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Slashed Exponentiated Rayleigh Distribution

Distribución Slash Rayleigh exponenciada

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Abstract

In this paper we introduce a new distribution for modeling positive data with high kurtosis. This distribution can be seen as an extension of the exponentiated Rayleigh distribution. This extension builds on the quotient of two independent random variables, one exponentiated Rayleigh in the numerator and $Beta(q, 1)$ in the denominator with $q > 0$. It is called the slashed exponentiated Rayleigh random variable. There is evidence that the distribution of this new variable can be more flexible in terms of modeling the kurtosis regarding the exponentiated Rayleigh distribution. The properties of this distribution are studied and the parameter estimates are calculated using the maximum likelihood method. An application with real data reveals good performance of this new distribution.

***Key words*:** Exponentiated Rayleigh Distribution, Kurtosis, Maximum Likelihood, Rayleigh Distribution, Slash Distribution.

Resumen

En este trabajo presentamos una nueva distribución para modelizar datos positivos con alta curtosis. Esta distribución puede ser vista como una extensión de la distribución Rayleigh exponenciada. Esta extensión se construye en base al cuociente de dos variables aleatorias independientes, una Raileigh exponenciada en el numerador y una $Beta(q, 1)$ en el denominador con $q > 0$. La llamaremos variable aleatoria slash Rayleigh exponenciada. Hay evidencias que la distribución de esta nueva variable puede ser más flexible en términos de modelizar la curtosis respecto a la distribución Rayleigh exponenciada. Se estudian las propiedades de esta distribución y se calculan

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las estimaciones de los parámetros utilizando el método de máxima verosimilitud. Una aplicación con datos reales revela el buen rendimiento de esta nueva distribución.

***Palabras clave*:** curtosis, distribución Rayleigh, distribución Rayleigh exponenciada, distribución Slash, máxima verosimilitud.

1. Introduction

An up to date and detailed review of the exponentiated Weibull (EW) distribution is presented in Nadarajah, Cordeiro & Ortega (2013) (see also, Cancho, Bolfarine & Achcar 1999). A random variable Z is said to follow the EW distribution if its cumulative distribution function and probability density function are given, respectively, by

$$F_Z(z;\alpha,\beta,\theta) = (1 - e^{-(\beta z)^\theta})^\alpha$$

and

$$f_Z(z;\alpha,\beta,\theta) = \theta\alpha\beta^\theta z^{\theta-1} e^{-(\beta z)^\theta}(1 - e^{-(\beta z)^\theta})^{\alpha-1}$$

for $z > 0$, $\alpha > 0$, $\beta > 0$ and $\theta > 0$. The particular case $\theta = 2$ is the Burr type X distribution studied by various authors. In this case, one obtains by scale transformation the distribution of a random variable X that follows an exponentiated Rayleigh (ER) distribution if its density function is given by

$$F_X(x;\alpha,\lambda) = (1 - e^{-\lambda x^2})^\alpha$$

and

$$f_X(x;\alpha,\lambda) = 2\alpha\lambda x e^{-\lambda x^2}(1 - e^{-\lambda x^2})^{\alpha-1} \tag{1}$$

respectively, for $x > 0$, $\lambda > 0$ and $\alpha > 0$. We write $X \sim ER(\alpha,\lambda)$.

In Gómez, Quintana & Torres (2007), of slash elliptical distributions was introduced. This class of distributions can be seen as an extension of the class of elliptical distributions studied in Fang, Kotz & Ng (1990). Genc (2007) derived the univariate slash by a scale mixture of the exponential power distribution and investigated asymptotically the bias of the estimators. Wang & Genton (2006) proposed the multivariate skew version of this distribution and examined its properties and inferences.

A random variable Y follows a slash elliptical distribution with location parameter μ and scale parameter σ, denoted by $Y \sim SEl(t;\mu,\sigma,g)$, if it can be represented as

$$Y = \sigma\frac{X}{U^{1/q}} + \mu,$$

where $X \sim El(0,1,g)$ and $U \sim U(0,1)$ are independent and $q > 0$. If $Y \sim SEl(0,1,q)$, then the density function of Y is given by

$$f_Y(y;0,1,q) = \begin{cases} \frac{q}{2|y|^{q+1}}\int_0^{y^2} v^{\frac{q-1}{2}} g(v)dv, & \text{if } y \neq 0, \\ \frac{q}{1+q} g(0), & \text{if } y = 0. \end{cases} \tag{2}$$

In the canonic case, namely $q = 1$, (2) reduces to

$$f_Y(y; 0, 1, 1) = \begin{cases} \frac{G(y^2)}{2y^2}, & \text{if } \ y \neq 0, \\ \frac{1}{2} g(0), & \text{if } \ y = 0, \end{cases}$$

where $G(x) = \int_0^x g(v)\, dv$.

The canonic case with $g = \phi$ where ϕ density function for the standard normal distribution, was studied by Rogers & Tukey (1972) and by Mosteller & Tukey (1977). Arslan (2008) discussed asymmetric versions of the slash elliptical family of distributions. Gómez, Olivares-Pacheco & Bolfarine (2009) considered the slash elliptical family of distributions to extend the Birnbaum-Saunders family of distributions (Leiva, Soto, Cabrera & Cabrera 2011). Olmos, Varela, Gómez & Bolfarine (2012) made use of the slash elliptical family of distributions to extend the half-normal distribution. Olivares-Pacheco, Cornide-Reyes & Monasterio (2010) used the slash elliptical family to extend the Weibull distribution. Iriarte, Gómez, Varela & Bolfarine (2015) use the family of slash elliptical distributions to extend the Rayleigh distribution.

The rest of this paper is organized as follows. In Section 2, we propose the new slash distribution and investigate its properties, including a stochastic representation. Section 3 discusses inference for model parameters, including moments and maximum likelihood estimation (MLE) for the parameters. Simulation studies are performed in Section 4 revealing good performance of the MLE. Section 5 gives a real illustrative application and reports the results indicating good performance in applied scenarios. Section 6 concludes our work.

2. Slashed Exponentiated Rayleigh Distribution

2.1. Stochastic Representation

Definition 1. A random variable T has slashed exponentiated Rayleigh distribution if it can be represented as the ratio

$$T = \frac{X}{W}, \tag{3}$$

where $X \sim ER(\alpha, \lambda)$ defined in (1) and $W \sim Beta(q, 1)$ are independent, $\alpha > 0$, $\lambda > 0$, $q > 0$. We denote it as $T \sim SER(\alpha, \lambda, q)$

Proposition 1. *Let $T \sim SER(\alpha, \lambda, q)$. Then, the density function of T is given by*

$$f_T(t; \alpha, \lambda, q) = \frac{\alpha q}{\lambda^{q/2}} t^{-(q+1)} H(\lambda t^2; \alpha, q) \qquad t \geq 0,$$

where $\alpha > 0$, $\lambda > 0$, $q > 0$ and $H(x; \alpha, q)$ is defined as

$$H(x; \alpha, q) = \int_0^x u^{q/2} e^{-u} \left(1 - e^{-u}\right)^{\alpha - 1} \, du$$

***Proof*.** Using the representation given in (3) and the Jacobian method, the density function associated with T is given by

$$f_T(t;\alpha,\lambda,q) \quad = \quad 2q\alpha\lambda\int_0^1 tw^{q+1}e^{-\lambda t^2w^2}\left(1-e^{-\lambda t^2w^2}\right)^{\alpha-1}dw$$

and considering the change of variables $u=\lambda t^2w^2$ the result follows. □

***Note* 1.** Particularly, if $\alpha=\lambda=q=1$, one obtains the exponentiated canonic slashed Rayleigh distribution, and is denoted as $T\sim SER(1,1,1)$. Then, the density function of the random variable T is given by

$$f_T(t)=\frac{\sqrt{\pi}}{2}t^{-2}G(t^2,3/2,1),\quad t\geq 0.$$

where $G(x,\alpha,\beta)=\int_0^x\frac{\beta^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-\beta u}\,du$ is the cumulative distribution function of the gamma distribution.

Figure 1 shows some density functions of the slashed exponentiated Rayleigh distribution with various parameters.

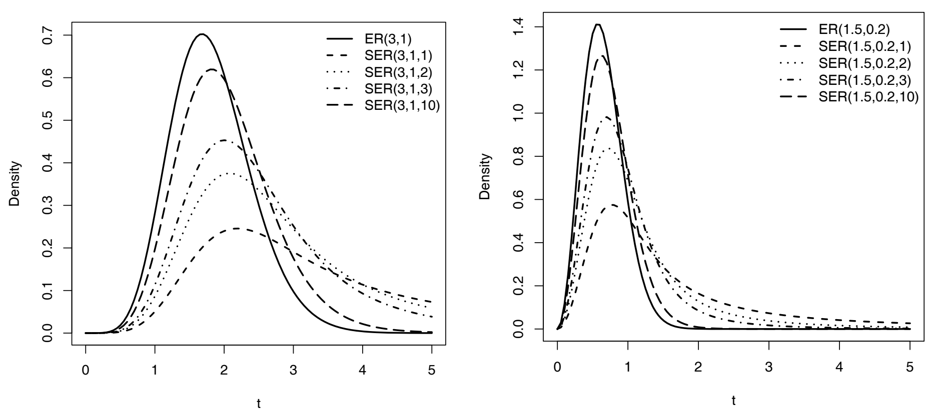

FIGURE 1: Slashed exponentiated Rayleigh density for different values of its parameters

2.2. Properties

In this subsection we deliver some basic properties of the slashed exponentiated Rayleigh distribution.

Let $T\sim SER(\alpha,\lambda,q)$, then

1. $\displaystyle\lim_{(\alpha,q)\to(1,\infty)} f_T(t;\alpha,1/(2\sigma^2),q)=\frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}}$

2. $\displaystyle\lim_{\alpha\to 1} f_T(t;\alpha,1/(2\sigma),q)=\frac{q(2\sigma)^{q/2}}{t^{q+1}}\Gamma\left(\frac{q+2}{2}\right)G\left(\frac{t^2}{2\sigma},\frac{q+2}{2},1\right)$

3. $\lim_{q\to\infty} f_T(t;\alpha,\lambda,q) = 2\alpha\lambda t e^{-\lambda t^2}\left(1-e^{-\lambda t^2}\right)^{\alpha-1}$
4. $F_T(k;\alpha,\lambda,q) = P(T<k) = (1-e^{-\lambda k^2})^{\alpha} - \frac{k}{q} f_T(k;\alpha,\lambda,q)$

where $\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}\,du$ is the gamma function.

***Note* 2.** Property 1 shows that as $(\alpha,q)\to(1,\infty)$ and $\lambda=1/(2\sigma^2)$, the slashed exponentiated Rayleigh converges to the ordinary Rayleigh distribution, i.e. $T\sim R(\sigma^2)$ (see Johnson, Kotz & Balakrishnan 1994). Property 2 shows that as $\alpha\to 1$ and $\lambda=1/(2\sigma)$ the slashed exponentiated Rayleigh distribution converges to the slashed Rayleigh distribution (Iriarte et al. 2015). Property 3 shows that as $q\to\infty$, the slashed exponentiated Rayleigh converges to the exponentiated Rayleigh distribution.

2.3. Moments

Proposition 2. *Let $T\sim SER(\alpha,\lambda,q)$, then the r^{th} moments are given by*

$$\mu_r = E(T^r) = \frac{\alpha q}{\lambda^{r/2}(q-r)} H(\alpha,r), \qquad q>r, \tag{4}$$

for $r=1,2,\ldots$ and $H(\alpha,r) := H(\infty;\alpha,r) = \int_0^\infty u^{r/2}e^{-u}\left(1-e^{-u}\right)^{\alpha-1}\,du$.

***Proof*.** Using the stochastic representation given in (3), we have that

$$\mu_r = E\left(\left(\frac{X}{W}\right)^r\right) = E(X^r W^{-r}) = E(X^r)E(W^{-r})$$

from where it follows that $E(X^r) = \frac{\alpha}{\lambda^{r/2}}\int_0^\infty u^{r/2}e^{-u}\left(1-e^{-u}\right)^{\alpha-1}du$ are the moments of the $ER(\alpha,\lambda)$ distribution and $E(W^{-r}) = q/(q-r)$ with $q>r$. □

Corollary 1. *Let $T\sim SER(\alpha,\lambda,q)$, so that*

$$E(T) = \frac{\alpha H(\alpha,1)}{\sqrt{\lambda}}\frac{q}{(q-1)},\ q>1$$

and

$$Var(T) = \frac{\alpha q}{\lambda}\left(\frac{H(\alpha,2)}{q-2} - \frac{\alpha q H^2(\alpha,1)}{(q-1)^2}\right),\ q>2$$

Corollary 2. *Let $T\sim SER(\alpha,\lambda,q)$, then the coefficients of asymmetry ($\sqrt{\beta_1}$) and kurtosis (β_2) for $q>3$ and $q>4$ are, respectively, given by*

$$\sqrt{\beta_1} = \frac{\frac{H(\alpha,3)}{q-3} - \frac{3\alpha H(\alpha,1)H(\alpha,2)q}{(q-1)(q-2)} + \frac{2\alpha^2 H^3(\alpha,1)q^2}{(q-1)^3}}{\sqrt{\alpha q}\left(\frac{H(\alpha,2)}{q-2} - \frac{\alpha q H^2(\alpha,1)}{(q-1)^2}\right)^{3/2}}$$

$$\beta_2 = \frac{\frac{H(\alpha,4)}{q-4} - \frac{4\alpha H(\alpha,1)H(\alpha,3)q}{(q-1)(q-3)} + \frac{6\alpha^2 H^2(\alpha,1)H(\alpha,2)q^2}{(q-1)^2(q-2)} - \frac{3\alpha^3 H^4(\alpha,1)q^3}{(q-1)^4}}{\alpha q\left(\frac{H(\alpha,2)}{q-2} - \frac{\alpha q H^2(\alpha,1)}{(q-1)^2}\right)^2}$$

***Note* 3.** Notice that as $(\alpha, q) \to (1, \infty)$, the asymmetry and kurtosis coefficients converge to $(\pi - 3)\sqrt{\frac{4\pi}{(4-\pi)^2}}$ y $\frac{32-3\pi^2}{(4-\pi)^2}$ respectively, which correspond to the corresponding coefficients for the Rayleigh distribution. As $q \to \infty$, the asymmetry and kurtosis coefficients converge to

$$\gamma_1 = \frac{H(\alpha,3) - 3\alpha H(\alpha,1)H(\alpha,2) + 2\alpha^2 H^3(\alpha,1)}{\sqrt{\alpha}\left(H(\alpha,2) - \alpha H^2(\alpha,1)\right)^{3/2}}$$

and

$$\gamma_2 = \frac{H(\alpha,4) - 4\alpha H(\alpha,1)H(\alpha,3) + 6\alpha^2 H^2(\alpha,1)H(\alpha,2) - 3\alpha^3 H^4(\alpha,1)}{\alpha(H(\alpha,2) - \alpha H^2(\alpha,1))^2}$$

the corresponding asymmetry and kurtosis coefficients for the exponentiated Rayleigh distribution, respectively.

Figures 2 and 3 depict graphs for the coefficients of asymmetry and kurtosis coefficients for variable SER for different values of the parameter q.

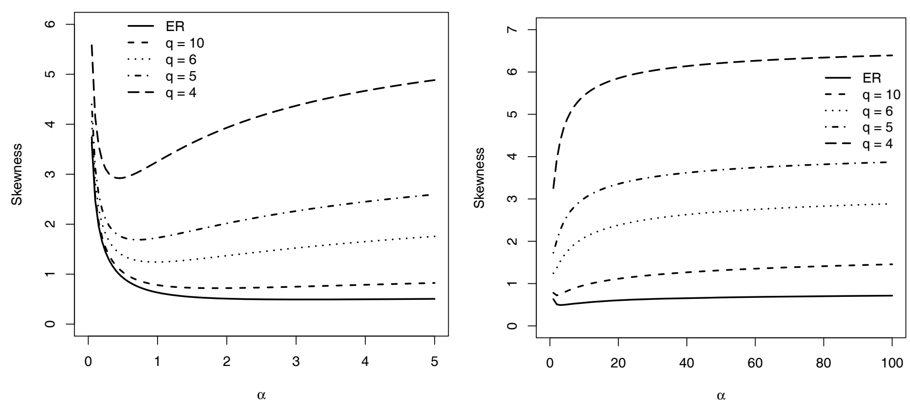

FIGURE 2: Asymmetry coefficient SER density for different values of q. The solid line corresponds to the asymmetry coefficient of ER density.

***Note* 4.** The function $H : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ can be approximated using power series about the point (α_0, r_0) as follows:

$$H(\alpha, r) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n_1+n_2=n} \frac{n!}{n_1!\,n_2!} \frac{\partial^n H(\alpha_0, r_0)}{\partial\alpha^{n_1}\partial r^{n_2}} (\alpha - \alpha_0)^{n_1} (r - r_0)^{n_2},$$

where $\frac{\partial^n H(\alpha_0,r_0)}{\partial\alpha^{n_1}\partial r^{n_2}}$ must be solved numerically.

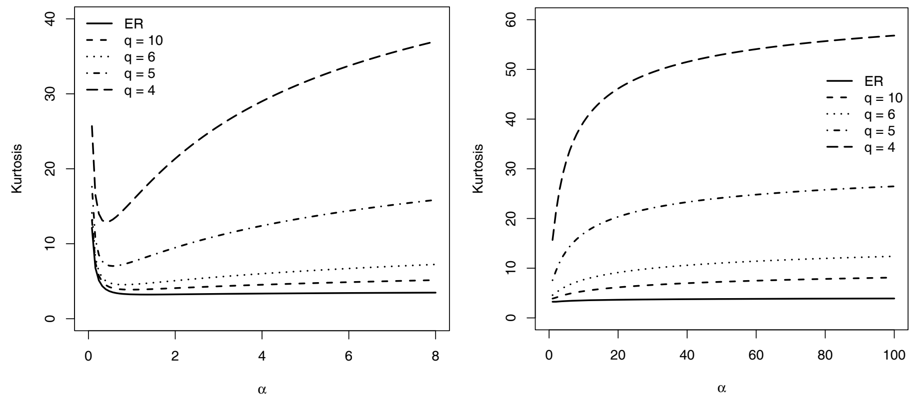

FIGURE 3: Kurtosis coefficient SER density for different values of q. The solid line corresponds to the kurtosis coefficient of ER density.

3. Inference

In this section, we study moment and maximum likelihood estimators for parameters α, λ and q for the exponentiated slashed Rayleigh distribution.

3.1. Moments Estimators (ME) Inference

The following proposition presents moment estimators for parameters α, λ and q.

Proposition 3. *Let $T_1, \dots, T_n$ a random sample from the distribution of the random variable $T \sim SER(\alpha, \lambda, q)$. Then, the moment estimators $(\widehat{\alpha}, \widehat{\lambda}, \widehat{q})$ for (α, λ, q) with $q > 3$ can be calculated numerically from the following expressions*

$$\widehat{\lambda} = \left(\frac{\widehat{\alpha}\widehat{q}}{\overline{T}(\widehat{q}-1)} \right)^2 H^2(\widehat{\alpha}, 1) \tag{5}$$

$$\widehat{q} = \frac{3(\widehat{d_1} - \widehat{d_2}) \pm \sqrt{\widehat{d_1}^2 - 10\widehat{d_1}\widehat{d_2} + 9\widehat{d_2}^2}}{2(\widehat{d_1} - \widehat{d_2})} \tag{6}$$

where $\widehat{d_1} = \frac{\widehat{\alpha}\,\overline{T^2} H^2(\widehat{\alpha},1)}{\overline{T}^2 H(\widehat{\alpha},2)}$, $\widehat{d_2} = \frac{\widehat{\alpha}^2\,\overline{T^3} H^3(\widehat{\alpha},1)}{\overline{T}^3 H(\widehat{\alpha},3)}$ and $\overline{T^k}$ is the k-th power sample mean, $k = 1, 2, 3$. The terms $\widehat{d_1}$ and $\widehat{d_2}$ depend only on the sample and the estimator $\widehat{\alpha}$.

Proof. Using (4) and replacing $E(T)$, $E(T^2)$ and $E(T^3)$ with $\overline{T}$, $\overline{T^2}$ and $\overline{T^3}$, respectively, it follows that

$$\overline{T} = \frac{\widehat{\alpha} H(\widehat{\alpha}, 1)}{\widehat{\lambda}^{1/2}} \frac{\widehat{q}}{\widehat{q} - 1}, \tag{7}$$

$$\overline{T^2} = \frac{\widehat{\alpha} H(\widehat{\alpha}, 2)}{\widehat{\lambda}} \frac{\widehat{q}}{\widehat{q} - 2}, \tag{8}$$

$$\overline{T^3} = \frac{\widehat{\alpha} H(\widehat{\alpha}, 3)}{\widehat{\lambda}^{3/2}} \frac{\widehat{q}}{\widehat{q} - 3}, \qquad \widehat{q} > 3, \tag{9}$$

From equation (7) is easy to obtain the expression for $\widehat{\lambda}$ in (5). Then, by replacing $\widehat{\lambda}$ in equations (8) and (9) we obtain

$$\frac{(\widehat{q} - 1)^2}{\widehat{q}(\widehat{q} - 2)} = \widehat{d_1}, \tag{10}$$

$$\frac{(\widehat{q} - 1)^3}{\widehat{q}^2(\widehat{q} - 3)} = \widehat{d_2}, \tag{11}$$

After an algebraic manipulation we get the equation $(\widehat{d_1} - \widehat{d_2})\widehat{q}^2 - 3(\widehat{d_1} - \widehat{d_2})\widehat{q} + 2\widehat{d_1} = 0$ whose solutions are given in (6) such that $\widehat{d_1}^2 - 10\widehat{d_1}\widehat{d_2} + 9\widehat{d_2}^2 > 0$. □

3.2. Maximum Likelihood (ML) Inference

In this section, we consider the maximum likelihood estimation for parameters $\boldsymbol{\theta} = (\alpha, \lambda, q)$ of the SER model. Suppose $t_1, t_2, \ldots, t_n$ is a random sample of size n from slashed exponentiated Rayleigh distribution. Then the log-likelihood function is given by

$$\log L(\boldsymbol{\theta}) = c(\boldsymbol{\theta}) - (q + 1) \sum_{i=1}^{n} \log(t_i) + \sum_{i=1}^{n} \log(H(t_i)), \tag{12}$$

where $H(t_i) := H\left(\lambda t_i^2; \alpha, q\right)$ and $c(\boldsymbol{\theta}) = n \log(\alpha) + n \log(q) - \frac{nq}{2} \log(\lambda)$.

Maximum likelihood estimators are obtained by maximizing the likelihood function, which can be obtained by differentiating the log-likelihood function and solving the corresponding (score) equations. The likelihood equations are given by

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \frac{H_1(t_i)}{H(t_i)} = 0,$$

$$-\frac{nq}{2\lambda} + \sum_{i=1}^{n} \frac{H_2(t_i)}{H(t_i)} = 0,$$

$$\frac{n}{q} - \frac{n}{2} \log(\lambda) - \sum_{i=1}^{n} \log(t_i) + \sum_{i=1}^{n} \frac{H_3(t_i)}{H(t_i)} = 0,$$

where

$$\begin{aligned}
H_1(t_i) &:= \frac{\partial}{\partial\alpha}H(t_i) = \int_0^{\lambda t_i^2} u^{q/2}e^{-u}(1-e^{-u})^{\alpha-1}\log(1-e^{-u})du,\\
H_2(t_i) &:= \frac{\partial}{\partial\lambda}H(t_i) = \lambda^{q/2}t_i^{q+2}e^{-\lambda t_i^2}(1-e^{-\lambda t_i^2})^{\alpha-1},\\
H_3(t_i) &:= \frac{\partial}{\partial q}H(t_i) = \frac{1}{2}\int_0^{\lambda t_i^2} u^{q/2}\log(u)\,e^{-u}(1-e^{-u})^{\alpha-1}du.
\end{aligned}$$

Therefore, numerical algorithms are required for solving the score equations. One possibility is to employ the subroutine `optim` with the R Core Team (2014).

It is well known that as the sample size increases, the distribution of the MLE tends (under regularity conditions) to the normal distribution with mean (α, λ, q) and covariance matrix equal to the inverse of the Fisher (expected) information matrix. Due to the complexity of the likelihood function it is not possible to obtain its analytical expression. It is possible, however, to work with the observed information matrix, which is a consistent estimator for the expected information matrix. The observed information matrix follows from the Hessian matrix by replacing unknown parameters by their MLEs. Some algebraic manipulation yield the following Hessian matrix:

$$I_n(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\alpha^2} & \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda\partial\alpha} & \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\alpha} \\ & \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda^2} & \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\lambda} \\ & & \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q^2} \end{pmatrix},$$

such that

$$\begin{aligned}
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\alpha^2} &= -\frac{n}{\alpha^2} + \sum_{i=1}^n \frac{H_{11}(t_i)}{H(t_i)} - \sum_{i=1}^n \left(\frac{H_1(t_i)}{H(t_i)}\right)^2,\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda\partial\alpha} &= \sum_{i=1}^n \frac{H_{12}(t_i)}{H(t_i)} - \sum_{i=1}^n \frac{H_1(t_i)H_2(t_i)}{H^2(t_i)},\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\alpha} &= \sum_{i=1}^n \frac{H_{13}(t_i)}{H(t_i)} - \sum_{i=1}^n \frac{H_1(t_i)H_3(t_i)}{H^2(t_i)},\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\alpha\partial\lambda} &= \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda\partial\alpha},\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda^2} &= \frac{nq}{2\lambda^2} + \sum_{i=1}^n \frac{H_{22}(t_i)}{H(t_i)} - \sum_{i=1}^n \left(\frac{H_2(t_i)}{H(t_i)}\right)^2\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\lambda} &= -\frac{n}{2\lambda} + \sum_{i=1}^n \frac{H_{23}(t_i)}{H(t_i)} - \sum_{i=1}^n \frac{H_2(t_i)H_3(t_i)}{H^2(t_i)}\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\alpha\partial q} &= \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\alpha},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial\lambda\partial q} &= \frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q\partial\lambda},\\
\frac{\partial^2 \log L(\boldsymbol{\theta})}{\partial q^2} &= -\frac{n}{q^2} + \sum_{i=1}^{n}\frac{H_{33}(t_i)}{H(t_i)} - \sum_{i=1}^{n}\left(\frac{H_3(t_i)}{H(t_i)}\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
H_{11}(t_i) &:= \frac{\partial}{\partial\alpha}H_1(t_i) = \int_0^{\lambda t_i^2} u^{q/2}e^{-u}(1-e^{-u})^{\alpha-1}(\log(1-e^{-u}))^2 du,\\
H_{12}(t_i) &:= \frac{\partial}{\partial\lambda}H_1(t_i) = \lambda^{q/2}t_i^{q+2}e^{-\lambda t_i^2}(1-e^{-\lambda t_i^2})^{\alpha-1}\log(1-e^{-u})du,\\
H_{13}(t_i) &:= \frac{\partial}{\partial q}H_1(t_i) = \frac{1}{2}\int_0^{\lambda t_i^2} u^{q/2}\log(u)e^{-u}(1-e^{-u})^{\alpha-1}\log(1-e^{-u})du,\\
H_{22}(t_i) &:= \frac{\partial}{\partial\lambda}H_2(t_i)\\
&= \frac{\lambda^{q/2}t_i^{q+2}e^{-\lambda t_i^2}(1-e^{-\lambda t_i^2})^{\alpha-1}(q - qe^{-\lambda t_i^2} - 2\lambda t_i^2 + 2\alpha\lambda t_i^2 e^{-\lambda t_i^2})}{2\lambda(1-e^{-\lambda t_i^2})},\\
H_{23}(t_i) &:= \frac{\partial}{\partial q}H_2(t_i) = \frac{1}{2}\lambda^{q/2}t_i^{q+2}e^{-\lambda t_i^2}(1-e^{-\lambda t_i^2})^{\alpha-1}(\log(\lambda)+2\log(t_i)),\\
H_{33}(t_i) &:= \frac{\partial}{\partial q}H_3(t_i) = \frac{1}{4}\int_0^{\lambda t_i^2} u^{q/2}(\log(u))^2 e^{-u}(1-e^{-u})^{\alpha-1}du.
\end{aligned}
$$

4. Simulation study

In this section, we conduct a small scale simulation study illustrating the MLEs behavior for parameters α, λ and q in small and moderate sample sizes. One thousand random samples of sizes n =100, 200 and 500 were generated from model $SER(\boldsymbol{\theta})$ for fixed parameters values. To generate $T \sim SER(\boldsymbol{\theta})$ the following algorithm was used:

1. Generate $U \sim U(0,1)$
2. Compute $X = \sqrt{-\frac{\log(1-U^{1/\alpha})}{\lambda}}$
3. Generate $W \sim Beta(q,1)$
4. Compute $T = XW^{-1}$

MLEs can be obtained as described above using R Core Team (2014). Empirical means and standard deviations are reported in Table 1 indicating good performances.

TABLE 1: Empirical means and SD's for the MLE's of α, λ and q.

α	λ	q	$\widehat{\alpha}$ (SD)	$\widehat{\lambda}$ (SD)	$\widehat{q}$ (SD)
			$n = 100$		
1.0	1.0	1.0	1.077 (0.314)	1.089 (0.563)	1.041 (0.155)
		1.5	1.065 (0.262)	1.083 (0.410)	1.569 (0.309)
		2.0	1.047 (0.206)	1.054 (0.361)	2.224 (1.084)
2.0	3.0	1.5	2.248 (1.178)	3.217 (1.154)	1.548 (0.249)
		2.0	2.223 (0.975)	3.172 (1.015)	2.106 (0.422)
		2.5	2.182 (0.650)	3.189 (0.952)	2.648 (0.748)
5.0	4.0	2.0	6.853 (10.173)	4.349 (1.383)	2.060 (0.340)
		2.5	6.088 (3.874)	4.243 (1.220)	2.610 (0.524)
		3.0	6.031 (3.293)	4.253 (1.104)	3.153 (0.924)
			$n = 200$		
α	λ	q	$\widehat{\alpha}$ (SD)	$\widehat{\lambda}$ (SD)	$\widehat{q}$ (SD)
1.0	1.0	1.0	1.021 (0.174)	1.025 (0.313)	1.024 (0.110)
		1.5	1.030 (0.152)	1.036 (0.270)	1.541 (0.193)
		2.0	1.029 (0.142)	1.036 (0.242)	2.063 (0.331)
2.0	3.0	1.5	2.105 (0.434)	3.121 (0.686)	1.515 (0.161)
		2.0	2.078 (0.407)	3.085 (0.663)	2.051 (0.288)
		2.5	2.072 (0.361)	3.071 (0.598)	2.559 (0.388)
5.0	4.0	2.0	5.499 (1.844)	4.125 (0.798)	2.030 (0.220)
		2.5	5.362 (1.477)	4.094 (0.699)	2.551 (0.301)
		3.0	5.285 (1.387)	4.045 (0.638)	3.098 (0.412)
			$n = 500$		
α	λ	q	$\widehat{\alpha}$ (SD)	$\widehat{\lambda}$ (SD)	$\widehat{q}$ (SD)
1.0	1.0	1.0	0.991 (0.115)	0.968 (0.197)	1.018 (0.070)
		1.5	1.006 (0.091)	1.008 (0.156)	1.514 (0.118)
		2.0	1.010 (0.084)	1.010 (0.147)	2.027 (0.188)
2.0	3.0	1.5	2.025 (0.248)	3.014 (0.392)	1.508 (0.102)
		2.0	2.032 (0.230)	3.042 (0.370)	2.015 (0.150)
		2.5	2.032 (0.214)	3.034 (0.367)	2.521 (0.231)
5.0	4.0	2.0	5.494 (1.908)	4.113 (0.793)	2.031 (0.221)
		2.5	5.152 (0.822)	4.052 (0.430)	2.514 (0.182)
		3.0	5.148 (0.824)	4.038 (0.438)	3.029 (0.254)

5. Real Data Illustration

We consider a data set to the life of fatigue fracture of Kevlar 49/epoxy which is subject to constant pressure at the 90% stress level until all failed, so we have complete data with the exact times of failure. For previous studies with this data set; see, Andrews & Herzberg (1985) and Barlow, Toland & Freeman (1984). Using results in Section 3.1, the following moment estimators were computed: $\widehat{\alpha}_M = 0.257$, $\widehat{\lambda}_M = 0.688$ and $\widehat{q}_M = 6.587$, which were used as starting values for MLEs. Table 2 presents descriptive statistics where $\sqrt{b_1}$ are b_2 are the asymmetry and kurtosis coefficients, respectively. We note that the data set presents high positive asymmetry and also high kurtosis. Table 3 shows maximum likelihood estimators for the parameters of the exponentiated Rayleigh and slashed exponentiated Rayleigh distributions. The usual Akaike information criterion (AIC) introduced by Akaike (1974) and Bayesian information criterion (BIC) proposed by Schwarz (1978) to measure the goodness of fit are also computed. It is known that AIC=$2k - 2\text{loglik}$ and BIC=$k \log n - 2\text{loglik}$ where k is the number of parameters in the model, n is the sample size and loglik is the maximized value of the likelihood function. For the ER model, AIC= $2(2) - 2(-107.697) = 219.394$ and BIC= $2\log(101) - 2(-107.697) = 224.624$. Similarly, for the SER model, AIC= $2(3) - 2(-100.594) = 207.188$ and BIC= $3\log(101) - 2(-100.594) = 215.033$. In both cases, the SER model has the lowest values of AIC and BIC. Thus, the results show that the SER model fits better the data set. Figure 4 displays the fitted models using the MLEs.

Table 2: Summary for stress-rupture data set.

sample size	mean	variance	$\sqrt{b_1}$	b_2
101	1.025	1.253	3.047	18.475

Table 3: Maximum likelihood parameter estimates (with (SD)) of the ER and SER models for stress-rupture data set.

Model	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{q}$	loglik	AIC	BIC
ER	0.312(0.035)	0.174(0.033)		−107.697	219.394	224.624
SER	0.382(0.047)	0.686(0.230)	2.759(0.828)	−100.594	207.188	215.033

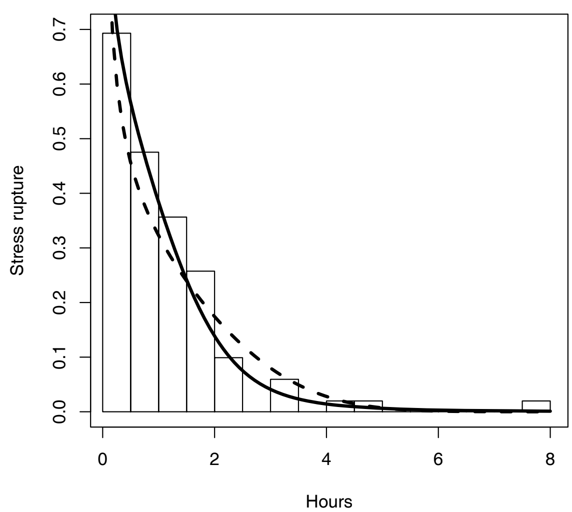

Figure 4: Histogram and models fitted for stress-rupture data set: SER (solid line) and ER (dashed line).

6. Concluding Remarks

In this paper the slashed exponentiated Rayleigh distribution (SER) is studied. A random variable SER is the quotient between two independent random variables,

an exponentiated Rayleigh and the $Beta(q, 1)$ random variable with $q > 0$. This proposal generalizes the exponentiated Rayleigh family and the Rayleigh family, among others. This generalization can be used for modeling positive data with high kurtosis. Moments and maximum likelihood estimation is discussed. A real data illustration revealed that the proposed model can be very useful in practical scenarios.

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