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A Bimodal Extension of the Generalized Gamma Distribution

Una extensión bimodal de la distribución gamma generalizada

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Abstract

A bimodal extension of the generalized gamma distribution is proposed by using a mixing approach. Some distributional properties of the new distribution are investigated. The maximum likelihood (ML) estimators for the parameters of the new distribution are obtained. Real data examples are given to show the strength of the new distribution for modeling data.

***Key words*:** Bimodality, Exponential Power Distribution, Generalized Gamma, Skewness.

Resumen

Una extensión bimodal de la distribución gamma generalizada es propuesta a través de un enfoque de mixturas. Algunas propiedades de la nueva distribución son investigadas. Los estimadores máximo verosímiles (ML por sus siglas en inglés) de los parámetros de la nueva distribución son obtenidos. Algunos ejemplos con datos reales son utilizados con el fin de mostrar las fortalezas de la nueva distribución en la modelación de datos.

***Palabras clave*:** bimodalidad, distribución potencia exponencial, gamma generalizada, sesgo.

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1. Introduction

It is not known how the real data behaves. In order to model the real data sets, a parametric model which is flexible enough to capture the data features is needed. In this paper, we propose a family of distribution with two important properties. One of these properties is bimodality and the other is skewness. The data sets, which may have bimodality and/or skewness, can be efficiently modeled with these two properties.

Hassan & Hijazi (2010) define a bimodal exponential power distribution, but their bimodal distribution has the same level of peaks and it is symmetric. Therefore, their distribution may not be very useful for data sets that have two modes with different frequencies of the observations and the asymmetry.

When exploring the literature on bimodal and skew distributions, there are many different proposals. Some of these works are (Eugene, Lee & Famoye 2002, Famoye, Lee & Eugene 2004, Ahmed, Goria & Hussein 2008, Sanhueza, Leiva & Balakrishnan 2008, Elal-Olivero 2010, Arellano-Valle, Cortés & Gómez 2010, Jamalizadeh, Arabpour & Balakrishnan 2011, Gómez, Elal-Olivero, Salinas & Bolfarine 2011, Sanhueza, Leiva & López-Kleine 2011, Rêgo, Cintra & Cordeiro 2012, Pereira, Marques & da Costa 2012, Torres-Avilés, Icaza & Arellano-Valle 2012, Genc 2013, Shams & Alamatsaz 2013, Rocha, Loschi & Arellano-Valle 2013, Cooray 2013, Martínez-Flórez, Vergara-Cardozo & González 2013, Salinas, Martínez-Flórez & Moreno-Arenas 2013, Abdulah & Elsalloukh 2013, Gui 2014, Gómez, Bolfarine & Gómez 2014, Celik, Senoglu & Arslan 2015, Iriarte, Gómez, Varela & Bolfarine 2015). The model proposed by Abdulah & Elsalloukh (2014) has a bimodality with the same height, which is not flexible enough to model bimodal data with a different number of observations in each group.

In this study, we define a new distribution as a scale mixture of the generalized gamma distribution. The resulting distribution has six parameters. Two of these parameters are the shape parameters which control the height of peaks. The other four parameters regulate the peakness, the skewness and the tail thickness. With these parameters, the model is more flexible than the previously proposed bimodal distributions for modeling bimodal data sets which may have skewness in each group.

The paper is organized as follows. In Section 2, we define the new distribution and give some distributional properties. Maximum likelihood estimations are given in Section 3. In Section 4, we give the real data examples. Finally, in the last section we give some conclusions and remarks.

2. Bimodal Generalized Gamma Distribution

It is easy to show that if $W \sim G(\frac{\delta+1}{\alpha\beta}, \eta^{\beta})$, $\delta > 0, \alpha > 0, \beta > 0$, and $\eta > 0$, then the random variable $Y = W^{1/\beta}$ will have a generalized gamma (GG) distribution with the density function

$$g(y) = \frac{\beta}{\eta^{\frac{\delta+1}{\alpha}}\Gamma(\frac{\delta+1}{\alpha\beta})} y^{\frac{\delta+1}{\alpha}-1} \exp\{-(y/\eta)^{\beta}\}. \tag{1}$$

Theorem 1. *Let Y be a continuous random variable distributed as a $GG(\beta, \eta, \frac{\delta+1}{\alpha\beta})$ with the parameters β, η and $\frac{\delta+1}{\alpha\beta}$. Let T be a discrete random variable with the following values and the corresponding probabilities,*

$$T = \begin{cases} -(1+\varepsilon), & \frac{1+\varepsilon}{2} \\ 1-\varepsilon, & \frac{1-\varepsilon}{2} \end{cases} \tag{2}$$

where $\varepsilon \in (-1, 1)$. Assume that Y and T are independent. Then, the distribution of the random variable

$$X = Y^{1/\alpha} T \tag{3}$$

will have the following density function

$$f(x) = \begin{cases} \frac{\alpha\beta}{2\eta^{\frac{\delta_1+1}{\alpha}}(1+\varepsilon)^{\delta_1}\Gamma(\frac{\delta_1+1}{\alpha\beta})}(-x)^{\delta_1}\exp\{-\frac{(-x)^{\alpha\beta}}{\eta^{\beta}(1+\varepsilon)^{\alpha\beta}}\}, & x < 0 \\ \frac{\alpha\beta}{2\eta^{\frac{\delta_0+1}{\alpha}}(1-\varepsilon)^{\delta_0}\Gamma(\frac{\delta_0+1}{\alpha\beta})}x^{\delta_0}\exp\{-\frac{x^{\alpha\beta}}{\eta^{\beta}(1-\varepsilon)^{\alpha\beta}}\}, & x \geq 0 \end{cases} \tag{4}$$

with the parameters $\alpha > 0, \beta > 0, \delta_0 > 0, \delta_1 > 0, \eta > 0$ and ε.

***Proof*.** For $x < 0$,

$$\begin{aligned} F_1(x) = P(X < x) &= \frac{1+\varepsilon}{2} P(Y > (\frac{-x}{1+\varepsilon})^{\alpha}) = \frac{1+\varepsilon}{2}[1 - P(Y < (\frac{-x}{1+\varepsilon})^{\alpha})] \\ &= \frac{1+\varepsilon}{2}[1 - \int_0^{(\frac{-x}{1+\varepsilon})^{\alpha}} \frac{\beta}{\eta^{\frac{\delta+1}{\alpha}}\Gamma(\frac{\delta+1}{\alpha\beta})} y^{\frac{\delta+1}{\alpha}-1} exp\{-(\frac{y}{\eta})^{\beta}\}dy]. \end{aligned}$$

For $x \geq 0$,

$$\begin{aligned} F_0(x) = P(X < x) &= \frac{1-\varepsilon}{2} P(Y < (\frac{x}{1-\varepsilon})^{\alpha}) \\ &= \frac{1-\varepsilon}{2} \int_0^{(\frac{x}{1-\varepsilon})^{\alpha}} \frac{\beta}{\eta^{\frac{\delta+1}{\alpha}}\Gamma(\frac{\delta+1}{\alpha\beta})} y^{\frac{\delta+1}{\alpha}-1} \exp\{-(\frac{y}{\eta})^{\beta}\}dy. \end{aligned}$$

Then, the derivatives of $F_1(x)$ and $F_0(x)$ give the density function $f(x)$ with the aid of the Leibniz integral rule. When we plot this, we observe that both peaks have the same height. To make this density function more flexible, we can reparametrize it by taking $\delta = \delta_1$ for $x < 0$, $\delta = \delta_0$ for $x \geq 0$. If we do so, we can get the $f(x)$ function given equation (4). To show that $f(x)$ is a density function, we have to prove that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1. \tag{5}$$

For the first integration, let $u = \frac{1}{\eta^\beta}(\frac{-x}{1+\varepsilon})^{\alpha\beta}$. Using this, we can easily see that

$$\int_{-\infty}^{0} f(x)dx = \frac{1+\varepsilon}{2}.$$

Similarly, for the second part of equation (5), let $u = \frac{1}{\eta^\beta}(\frac{x}{1-\varepsilon})^{\alpha\beta}$ then it will be easily seen that

$$\int_{0}^{\infty} f(x)dx = \frac{1-\varepsilon}{2}.$$

Thus, the desired result is obtained. □

Definition 1. The distribution of the random variable X with the density function given in equation (4) is called a bimodal extended generalized gamma ($BEGG$) distribution.

We can also define the location-scale form of this distribution.

Proposition 1. *Suppose that $Z \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. Then, the random variable $X = \mu + \sigma Z$, $\mu \in \mathbb{R}, \sigma > 0$ will have $BEGG$ distribution with the following density function ($X \sim BEGG(\mu, \sigma, \alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$)*

$$g(x) = \begin{cases} \frac{\alpha\beta}{2\sigma\eta^{\frac{\delta_0+1}{\alpha}}(1-\varepsilon)^{\delta_0}\Gamma(\frac{\delta_0+1}{\alpha\beta})}(\frac{x-\mu}{\sigma})^{\delta_0}\exp\{-\frac{(x-\mu)^{\alpha\beta}}{\eta^\beta((1-\varepsilon)\sigma)^{\alpha\beta}}\}, & x \geq \mu \\ \frac{\alpha\beta}{2\sigma\eta^{\frac{\delta_1+1}{\alpha}}(1+\varepsilon)^{\delta_1}\Gamma(\frac{\delta_1+1}{\alpha\beta})}(\frac{\mu-x}{\sigma})^{\delta_1}\exp\{-\frac{(\mu-x)^{\alpha\beta}}{\eta^\beta((1+\varepsilon)\sigma)^{\alpha\beta}}\}, & x < \mu \end{cases} \quad (6)$$

where μ and σ are the location and the scale parameters, respectively.

Proof. Let $Z \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. If we replaced Z by $\frac{X-\mu}{\sigma}$ with the Jacobian $1/\sigma$ in the density function of Z, then we get the probability density function given in equation (6). □

2.1. Some Properties

Proposition 2. *Let $X \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. Then, the cumulative distribution function (cdf) of X is*

$$F(x) = \begin{cases} F_1(x) = \int_{-\infty}^{x} f_1(u)du = \frac{1+\varepsilon}{2\Gamma(\frac{\delta_1+1}{\alpha\beta})}\Gamma(\frac{\delta_1+1}{\alpha\beta}, \frac{(-x)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}), & x < 0 \\ F_0(x) = \int_{0}^{x} f_0(u)du = \frac{1-\varepsilon}{2\Gamma(\frac{\delta_0+1}{\alpha\beta})}\gamma(\frac{\delta_0+1}{\alpha\beta}, \frac{x^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}), & x \geq 0 \end{cases} \quad (7)$$

where γ is the incomplete gamma function.

Proof. For $X < 0$, $\int_{-\infty}^{x} f_1(t)dt = F_1(x)$, let $\frac{(-t)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}$ be u, then $du = \frac{-\alpha\beta(-t)^{\alpha\beta-1}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}dt$. For $X \geq 0$, $\int_0^x f_0(t)dt = F_0(x)$, let $\frac{t^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}$ be u, then $du = \frac{\alpha\beta t^{\alpha\beta-1}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}dt$. □

Proposition 3. *Let $X \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. The rth, $r \in \mathbb{R}$, noncentral moments are given by*

$$\mathbb{E}(X^r) = \frac{(-1)^r \eta^{r/\alpha}(1+\varepsilon)^{r+1}}{2} \frac{\Gamma(\frac{\delta_1+r+1}{\alpha\beta})}{\Gamma(\frac{\delta_1+1}{\alpha\beta})} + \frac{\eta^{r/\alpha}(1-\varepsilon)^{r+1}}{2} \frac{\Gamma(\frac{\delta_0+r+1}{\alpha\beta})}{\Gamma(\frac{\delta_0+1}{\alpha\beta})}. \tag{8}$$

***Proof*.** For $X < 0$, $\mathbb{E}_1(X^r) = \int_{-\infty}^0 x^r f_1(x)dx$, let $\frac{(-x)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}$ be u, then $du = \frac{-\alpha\beta(-x)^{\alpha\beta-1}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}dx$. For $X \geq 0$, $\mathbb{E}_0(X^r) = \int_0^\infty x^r f_0(x)dx$, let $\frac{x^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}$ be u, then $du = \frac{\alpha\beta x^{\alpha\beta-1}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}dx$. As a result, $\mathbb{E}(X^r) = \mathbb{E}_1(X^r) + \mathbb{E}_0(X^r)$. □

Corollary 1. *Let $X \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. The expected value of X is*

$$\mathbb{E}(X) = \frac{-\eta^{1/\alpha}(1+\varepsilon)^2\Gamma(\frac{\delta_1+2}{\alpha\beta})}{2\Gamma(\frac{\delta_1+1}{\alpha\beta})} + \frac{\eta^{1/\alpha}(1-\varepsilon)^2\Gamma(\frac{\delta_0+2}{\alpha\beta})}{2\Gamma(\frac{\delta_0+1}{\alpha\beta})}$$

and the variance of X is

$$\begin{aligned} V(X) = {} & \frac{\eta^{2/\alpha}(1-\varepsilon)^3\Gamma(\frac{\delta_0+3}{\alpha\beta})}{2\Gamma(\frac{\delta_0+1}{\alpha\beta})} + \frac{\eta^{2/\alpha}(1+\varepsilon)^3\Gamma(\frac{\delta_1+3}{\alpha\beta})}{2\Gamma(\frac{\delta_1+1}{\alpha\beta})} \\ & - \left[\frac{-\eta^{1/\alpha}(1+\varepsilon)^2\Gamma(\frac{\delta_1+2}{\alpha\beta})}{2\Gamma(\frac{\delta_1+1}{\alpha\beta})} + \frac{\eta^{1/\alpha}(1-\varepsilon)^2\Gamma(\frac{\delta_0+2}{\alpha\beta})}{2\Gamma(\frac{\delta_0+1}{\alpha\beta})}\right]^2. \end{aligned}$$

***Proof*.** If $r = 1$, then $\mathbb{E}(X)$ is the first moment. If $r = 2$, then $\mathbb{E}(X^2)$ is the second moment. Thus, $V(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$. □

Proposition 4. *Let $X \sim BEGG(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon)$. Then, the hazard function of X is obtained as*

$$h(x) = \begin{cases} \dfrac{\frac{\alpha\beta}{2\eta^{\frac{\delta_1+1}{\alpha}}(1+\varepsilon)^{\delta_1}\Gamma(\frac{\delta_1+1}{\alpha\beta})}(-x)^{\delta_1}\exp\{-\frac{(-x)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}\}}{1-\frac{1+\varepsilon}{2}\{1-\frac{\gamma(\frac{\delta_1+1}{\alpha\beta}, \frac{(-x)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}})}{\Gamma(\frac{\delta_1+1}{\alpha\beta})}\}}, & x < 0 \\ \dfrac{\alpha\beta x^{\delta_0}\exp\{\frac{-x^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}\}}{2\eta^{\frac{\delta_0+1}{\alpha}}(1-\varepsilon)^{\delta_0}\{2-(1-\varepsilon)\gamma(\frac{\delta_0+1}{\alpha\beta}, \frac{x^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}})\}}, & x \geq 0. \end{cases} \tag{9}$$

***Proof*.** Recall that the Hazard function has the form $h(x) = \frac{f(x)}{1-F(x)}$. Using this formulae, we can easily get the Hazard function given in equation (9). Note that since the probability density function and the cumulative density function come in two parts, the Hazard function also has two parts. □

Figures 1 and 2 display some examples of the density function and corresponding cdfs of the *BEGG* distribution for some values of parameters. From these figures, we can see bimodality and skewness and we can also observe that if we take different values of δ_0 and δ_1 we can get the modes with different heights.

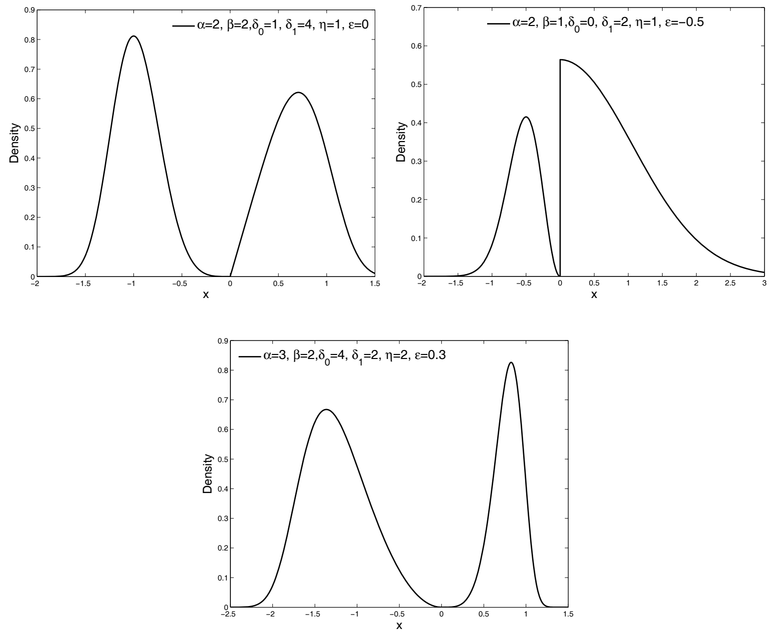

FIGURE 1: Examples of the density function of the *BEGG* distribution for the different values of parameters.

The parameters α and β control the kurtosis. The distribution is leptokurtic for $\alpha \in (0, 2)$ and $\beta = 1$, and it is platikurtic for $\alpha \in (2, \infty)$ and $\beta = 1$. The parameters δ_0 and δ_1 control the bimodality. The parameter ε and η control the skewness and the tail thickness, respectively.

2.2. Special Cases

- If $\delta_0 = \delta_1$, the density function will have two modes with the same height. If $\delta_0 = \delta_1 = 0$, the distribution will be a unimodal.
- When $\varepsilon = 0$, the distribution will be symmetric with two modes with different height.
- When $\alpha = 2$, $\beta = 1$, $\delta_0 = \delta_1 = 0$, $\eta = 2$, and $\varepsilon = 0$, the distribution will be a standard normal distribution. Location μ and scale σ case of *BEGG* distribution is defined in equation (6).
- If $\alpha = 1$, $\beta = 1$,$\delta_0 = \delta_1 = 0$, $\eta = 1$, and $\varepsilon = 0$, the distribution is the Laplace distribution with the parameters location μ and scale σ in equation (6).
- If $\beta = 1, \delta_0 = \delta_1 = \delta, \eta = 1, \varepsilon = 0$, the distribution is the bimodal exponential power distribution proposed by Hassan & Hijazi (2010).

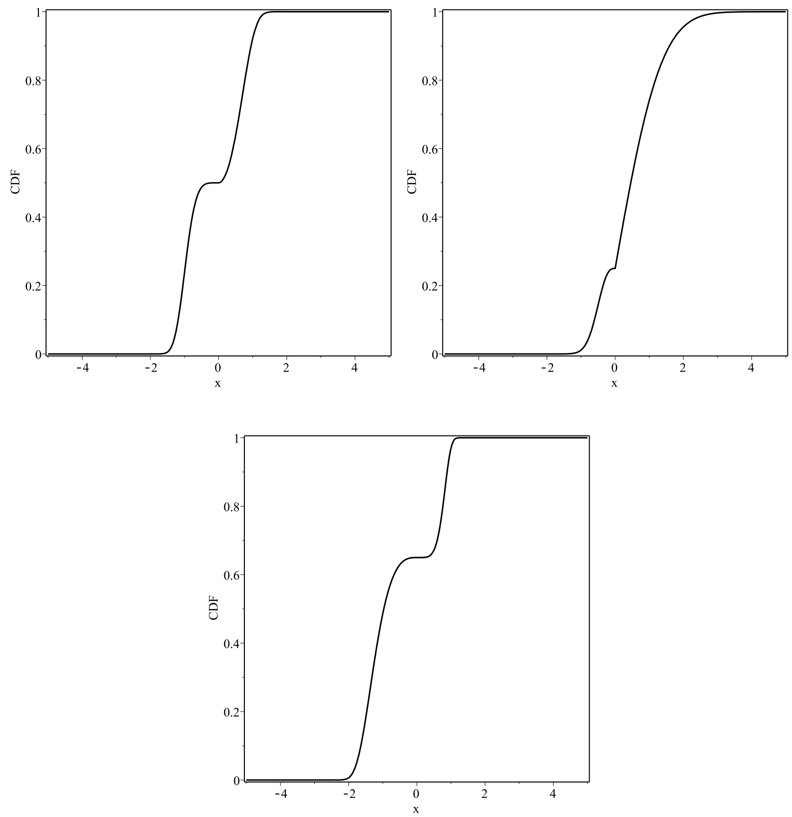

FIGURE 2: The cdf of the *BEGG* distribution given in Figure 1.

- The ε-skew exponential power distribution proposed by Elsalloukh, Guardiola & Young (2005) is a special case of this family for $\beta = 1, \delta_0 = \delta_1 = 0$ and $\eta = 2^{\alpha/2}$.
- For the case $\delta_0 = \delta_1 = k - 1$, $\alpha = 1$, $\beta = 1$, the *BEGG* distribution becomes ε-skew gamma distribution proposed by Abdulah & Elsalloukh (2013).
- For the case $\alpha = 2$, $\beta = 1$, $\delta_0 = \delta_1 = 0$, $\eta = 2$, the distribution becomes the extended skew normal distribution proposed by Arellano-Valle et al. (2010).
- When $\alpha = 2, \beta = 1, \delta_0 = \delta_1 = 0$ and $\eta = 2$, the distribution is the ε-skew normal distribution proposed by Mudholkar & Hutson (2000).

3. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a $BEGG$ distributed population. We would like to estimate the unknown parameters $\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon$. The log-likelihood function is

$$
\begin{aligned}
l = & n_0[\log(\alpha) + \log(\beta) - \log(2) - \frac{\delta_0+1}{\alpha}\log(\eta) - \delta_0\log(1-\varepsilon) \\
& - \log(\Gamma(\frac{\delta_0+1}{\alpha\beta}))] + \delta_0\sum_{i=1}^{n_0}\log(x_i) - \sum_{i=1}^{n_0}\frac{x_i^{\alpha\beta}}{\eta^\beta(1-\varepsilon)^{\alpha\beta}} \\
& + n_1[\log(\alpha) + \log(\beta) - \log(2) - \frac{\delta_1+1}{\alpha}\log(\eta) - \delta_1\log(1+\varepsilon) \\
& - \log(\Gamma(\frac{\delta_1+1}{\alpha\beta}))] + \delta_1\sum_{i=1}^{n_1}\log(-x_i) - \sum_{j=1}^{n_1}\frac{(-x_i)^{\alpha\beta}}{\eta^\beta(1+\varepsilon)^{\alpha\beta}},
\end{aligned}
\tag{10}
$$

where n_0 is the number of non-negative observations and n_1 is the number of negative observations.

The maximum likelihood estimates of the parameters $\alpha, \beta, \delta_0, \delta_1, \eta$ and ε will be the solution of the following equations

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha} = & \frac{n_0+n_1}{\alpha} + \frac{\log(\eta)}{\alpha^2}[n_0(\delta_0+1) + n_1(\delta_1+1)] \\
& + [\psi(\frac{\delta_0+1}{\alpha\beta})n_0(\delta_0+1) + \psi(\frac{\delta_1+1}{\alpha\beta})n_1(\delta_1+1)]/(\alpha^2\beta) \\
& - \frac{\beta}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}\sum_{i=1}^{n_0}(x_i^{\alpha\beta}\log(x_i) - x_i^{\alpha\beta}\log(1-\varepsilon)) \\
& - \frac{\beta}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}\sum_{j=1}^{n_1}((-x_i)^{\alpha\beta}\log(-x_i) - (-x_i)^{\alpha\beta}\log(1+\varepsilon)) = 0,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \beta} = & \frac{n_0+n_1}{\beta} + [\psi(\frac{\delta_0+1}{\alpha\beta})n_0(\delta_0+1) + \psi(\frac{\delta_1+1}{\alpha\beta})n_1(\delta_1+1)]/(\alpha\beta^2) \\
& - \sum_{i=1}^{n_0}\{x_i^{\alpha\beta}[\log(x_i^\alpha) - \log(\eta(1-\varepsilon)^\alpha)]\}/(\eta^\beta(1-\varepsilon)^{\alpha\beta}) \\
& - \sum_{j=1}^{n_1}\{(-x_i)^{\alpha\beta}[\log((-x_i)^\alpha) - \log(\eta(1+\varepsilon)^\alpha)]\}/(\eta^\beta(1+\varepsilon)^{\alpha\beta}) = 0,
\end{aligned}
\tag{12}
$$

$$\begin{aligned}\frac{\partial l}{\partial \delta_0} &= \frac{-n_0}{\alpha}\log(\eta) - n_0\log(1-\varepsilon) - \frac{n_0}{\alpha\beta}\psi(\frac{\delta_0+1}{\alpha\beta}) \\ &+ \sum_{i=1}^{n_0}\log(x_i) = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial l}{\partial \delta_1} &= \frac{-n_1}{\alpha}\log(\eta) - n_1\log(1+\varepsilon) - \frac{n_1}{\alpha\beta}\psi(\frac{\delta_1+1}{\alpha\beta}) \\ &+ \sum_{j=1}^{n_1}\log(-x_i) = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial l}{\partial \eta} &= \frac{n_0(\delta_0+1)+n_1(\delta_1+1)}{-\alpha\eta} \\ &+ \frac{\beta}{\eta^{\beta+1}}[\sum_{i=1}^{n_0} x_i^{\alpha\beta}/(1-\varepsilon)^{\alpha\beta} + \sum_{j=1}^{n_1}(-x_i)^{\alpha\beta}/(1+\varepsilon)^{\alpha\beta}] = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial l}{\partial \varepsilon} &= \frac{n_0\delta_0}{1-\varepsilon} - \frac{n_1\delta_1}{1+\varepsilon} \\ &- \frac{\alpha\beta}{\eta^{\beta}}\{\sum_{i=1}^{n_0} x_i^{\alpha\beta}/(1-\varepsilon)^{\alpha\beta+1} - \sum_{j=1}^{n_1}(-x_i)^{\alpha\beta}/(1+\varepsilon)^{\alpha\beta+1}\} = 0.\end{aligned} \tag{16}$$

Since these equations cannot be solved analytically, the numerical methods should be used to obtain the ML estimates. Since the random variable has scale mixture format the EM algorithm can be used to obtain the ML estimates. In this paper, we will use the R package called BB proposed by Varadhan & Gilbert (2009) to get the solutions of these equations. It should be noted that the BB package also uses the EM algorithm to solve the system of nonlinear equations like these equations.

4. Real Data Examples

In this section real data sets will be used to illustrate the modeling capability of the proposed distribution. We used two data sets. Here, data sets will be modeled with the $BEGG$ distribution. We first standardize the data set to get rid of estimating the location and the scale.

Example 1. The data set, which is called duration of Geyser data, is given in $MASS$ package in R. This data set is also used by Abdulah & Elsalloukh (2014). It consists of $n = 299$ observations, and preliminary examination of this data set shows bimodality (see Figure 3). Figure 3 shows the histogram of the data set with fitted densities from $BEGG$, $ESIG$ (Epsilon Skew Inverted Gamma) and BEP (Bimodal Exponential Power) distributions. In Table 1 and 2, the estimates of the parameters and the log-likelihood, AIC, BIC are given, respectively. We can see that the proposed distribution can capture the bimodality and accurately model the data. It has the smallest AIC and BIC among these three distributions.

Table 1: MLE of parameters for the duration of geyser data.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\eta}$	$\hat{\varepsilon}$	$\hat{k}$	$\hat{b}$
$BEGG$	2.45979	1.85121	1.00344	2.60223	1.33729	0.22032	-	-
BEP	2.36511	-	1.43577	$\hat{\delta}_0 = \hat{\delta}_1$	-	-	-	-
$ESIG$	-	-	-	-	-	-0.13725	1.39692	0.73039

Table 2: Log-likelihood, AIC and BIC values.

	Log(L)	AIC	BIC
$BEGG$	-6.54	25.08	47.29
BEP	-357.46	718.91	726.31
$ESIG$	-542.10	1090.12	1101.23

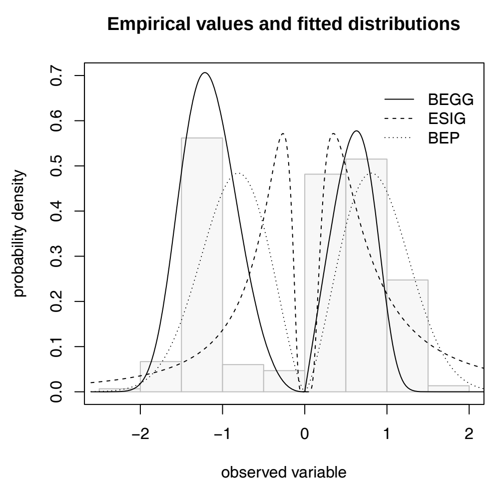

Figure 3: Histogram of the geyser data set together with the fitted three distributions.

Example 2. In this example we will use the height data set which consists of height of 126 students from the University of Pennsylvania (Cruz-Medina 2001). The same data set is also considered by Hassan & Hijazi (2010). In this paper we used the $BEGG$ distribution to model the data set. In Table 3 and 4, the estimates for the parameters, the log-likelihood, $AICs$ and $BICs$ are given for the $BEGG$, $ESIG$ and BEP distributions, respectively. From AIC and BIC, we observe that $BEGG$ distribution again has the smallest AIC and BIC. In Figure 4, the histogram of the data set and fitted densities from the above distributions are displayed. This figure also confirms that the *BEGG* distribution provides a better fit than the other distributions in terms of capturing the bimodality. Note that for this data set the estimate for the skewness parameter is 0.026275 which indicates that the skewness is not a serious problem.

Table 3: MLE of parameters for the height data.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\eta}$	$\hat{\varepsilon}$	$\hat{k}$	$\hat{b}$
BEGG	2.632853	1.285872	0.662026	0.498223	2.481848	0.026275	-	-
BEP	1.59198	-	0.42346	$\hat{\delta}_0 = \hat{\delta}_1$	-	-	-	-
ESIG	-	-	-	-	-	0.09501	1.30702	0.52757

Table 4: Log-likelihood, *AIC* and *BIC* values.

	Log(L)	AIC	BIC
BEGG	-85.39	182.78	199.79
BEP	-174.8047	353.60	359.28
ESIG	-204.89	415.79	424.29

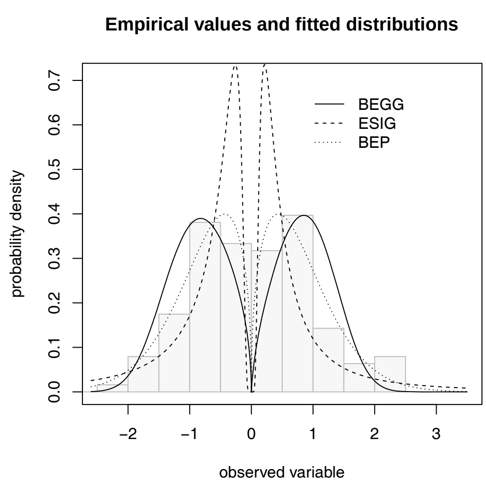

Figure 4: Histogram of the height data set together with the fitted three distributions.

5. Conclusions

We have proposed a new family of bimodal distributions. The advantage of the new family is that the data sets that may have bimodal empirical distribution with different numbers of observations in each mode can be easily modeled using the distributions in this family. We have shown that many of the well known distributions are special or limiting cases of this family. Therefore, the new family can be considered as a unified family of the bimodal distributions defined in this fashion.

We have provided some examples to show the strength of this family for modeling bimodality and skewness. We have observed from these examples that the distributions that belong to the new family can provide alternative distributions to model bimodality.

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