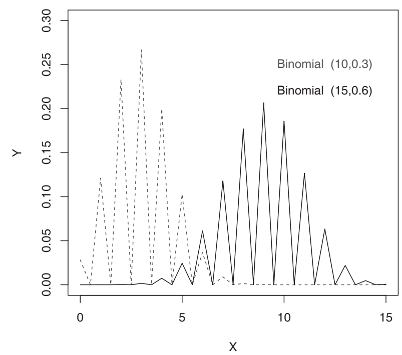

FIGURE 1: The Dot-Dot line shows the Binomial distribution with parameters $n_1 = 10$ and $p_1 = 0.3$ and the stretch shows the Binomial distribution with parameters $n_2 = 15$ and $p_2 = 0.6$.

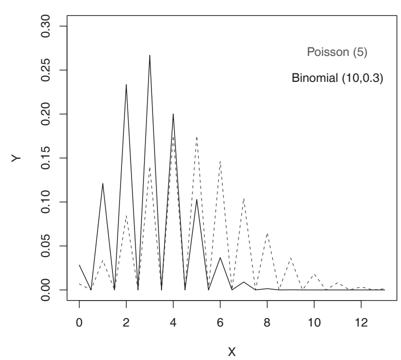

FIGURE 2: The Dot-Dot line shows the Poisson distribution with parameter $\lambda = 5$ and the stretch shows the Binomial distribution with parameters $n = 10$ and $p = 0.3$.

3. Discrete Proportional Likelihood Ratio Order for Power Series Distribution Family

Ramos-Romero & Sordo-Diaz (2001) studied proportional likelihood ratio order as extension of the likelihood ratio order for non-negative absolutely continuous random variables. They obtained various properties and applications of the proportional likelihood ratio order. In this section, discrete proportional likelihood ratio order is studied. Also, we looked the conditions under which this ordering is hold for PSD.

Definition 2. For two discrete non-negative random variables X and Y with probability functions $P_X(x)$ and $P_Y(x)$ respectively, if

$$\frac{P_Y([\lambda x])}{P_X(x)} \text{ is increasing in } x \in N \tag{4}$$

where $\lambda \leq 1$ is any positive constant and $[\cdot]$ denote the integer part function. Then, we say that X is smaller than Y in the discrete proportional likelihood ratio order (denoted by $X \leq_{plr} Y$).

Definition 3. We say that the discrete non-negative random variables X has increasing likelihood ratio order (denoted by $X \in IPLR$) if $\frac{p_X([\lambda x])}{p_X(x)}$ for $0 \leq \lambda \leq 1$ in increasing.

Theorem 1. *Let X and Y be two discrete non-negative random variables with probability functions $P_X(x)$ and $P_Y(x)$ respectively. If $X \leq_{lr} Y$ and $Y \in IPLR$, then $X \leq_{plr} Y$.*

Proof. Since
$$\frac{p_Y([\lambda x])}{p_X(x)} = \frac{p_Y(x)}{p_X(x)} \frac{p_Y([\lambda x])}{p_Y(x)}$$
the proof is clear. $\square$

Let X and Y be discrete non-negative random variables with probability functions $P(X = x) = \frac{\alpha_x \theta_1^x}{b(\theta_1)}$ and $P(Y = x) = \frac{\beta_x \theta_2^x}{b(\theta_2)}$ respectively. So, using Definition 2, $X \leq_{plr} Y$ if and only if
$$\left(\frac{\alpha_{[\lambda x+\lambda]}}{\alpha_{[\lambda x]}}\right)\left(\frac{\beta_x}{\beta_{x+1}}\right) \geq \frac{\theta_2}{{\theta_1}^{[\lambda x+\lambda]-[\lambda x]}}. \tag{5}$$

Geometric Distribution: Let X and Y having geometric distribution with parameters p_1 and p_2 respectively, using (5), we have $X \leq_{plr} Y$ if
$$\frac{P_Y([\lambda x])}{P_X(x)} = \frac{q_2^{[\lambda x]-1} p_2}{q_1^{x-1} p_1}$$
is increasing in x. That is
$$\frac{q_2^{[\lambda x]-1} p_2}{q_1^{x-1} p_1} \leq \frac{q_2^{[\lambda x+\lambda]-1} p_2}{q_1^{x} p_1}$$
that is equivalent to $q_1 \leq q_2^{[\lambda x+\lambda]-[\lambda x]}$. If $[\lambda x + \lambda] = [\lambda x]$, then $q_1 \leq 1$. If $[\lambda x + \lambda] = [\lambda x] + 1$, then $q_1 \leq q_2$. So, $X \leq_{plr} Y$ if and only if $p_1 \geq p_2$.

Poisson Distribution: Let X having Poisson distribution with parameter θ. If
$$\frac{x!}{[\lambda x]!}\theta^{[\lambda x]-x} \leq \frac{(x+1)!}{[\lambda x+\lambda]!}\theta^{[\lambda x+\lambda]-x-1}$$
then,
$$\frac{P_X([\lambda x])}{P_X(x)} = \frac{x!}{[\lambda x]!}\theta^{[\lambda x]-x}$$
is increasing. If $[\lambda x + \lambda] = [\lambda x]$, then $x!\theta^{[\lambda x]-x} \leq (x+1)!\theta^{[\lambda x]-x-1}$, so, $\theta \leq x+1$, that by increasing $h(x) = x+1$, it implies that $\theta \leq 1$. But if $[\lambda x + \lambda] = [\lambda x] + 1$, then
$$\frac{x!}{[\lambda x]!}\theta^{[\lambda x]-x} \leq \frac{(x+1)!}{([\lambda x]+1)!}\theta^{([\lambda x]+1)-x-1}$$

that is $[\lambda x + 1] \leq x + 1$, which always is true. Therefore, if X and Y having Poisson distribution with parameters θ_1 and θ_2 respectively and $\theta_1 \leq \theta_2 \leq 1$, then $X \leq_{plr} Y$.

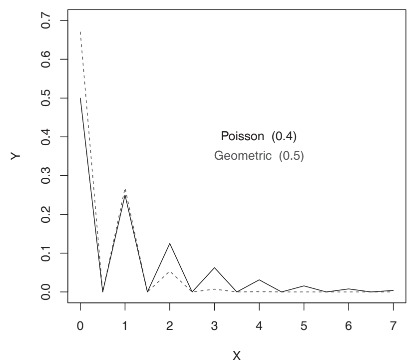

FIGURE 3: The Dot-Dot line shows the Geometric distribution with parameter $p = 0.5$ and the stretch shows the Poisson distribution with parameter $\lambda = 0.4$.

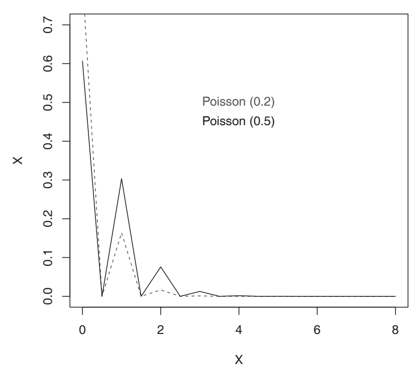

FIGURE 4: The Dot-Dot line shows the Poisson distribution with parameter $\lambda_1 = 0.2$ and the stretch shows the Poisson distribution with parameter $\lambda_2 = 0.5$.

Binomial Distribution: Consider X having binomial distribution with parameters n and p, then,

$$\frac{P_X([\lambda x])}{P_X(x)} = \frac{x!}{[\lambda x]!} \frac{(n-x)!}{(n-[\lambda x])!} \left(\frac{p}{q}\right)^{[\lambda x]-x}$$

is increasing in x if

$$\frac{x!}{[\lambda x]!} \frac{(n-x)!}{(n-[\lambda x])!} \left(\frac{p}{q}\right)^{[\lambda x]-x} \leq \frac{(x+1)!}{[\lambda x+\lambda]!} \frac{(n-x-1)!}{(n-[\lambda x+\lambda])!} \left(\frac{p}{q}\right)^{[\lambda x+\lambda]-x-1}$$

If $[\lambda x + \lambda] = [\lambda x]$, we have

$$\frac{x!}{(x+1)!} \frac{(n-x)!}{(n-x-1)!} \leq \frac{q}{p}$$

that means $\frac{n-x}{x+1} \leq \frac{q}{p}$. The function $h(x) = \frac{n-x}{x+1}$ is decreasing in x. So, $q \geq np$.

If $[\lambda x + \lambda] = [\lambda x] + 1$, then,

$$\frac{n-x}{n-[\lambda x]} \leq \frac{x+1}{[\lambda x]+1}$$

that is $n[\lambda x] - x \leq nx - [\lambda x]$ which always is true. Therefore, if X having binomial distribution with parameters n_1 and p_1 and Y having binomial distribution with parameters n_2 and p_2, which $n_1 < n_2$ respectively. If $\frac{n_1 p_1}{1-p_1} \leq \frac{n_2 p_2}{1-p_2} \leq 1$, then, $X \leq_{plr} Y$.

Table 2: Necessary conditions for establishment discrete proportional likelihood ratio order.

$X \leq_{plr} Y$	Conditions
$X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$	$\lambda_1 \leq \lambda_2 \leq 1$
$X \sim Bin(n_1, p_1)$ and $Y \sim Bin(n_2, p_2)$	$n_1 < n_2$ and $\frac{n_1 p_1}{1-p_1} \leq \frac{n_2 p_2}{1-p_2} \leq 1$
$X \sim Ge(p_1)$ and $Y \sim Ge(p_2)$	$p_1 \geq p_2$

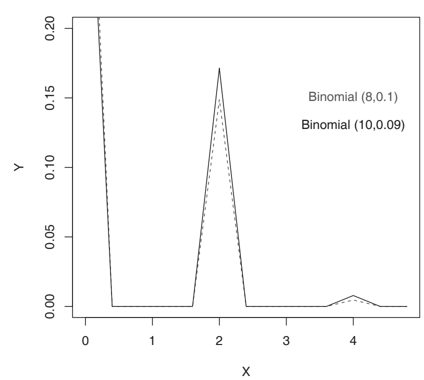

Figure 5: The Dot-Dot line shows the Binomial distribution with parameters $n_1 = 8$ and $p_1 = 0.1$ and the stretch shows the Binomial distribution with parameters $n_2 = 10$ and $p_2 = 0.09$.

At the end of paper and in order to better understand, some distributions of the PSD family are simulated satisfying in the above conditions.

4. Conclusions

In this paper, we compare some members of the PSD family due to discrete likelihood ratio order. Then we presented the discrete version of proportional likelihood ratio order as an extension of the discrete likelihood ratio order and studied it for the PSD family.

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