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Proportional Hazard Birnbaum-Saunders Distribution With Application to the Survival Data Analysis

Distribución de riesgo proporcional Birnbaum-Saunders con aplicación al análisis de datos de supervivencia

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Abstract

Birnbaum & Saunders (1969*b*) used a probability distribution to explain the lifetime data and stress produced in materials. Based on this distribution, we propose a generalization of the Birnbaum-Saunders distribution, referred to as the proportional hazard Birnbaum-Saunders distribution, which includes a new parameter that provides more flexibility in terms of skewness and kurtosis than existing models. We derive the main properties of the model. We discuss maximum likelihood estimation of the model parameters. As a natural step, we define the log-linear proportional hazard Birnbaum-Saunders regression model. An empirical application to a real data set is presented in order to illustrate the usefulness of the proposed model. The results showed that the proportional hazard Birnbaum-Saunders model can be used quite effectively in analyzing survival data, reliability problems and fatigue life studies.

***Key words*:** Birnbaum-Saunders Distribution, Proportional Hazard, Reliability, Survival Data.

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Resumen

Birnbaum & Saunders (1969*b*) presentaron una distribución de probabilidad para explicar los datos de supervivencia y estrés producidos sobre los materiales. Basados en esta distribución, proponemos una generalización de la distribución Birnbaum-Saunders, la cual llamamos distribución Birnbaum-Saunders de riesgo proporcional, incluyendo un nuevo parámetro que proporciona una mayor flexibilidad en términos de asimetría y curtosis comparado con los modelos existentes. Derivamos las principales propiedades del modelo. Discutimos la estimación de máxima verosimilitud de los parámetros del modelo. Como un paso natural, definimos el modelo de regresion log-lineal Birnbaum-Saunders de riesgo proporcional. Presentamos una aplicación con un conjunto de datos reales con el propósito de ilustrar la utilidad del modelo propuesto. Los resultados mostraron que el modelo Birnbaum-Saunders de riesgo proporcional puede ser utilizado efectivamente en el análisis de datos de supervivencia, problemas de confiabilidad y estudios de resistencia a la fatiga.

***Palabras clave*:** distribución Birnbaum-Saunders, riesgo proporcional, confiabilidad, datos de supervivencia.

1. Introduction

The Birnbaum-Saunders (BS) distribution was introduced by Birnbaum & Saunders (1969*b*) to explain survival time and the stress produced in materials due to the cumulative damage laws for fatigue. This model gives probabilistic interpretation for a physical fatigue process where dominant crack growth causes failure. A random variable T has a Birnbaum–Saunders distribution if it can be expressed as

$$T = \beta \left[\frac{\gamma}{2} Z + \sqrt{\left(\frac{\gamma}{2} Z \right)^2 + 1} \right]^2, \tag{1}$$

where Z is a random variable following the standard normal distribution, denoted by $Z \sim \mathrm{N}(0, 1)$. Its density function is

$$f_T(t) = \phi \left(\frac{1}{\gamma} \left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right] \right) \ \frac{t^{-3/2}(t+\beta)}{2\gamma\sqrt{\beta}}, \quad t > 0, \tag{2}$$

where $\phi(\cdot)$ is the standard normal density function, $\gamma > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. The parameter β is also the median of the distribution. The density function (2) is right skewed and the skewness decreases with γ. We have $kT \sim \mathrm{BS}(\gamma, k\beta)$ for any $k > 0$, that is, the BS distribution is closed under scale transformations. Some interesting results on statistical inference for the BS distribution may be revised in Wu & Wong (2004) and Lemonte, Cribari-Neto & Vasconcellos (2007).

The BS distribution was extended to other families using distributions with less or more asymmetry than the normal distribution. Díaz-García & Leiva-Sánchez (2005) generalize this model to the case of elliptical distributions. Extensions of the BS distribution to the asymmetric case have been given by several authors, including, Vilca-Labra & Leiva-Sánchez (2006) extend to the elliptical asymmetric distribution known as the Doubly Generalized Birnbaum-Saunders model, Leiva, Vilca, Balakrishnan & Sanhueza (2010) present the asymmetric BS distribution with five parameters, while Castillo, Gomez & Bolfarine (2011) considered the asymmetric epsilon-Birnbaum-Saunders model and Gómez, Elal-Olivero, Salinas & Bolfarine (2009) considered an extension based on the slash-elliptical family of distributions.

Recently Martínez-Flórez, Moreno-Arenas & Vergara-Cardozo (2013) have studied an other family of univariate asymmetric distributions which is called Proportional Hazard distribution. Its probability density function is given by

$$\varphi_F(z;\alpha)=\alpha f(z)\{1-F(z)\}^{\alpha-1},\quad z\in\mathbb{R},\tag{3}$$

where α is a positive real number and F is a continuous distribution function with continuous density function f. This is denoted by PHF(α). Its hazard function with respect to the density φ_F is

$$h_{\varphi_F}(X,\alpha)=\alpha\ h_f(x)$$

where $h_f=f/(1-F)$ is the hazard function with respect to the density f.

When $F=\Phi(\cdot)$ and $f=\phi(\cdot)$, where $\Phi(\cdot)$ is the standard normal cumulative function, called the *proportional hazard normal* distribution, denoted by $Z\sim$ PHN(α). Its density function is given by

$$\varphi_\Phi(z;\alpha)=\alpha\phi(z)\{1-\Phi(z)\}^{\alpha-1},\ z\in\mathbb{R}.\tag{4}$$

This model is also an alternative to accommodate data with asymmetry and kurtosis that are outside the ranges allowed by the normal distribution. Taking any values from α they find that the range of the asymmetry and kurtosis coefficients, $\sqrt{\beta_1}$ and β_2 of the variable $Z\sim$ PHN(α) belong to the intervals $(-1.1578, 0.9918)$ and $(1.1513, 4.3023)$, respectively. They are better in terms of asymmetry and kurtosis than the skew-normal distribution and the alpha-power normal distribution (Pewsey, Gómez & Bolfarine 2012).

In this paper we extend the BS model to the case of the family of Proportional Hazard distributions. This new family of distributions is a huge generalization of the BS model since it has a newer and more flexible family than the BS model to fit survival data, those related to material fatigue, and other data types in which the BS distribution has had wide applicability, for example, pollution air (Leiva et al. 2010).

The paper is organized as follows. Section 2 is dedicated to the development of an asymmetric proportional hazard BS model and some of its properties studied. Section 3 is dedicated to moments and maximum likelihood estimation for the new model. Section 4 is dedicated to the development of the log-linear regression proportional hazard BS model. Section 5 defines the generalized proportional hazard BS distribution. Section 6 is devoted to real data applications. It is revealed that the model proposed can perform well in applied scenarios. Finally, Section 7 closes the paper with some concluding remarks.

2. Proportional Hazard Birnbaum-Saunders Model

Given the characteristics that distribution PHN have, with respect to fits data with less negative asymmetry and more platykurtic than the SN and PN distributions do, it also fit distributions with a higher positive asymmetry than PN and more leptokurtic than SN. Additionally, it fits data with as much positive asymmetry as SN does and as much kurtosis as PN does. We now extend the BS model to the case of a family of PHN distributions. Thus it can be said that the random variable T follows the proportional hazard Birnbaum-Saunders model, with shape parameter γ, scale parameter β and parameter of asymmetry α, and can be written as (1) where $Z \sim \text{PHN}(\alpha)$. The probability density function of T is on the form:

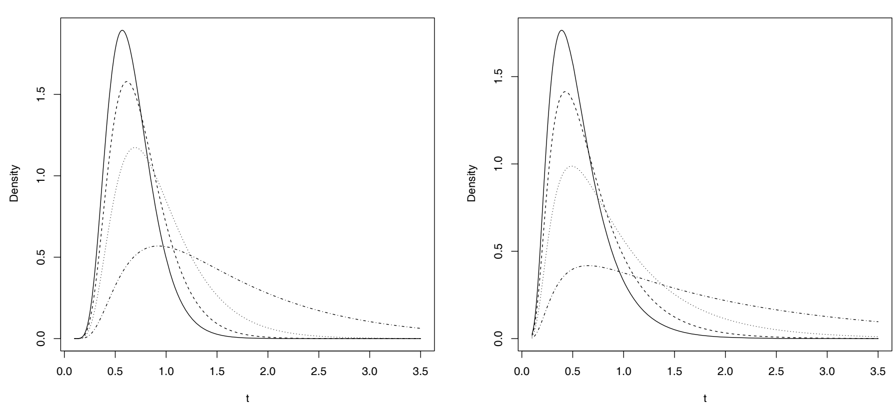
$$\varphi_T(t) = \alpha\phi(a_t)\left\{1 - \Phi(a_t)\right\}^{\alpha-1} A_t, \quad t > 0, \tag{5}$$

with $a_t = \dfrac{1}{\gamma}\left[\sqrt{\dfrac{t}{\beta}} - \sqrt{\dfrac{\beta}{t}}\right]$ and $A_t = \dfrac{t^{-3/2}(t+\beta)}{2\gamma\sqrt{\beta}}$. We use the notation $T \sim \text{PHBS}(\gamma, \beta, \alpha)$. The inclusion of α makes the proposed model more flexible than the previous extensions discussed above.

This model is a huge generalization of the BS model, since it can be applied to material fatigue data to explain the cumulative probability of stress in materials after some time, when the asymmetry and/or kurtosis of the data exceeds or is under permitted values of the BS model.

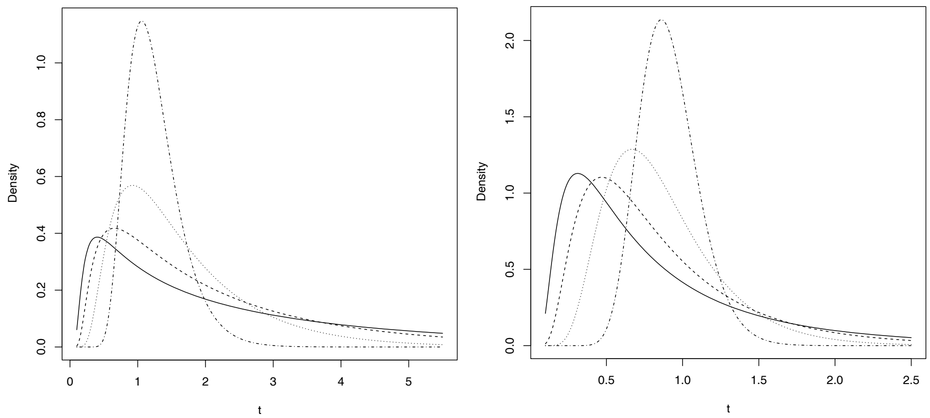
We can note that the PHBS model contains, as a special case, the BS model when $\alpha = 1$, and the Skew-BS model when $\gamma = -1$ and $\alpha = 2$.

Figures 1 and 2 depict the behavior of the distribution for some parameter values.



(a) (b)

Figure 1: PHBS distributions for $\alpha = 0.5$ (dashed and dotted line), 1.5 (dotted line), 2.5 (dashed line) and 3.5 (solid line) (a) $\gamma = 0.5$ and (b) $\gamma = 0.75$.



(a) (b)

FIGURE 2: PHBS distributions for $\gamma = 0.25$ (dashed and dotted line), 0.5 (dotted line), 0.75 (dashed line) and 1.0 (solid line) (a) $\alpha = 0.5$ and (b) $\alpha = 1.75$.

2.1. Properties

In this section we present the main properties of the PHBS model. Some of them come directly from those already known in the classic BS model.

Proposition 1. *Let* $T \sim \mathrm{PHBS}(\gamma, \beta, \alpha)$*, then the cumulative distribution function is given by*

$$\mathcal{F}_T(t;\alpha) = 1 - \{1 - \Phi(a_t)\}^{\alpha}, \ t \in \mathbb{R}^+. \tag{6}$$

The inversion method can be used to generate a random variable with PHBS distribution. Thus, if $U \sim \mathrm{U}(0,1)$, then generating the random variable $Z = 0.5\gamma\Phi^{-1}(1-(1-U)^{1/\alpha})$ with $Z_i \sim \mathrm{PHN}(0, 0.5\gamma, \alpha)$, for $i = 1, 2, \ldots, n$, where $\mathrm{PHN}(\mu, \sigma, \alpha)$ denotes the location-scale PHN model, see Martínez-Flórez et al. (2013). Thus, the random variable T with distribution $\mathrm{PHBS}(\gamma, \beta, \alpha)$ is obtained from $T = \beta\left(1 + 2Z^2 + 2Z\left(1+Z^2\right)^{1/2}\right)$.

Proposition 2. *Let* $T \sim \mathrm{PHBS}(\gamma, \beta, \alpha)$*, with* α*,* β *and* $\gamma \in \mathbb{R}^+$*. Then*

(i) $aT \sim \mathrm{PHBS}(\gamma, a\beta, \alpha)$ *for* $a > 0$.

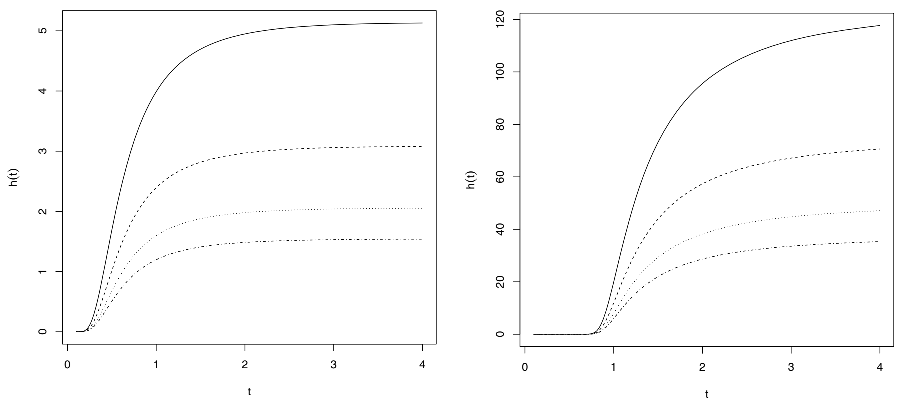
(ii) $\varphi_{T^{-1}}(t) = \alpha\phi(a_t)\left\{\Phi(a_t)\right\}^{\alpha-1} A_t$.

One of the large applications of the BS distributions is analyzing survival data. Survival functions, cumulative hazard rate and hazard function of the PHBS model are respectively show by:

$$S(t) = \{1 - \Phi(a_t)\}^{\alpha}, \quad H(t) = -\alpha\log[1 - \Phi(a_t)] \text{ and } h(t) = \alpha h_{BS}(t),$$

where $h_{BS}(t)$ is the hazard function of the BS model. That is, the hazard function of the PHBS model is proportional to the hazard function of the BS model under normality. It also has the same increase and decrease intervals.

That is, in the presence of asymmetry and/or kurtosis outside the permitted range of the BS distribution, $0 < \alpha < 1$ or $\alpha > 1$, the curve of the hazard function is above or below that of the BS model, as is illustrated in Figure 3.



(a) (b)

FIGURE 3: Plots for $h(t)$ for $\alpha = 0.75$ (dashed and dotted line), 1.0 (dotted line), 1.5 (dashed line) and 2.5 (solid line) (a) $\gamma = 0.5$ and (b) $\gamma = 0.1$.

Theorem 1. *If $h(t)$ the hazard function, then*

- $\lim_{t\to\infty} h(t) = \alpha(2\gamma^2\beta)^{-1}$
- *When $\gamma \to 0$ and $\alpha > 1$, h(t) tends to be a non-decreasing function.*

3. Moments

For the PHN(α) distribution, the r-th moment is given by

$$\mu_r = \alpha \int_0^1 \left\{\Phi^{-1}(y)\right\}^r (1-y)^{\alpha-1} dy,\ r = 0, 1, 2, \dots. \tag{7}$$

The following theorem guarantees the existence of moments for the PHBS model.

Theorem 2. *Let $T \sim \mathrm{PHBS}(\gamma, \beta, \alpha)$ and $Z \sim \mathrm{PHN}(\alpha)$. Hence, $E(T^r)$ exists if and only if,*

$$\mathbb{E}\left[\left(\frac{\gamma Z}{2}\right)^{k+l}\left(\left(\frac{\gamma Z}{2}\right)+1\right)^{\frac{k-l}{2}}\right] \tag{8}$$

exists for $k = 1, 2, \dots, r$ with $l = 0, 1, \dots, k$.

The r-th moment of the random variable T with PHBS distribution, denoted by $\mu_r = \mathbb{E}(T^r)$, can be obtained from the following theorem.

Theorem 3. *Let $T \sim \mathrm{PHBS}(\gamma, \beta, \alpha)$ and $Z \sim \mathrm{PHN}(\alpha)$. If $\mathbb{E}[Z^r]$ exists for $r = 1, 2, \ldots,$ then*

$$\frac{\mu_r}{\beta^r} = \sum_{[0 \le k \le r/2]} \left(\frac{1}{2}\right)^{2k} \left[\binom{r}{2k} \sum_{j=0}^{2k} \binom{2k}{j} \kappa_{1j} + \frac{1}{2} \binom{r}{2k+1} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \kappa_{2j} \right],$$

where $\kappa_{1j} = \mathbb{E}[(\gamma Z)^{4k-j}(\gamma^2 Z^2 + 4)^{j/2}]$, $\kappa_{2j} = \mathbb{E}[(\gamma Z)^{4k+2-j}(\gamma^2 Z^2 + 4)^{j/2}]$ and $[\cdot]$ index of the sum is the integer part function.

The central moments, for $r = 2, 3, 4$, can be obtained using the relations $\mu_2' = \mu_2 - \mu_1^2$, $\mu_3' = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$ and $\mu_4' = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4$. Then, the variance, the coefficient of variation, asymmetry and kurtosis can be obtained using the relationships: $\sigma_T^2 = \mu_2'$, $CV = \dfrac{\sigma_T}{\mu_1}$, $\sqrt{\beta_1} = \dfrac{\mu_3'}{[\mu_2']^{3/2}}$ and $\beta_2 = \dfrac{\mu_4'}{[\mu_2']^2}$.

If $\alpha = 1$ then $Z \sim \mathrm{N}(0, 1)$. We obtain that $\kappa_{11} = \kappa_{13} = \kappa_{15} = \kappa_{17} = 0$, $\kappa_{12} = 1$, $\kappa_{14} = 3$, $\kappa_{16} = 15$, $\kappa_{18} = 105$ and $\kappa_{21} = \kappa_{23} = \kappa_{25} = \kappa_{27} = 0$, from which we obtain

$$\sqrt{\beta_1(T)} = \frac{4\gamma(11\gamma^2 + 6)}{(5\gamma^2 + 4)^{3/2}} \quad \text{and} \quad \beta_2(T) = 3 + \frac{6\gamma^2(93\gamma^2 + 40)}{(5\gamma^2 + 4)^2},$$

which coincides with the results obtained by Ng, Kundu & Balakrishnan (2003) and Johnson, Kotz & Balakrishnan (1995) for the classical BS model.

3.1. Maximum Likelihood Estimators

The estimation of the $\mathrm{BS}(\alpha, \beta)$ model parameters has been directed in several ways. Birnbaum & Saunders (1969*a*) use EMV to estimate α and β, while Ng et al. (2003) study the estimators via modified moments. From & Li (2006) approach the estimation of the parameters of the model $\mathrm{BS}(\alpha, \beta)$ using some unconventional methods by using order statistics. Castillo & Hadi (1995) use the elemental percentile method, while Cisneiros, Cribari-Neto & Araújo (2008) use the technique of maximum likelihood profiled, and Farias, Moreno-Arenas & Patriota (2009) present an overview of these estimation methods. Other inferential results of the estimation process MLE in the BS model have been given by Engelhardt, Bain & Wright (1981), Lemonte et al. (2007) and Barros, Paula & Leiva (2008). Bias correction for maximum likelihood estimation (MLE) is discussed in Ng et al. (2003) and has been further investigated in Lemonte et al. (2007) where $O(n^{-1})$ bias corrected estimators are derived.

Estimation by the modified method of moments (MME) and maximum likelihood (MLE) are commonly used for the parameter estimation for the BS model. The MME estimators are given by

$$\hat{\beta}_M = \sqrt{sr}, \qquad \hat{\alpha}_M = \sqrt{2\left(\sqrt{\frac{s}{r}} - 1\right)}.$$

where $s = n^{-1}\sum_{i=1}^{n} t_i$ and $r = \left(n^{-1}\sum_{i=1}^{n}\frac{1}{t_i}\right)^{-1}$.

We now discuss the MLE for the parameter vector $\boldsymbol{\theta} = (\gamma, \beta, \alpha)^\top$ in PHBS model. Given n observations $t_1, t_2, \ldots, t_n$, with $T_i \sim \text{PHBS}(\gamma, \beta, \alpha)$, except for a constant, the log-likelihood function can be written as

$$\begin{aligned}\ell(\theta) &= n\left[\log(\alpha) - \log(\gamma) - \frac{1}{2}\log(\beta)\right] + \sum_{i=1}^{n}\log(t_i+\beta) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) \\ &- \frac{1}{2\gamma^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] + (\alpha-1)\sum_{i=1}^{n}\log(1-\Phi(a_{t_i})). \end{aligned} \tag{9}$$

The score function leading to the maximum likelihood estimators are given by

$$\sum_{i=1}^{n} a_{t_i}^2 - (\alpha-1)\sum_{i=1}^{n} a_{t_i}\frac{\phi(a_{t_i})}{1-\Phi(a_{t_i})} = n, \quad \alpha = -n^{-1}\sum_{i=1}^{n}\log\{1-\Phi(a_{t_i})\} \text{ and}$$

$$\sum_{i=1}^{n}\frac{\beta}{\beta+t_i} - \frac{\beta}{2\gamma^2}\sum_{i=1}^{n}\left[\frac{1}{t_i} - \frac{t_i}{\beta^2}\right] + \frac{\alpha-1}{2\gamma\beta^{1/2}}\sum_{i=1}^{n}\frac{t_i+\beta}{t_i^{1/2}}\frac{\phi(a_{t_i})}{1-\Phi(a_{t_i})} = \frac{n}{2}$$

The solution to the system of equations has to be obtained by numerical procedures such as the Newton-Raphson or quasi-Newton. These can be implemented using software statistical *R*.

Numerical approaches are required to solve the above system of equations. Hence, the maximum likelihood estimator for $\boldsymbol{\theta}$ can be obtained by implementing the following iterative procedure:

$$\hat{\boldsymbol{\theta}}^{(k+1)} = \hat{\boldsymbol{\theta}}^{(k)} + [J(\hat{\boldsymbol{\theta}}^{(k)})]^{-1}U(\hat{\boldsymbol{\theta}}^{(k)}), \tag{10}$$

where $J(\boldsymbol{\theta}) = -\dfrac{\partial^2\ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\,\partial\boldsymbol{\theta}^\top}$ is the observed information matrix. There are however, other numerical procedures based on the expected (Fisher) information matrix.

To initialize the MLE approach, the modified moments estimators (MME) of the BS distribution can be used. For α, when $Z = a_t$ and modified moments estimators are γ and β we obtain that

$$\hat{Z} = \frac{1}{\hat{\lambda}_M}\left(\sqrt{\frac{T}{\hat{\beta}_M}} - \sqrt{\frac{\hat{\beta}_M}{T}}\right),$$

where $\hat{\lambda}_M$ and $\hat{\beta}_M$ are the MME. Hence, using the elemental percentile approach, see Castillo & Hadi (1995), the percentiles estimator of α for the *i-th* order statistic $t_{(i)}$ is given by

$$\hat{\alpha}(i) = \frac{\log((n-i)+1) - \log(n+1)}{\log\left(1-\Phi\left(\hat{Z}\right)\right)}.$$

Then, calculating this estimator for m order statistics, $t_{(1)}, t_{(2)}, \ldots, t_{(m)}$, we get m estimates of this parameter, so a robust statistic such as the median, median least squares or the truncated mean can be used to obtain an estimate of α.

3.2. Observed and Expected Information Matrices

The elements of the observed information matrix are defined as minus the second derivative of the log-likelihood function with respect to the parameters, i.e.,

$$k_{\theta_j\theta_{j'}} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\theta_j \partial\theta_{j'}}, \ j, j' = 1, 2, 3,$$

with $\theta_1 = \gamma$, $\theta_2 = \beta$ and $\theta_3 = \alpha$.

These are written as:

$$k_{\gamma\gamma} = -\frac{n}{\gamma^2} + \frac{3}{\gamma^2}\sum_{i=1}^{n} a_{t_i}^2 + \frac{\alpha-1}{\gamma^2}\sum_{i=1}^{n} a_{t_i} C_{t_i}\left[2 + a_{t_i} B_{t_i}\right]$$

$$k_{\beta\gamma} = \frac{1}{\gamma^3}\sum_{i=1}^{n}\left[\frac{t_i}{\beta^2} - \frac{1}{t_i}\right] + \frac{\alpha-1}{2\beta^{3/2}\gamma^2}\sum_{i=1}^{n}\frac{t_i+\beta}{t_i^{1/2}} C_{t_i}\left[1 - a_{t_i} B_{t_i}\right],$$

$$\begin{aligned} k_{\beta\beta} &= -\frac{n}{2\beta^2} + \sum_{i=1}^{n}\frac{1}{(t_i+\beta)^2} + \frac{1}{\lambda^2\beta^3}\sum_{i=1}^{n} t_i \\ &+ \frac{\alpha-1}{4\beta^{5/2}\gamma}\sum_{i=1}^{n} C_{t_i}\left[\frac{3t_i+\beta}{t_i^{1/2}} - \frac{(t_i+\beta)^2}{\beta^{1/2}\gamma t} B_{t_i}\right], \end{aligned}$$

$k_{\alpha\gamma} = -\frac{1}{\gamma}\sum_{i=1}^{n} a_{t_i} C_{t_i}$, $k_{\alpha\beta} = \frac{1}{2\beta^{3/2}\gamma}\sum_{i=1}^{n}\frac{t_i+\beta}{t_i^{1/2}} C_{t_i}$, $k_{\alpha\alpha} = \frac{n}{\alpha^2}$, where $C_t = \frac{\phi(a_t)}{1-\Phi(a_t)}$ and $B_t = a_t - C_t$.

The Fisher (expected) information matrix follows by computing the expected values of the above second derivatives. This means that in this matrix $\alpha = 1$, and, $T \sim \mathrm{BS}(\gamma, \beta)$. Therefore,

$$I(\boldsymbol{\theta}) = \begin{pmatrix} \frac{2}{\gamma^2} & 0 & -\frac{0.5956}{\gamma} \\ 0 & \gamma^{-2}\beta^{-2}\left(1 + \frac{\gamma q(\gamma)}{\sqrt{2\pi}}\right) & D_1(t) \\ -\frac{0.5956}{\gamma} & D_1(t) & 1 \end{pmatrix},$$

where $D_1(t) = \mathbb{E}\left(\frac{t+\beta}{t^{1/2}} C_t\right)$, $q(\gamma) = \gamma\sqrt{\frac{2}{\pi}} - \frac{\pi \exp^{2/\gamma^2}}{2}\operatorname{erfc}\left(\frac{2}{\gamma}\right)$, with $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp^{-t^2} dt$ being the complementary error function, see Gradshteyn & Ryzhik (2007). It can be shown that $|I(\boldsymbol{\theta})| \neq 0$, so that the Fisher information matrix is not singular at $\alpha = 1$.

Hence, for large samples, the MLE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is asymptotically normal, that is

$$\hat{\boldsymbol{\theta}} \xrightarrow{A} N_3(\boldsymbol{\theta}, I_F(\boldsymbol{\theta})^{-1}).$$

The result is that the asymptotic variance of the $\hat{\boldsymbol{\theta}}$ is the inverse of $I(\boldsymbol{\theta})$. Confidence intervals for model parameters can be obtained from the above results.

Ng et al. (2003) propose Jackknife estimators (Efron 1982), to estimate the parameters of a classic Birnbaum-Saunders distribution. This same idea can be used in the PHBS distribution. Even in the classic case, the idea is to remove observation t_j in the random sample $T = \{t_1, t_2, \dots, t_n\}$. This estimates the parameters based on $n-1$ observations, following Ng et al. (2003) and naming $u_i = \log\{1-\Phi(a_{t_i})\}$, $w_i = \frac{\phi(a_{t_i})}{1-\Phi(a_{t_i})}$, $m_{t_i} = \frac{(t_i+\beta)}{t_i^{1/2}}$, $\overline{a_t w} = \sum_{i=1}^n n^{-1} a_{t_i} w_i$, $\overline{m_t w} = \sum_{i=1}^n n^{-1} m_{t_i} w_i$, $k(\beta) = n\left[\sum_{i=1}^n (\beta+t_i)^{-1}\right]^{-1}$, $i = 1, \dots, n$. We have the following expressions for α, γ and β in the PHBS model: $\alpha_{(j)} = -\frac{1}{\overline{u}_{(j)}}$ with $\overline{u}_{(j)} = \frac{1}{n-1}\sum_{i=1, i\neq j}^{n} u_i - \frac{1}{n-1}u_j = \frac{n\overline{u}-u_j}{n-1}$, $\gamma_{(j)} = \left\{\frac{\frac{s_{(j)}}{\beta_{(j)}} + \frac{\beta_{(j)}}{r_{(j)}} - 2}{1+(\alpha_{(j)}-1)\overline{(a_t w_t)}_{(j)}}\right\}^{1/2}$ with $\overline{(a_t w_t)}_{(j)} = \frac{n\overline{a_t w_t} - a_{t_j} w_{t_j}}{n-1}$, $s_{(j)} = \frac{ns-t_j}{n-1}$, and $r_{(j)} = \left[\frac{nr^{-1}-t_j^{-1}}{n-1}\right]^{-1}$ even so, $\beta_{(j)} = \left\{\frac{2\gamma_{(j)} k_{(j)}(\beta) - r_{(j)})^{-1}}{\gamma_{(j)}^2 - s_{(j)} - (\alpha_{(j)}-1)\overline{(h_t w_t)}_{(j)}}\right\}^{1/2}$ where $k_{(j)}(\beta) = \left[\frac{nk^{-1}(\beta)-(\beta+t_j)^{-1}}{n-1}\right]^{-1}$ and $\overline{(h_t w_t)}_{(j)} = \frac{n\overline{h_t w_t} - h_{t_j} w_{t_j}}{n-1}$.

Thus the Jackknife estimators are: $\alpha_{JK} = n^{-1}\sum_{i=1}^n \alpha_{(j)}$, $\lambda_{JK} = n^{-1}\sum_{i=1}^n \lambda_{(j)}$ and $\beta_{JK} = n^{-1}\sum_{i=1}^n \beta_{(j)}$. In this paper we study statistic properties of the MLE and Jackknife.

4. Log-PHBS Model

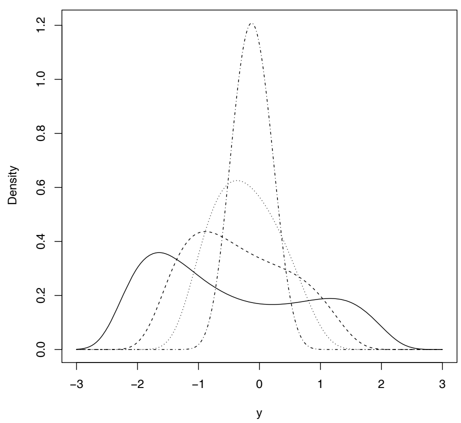
The sinh-normal model was introduced by Rieck & Nedelman (1991), and was based on a nonlinear transformation of a normal variable. This model is also known as a log-BS model, since the logarithm of a random variable with BS generates a sinh-normal variable. Different extensions of this model have been performed assuming certain types of distributions, for example, the sinh-normal model using an asymmetric setup was studied in Leiva et al. (2010), which developed a skew-sinh-normal model. Some other asymmetric extensions of the sinh-normal models are reported in Lemonte (2012) and Santana, Vilca & Leiva (2011), which report a study on influence of observations.

4.1. The Proportional Hazard Sinh-Normal Model

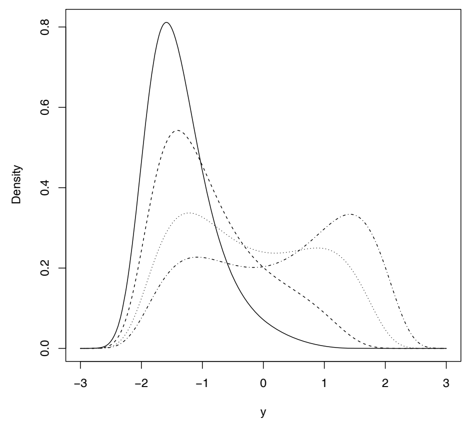
As in the log-Birnbaum-Saunders model (commonly known as sinh-normal model, see Rieck & Nedelman (1991)), the log-proportional hazard Birnbaum-Saunders model comes from the transformation, $Y = \operatorname{arcsinh}(\gamma Z/2)\sigma + \mu$ with $Z \sim \text{PHN}(\alpha)$ where $\gamma \in \mathbb{R}^+$ is shape parameter, $\alpha \in \mathbb{R}^+$ is a parameter of asymmetry, $\mu \in \mathbb{R}$ is a location parameter and $\sigma > 0$ is a scale parameter. The density function of Y is given by

$$\varphi(y) = \alpha \frac{\frac{2}{\gamma}\cosh\left(\frac{y-\mu}{\sigma}\right)}{\sigma} \phi\left(\frac{2}{\gamma}\sinh\left(\frac{y-\mu}{\sigma}\right)\right)\left\{1-\Phi\left(\frac{2}{\gamma}\sinh\left(\frac{y-\mu}{\sigma}\right)\right)\right\}^{\alpha-1}. \quad (11)$$

We denote $Y \sim \text{PHSN}(\gamma, \mu, \sigma, \alpha)$. Notice that when $\alpha = 1$, we have the sinh-normal model. Figure 4 shows the behavior of the PHSN density for some values of the parameters.



(a)



(b)

FIGURE 4: PHSN density for $\mu = 0$ and $\sigma = 1$ (a) $\alpha = 1.5$ and $\gamma = 0.75$ (dashed and dotted line), 1.5 (dotted line), 2.5 (dashed line) and 5 (solid line), (b) $\gamma = 3.5$ and $\alpha = 0.75$ (dashed and dotted line), 1.25 (dotted line), 2.5 (dashed line) and 5 (solid line).

We can observe big values of γ in Figure 4 and that the PHSN distribution can adjust data with bimodal behavior.

The cumulative density function of $Y \sim \text{PHSN}(\gamma, \mu, \sigma, \alpha)$ is given by

$$\mathbb{F}(y) = 1 - \left\{1 - \Phi\left[\frac{2}{\gamma}\sinh\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha}. \quad (12)$$

By the inversion methods we can obtain

$$Y = \mu + \sigma\left[\operatorname{arcsinh}\left\{\frac{\gamma}{2}\Phi^{-1}(1-(1-U)^{1/\alpha})\right\}\right],$$

where $U \sim \text{U}(0,1)$, is distributed according to the PHSN distribution with parameters γ, μ, σ and α.

It can be shown that if $Y \sim \text{PHSN}(\gamma, \mu, \sigma, \alpha)$ then the random variable $Z = \frac{2(Y-\mu)}{\gamma\sigma}$, converges in distribution to a random variable with distribution $\text{PHN}(\alpha)$, when $\gamma \longrightarrow 0$.

Theorem 4 is a generalization of the Theorem 1.1, as shown in Rieck & Nedelman (1991), which relates the sinh-normal model with the BS distribution. This theorem is very important in extending the log-linear BS model to the case of hazard proportional family distributions.

Theorem 4. *If* $T \sim \text{PHBS}(\gamma, \beta, \alpha)$*, then* $\log(T) \sim \text{PHSN}(\gamma, \log(\beta), 2, \alpha)$.

4.2. Log-Proportional Hazard Birnbaum-Saunders Model

We will now we define the log-proportional hazard Birnbaum-Saunders, linear regression model. Based on same considerations in Rieck & Nedelman (1991), and assuming that $Y_i = \log(T_i)$, and that the distribution of T_i is independent of a set of p explanatory variables, and can be denoted by $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$, it is a p-dimensional vector of unknown parameters. We can define the following regression model as

$$y_i = \boldsymbol{x}_i^\top \boldsymbol{\theta} + \epsilon_i, \; i = 1, \ldots, n, \tag{13}$$

where $\epsilon_i \sim \text{PHSN}(\gamma, 0, 2, \alpha)$, for $i = 1, \ldots, n$ and y_i is the log-survival for the i-th individual. This model can be denoted by $\text{LPHBS}(\gamma, \theta_1, \theta_2^*, \alpha)$, where θ_2^* is a vector of $p-1$ parameters. When $\alpha = 1$ is the log-BS model, $\text{LBS}(\gamma, \theta_1, \theta_2^*)$, then the LPHBS model is more flexible than the LBS model in terms of asymmetry and Kurtosis.

We assume that the explanatory variables are independent of the shape parameters. Then, given the above results, we can conclude that $Y_i \sim \text{PHSN}(\gamma, \boldsymbol{x}_i^\top \boldsymbol{\theta}, 2, \alpha)$ for $i = 1, \ldots, n$. It can be shown that $\mathbb{E}(Y_i) \neq \boldsymbol{x}_i^\top \boldsymbol{\theta}$, so that the intercept has to be corrected so that Y_i becomes unbiased for its expectation.

Therefore, making $\theta_1^* = \theta_1 + 2w_1(\gamma, \alpha)$, where

$$w_1(\gamma, \alpha) = \alpha \int_{-\infty}^{\infty} arcsinh\left(\frac{\gamma z}{2}\right) \phi(z) \left\{1 - \Phi(z)\right\}^{\alpha - 1} dz,$$

we obtain $\mathbb{E}(y_i) = \boldsymbol{x}_i^\top \boldsymbol{\theta}^*$, so that a linear estimator for $\boldsymbol{\theta}^* = (\theta_1^*, \boldsymbol{\theta}_2^{*\top})^\top$ can be obtained using the ordinary least squares approach, with the solution given by

$$\hat{\boldsymbol{\theta}}^* = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y},$$

and covariance matrix

$$Cov(\hat{\boldsymbol{\theta}}^*) = 4w_2(\lambda, \alpha)(\boldsymbol{X}^\top \boldsymbol{X})^{-1},$$

with $w_2(\lambda, \alpha) = Var(\epsilon)/4$.

For the vector $(\boldsymbol{\theta}^\top, \gamma, \alpha)^\top$, the log-likelihood function corresponding to the random sample $y_1, y_2, \ldots, y_n$ is:

$$\ell(\boldsymbol{\theta}^\top, \gamma, \alpha) = n\log(\alpha) + \sum_{i=1}^{n} \log(\xi_{i1}) - \frac{1}{2}\sum_{i=1}^{n} \xi_{i2}^2 + (\alpha - 1)\sum_{i=1}^{n} \xi_{i3}, \tag{14}$$

where $\xi_{i1} = 2\gamma^{-1}\cosh(z_i)$, $\xi_{i2} = 2\gamma^{-1}\sinh(z_i)$ and $\xi_{i3} = \log\left[1 - \Phi\left\{2\gamma^{-1}\sinh(z_i)\right\}\right]$, with $z_i = \dfrac{y_i - \boldsymbol{x}_i^\top \boldsymbol{\theta}}{2}$, $i = 1, 2, \ldots, n$.

The score function is given by

$$U(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n} \xi_{i3}, \qquad U(\gamma) = -\frac{n}{\gamma} + \frac{1}{\gamma}\sum_{i=1}^{n} \xi_{i2}^2 + \frac{\alpha - 1}{\gamma}\sum_{i=1}^{n} \Delta_i \xi_{i2},$$

$$U(\beta_j) = \frac{1}{2}\sum_{i=1}^{n} x_{ij}\left(\xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}}\right) + \frac{\alpha - 1}{2}\sum_{i=1}^{n} x_{ij}\Delta_i \xi_{i1}, \quad j = 1, 2, \ldots, p,$$

where $\Delta_i = \phi(\xi_{i2})/\left(1 - \Phi(\xi_{i2})\right)$. Maximum likelihood estimators for $\theta_1, \theta_2, \ldots, \theta_p$, α and γ are the solutions to the equations $U(\beta_j) = 0$, $j = 1, 2, \ldots, p$, $U(\alpha) = 0$ and $U(\gamma) = 0$, which require numerical procedures.

The information matrix can be obtained as minus the second derivative of the log-likelihood function.

5. Generalized PHBS Distribution

We now we extend the BS model to the PHF family, which is achieved assuming that the given random variable (1), $Z \sim \text{PHF}(\alpha)$. Then, the density function is of the form

$$\varphi_T(t) = \alpha f(a_t)\left\{1 - F(a_t)\right\}^{\alpha - 1} A_t, \quad t > 0, \tag{15}$$

with a_t and A_t as defined above. We use the notation $T \sim \text{GPHBS}_F(\gamma, \beta, \alpha)$. Some particular cases of the GPHBS_F, which are asymmetric-type distributions for some widely known elliptic models, are given below.

5.1. Proportional Hazard Logistic BS Distribution

The *proportional hazard logistic BS* distribution, denoted by $\text{PHBS}_L(\alpha)$, is defined by the probability density function

$$\varphi_L(t; \alpha) = \alpha A_t \exp(a_t)\left\{\frac{1}{1 + \exp(a_t)}\right\}^{\alpha + 1}. \tag{16}$$

5.2. Proportional Hazard t-Student BS Distribution

The *proportional hazard t-student BS* distribution is defined by the probability density function

$$\varphi_T(t; \alpha, v) \quad = \frac{\alpha\Gamma(\frac{v+1}{2})}{(v\pi)^{1/2}\Gamma(\frac{v}{2})}\left[1 + \frac{a_t^2}{v}\right]^{-(v+1)/2}\{1 - F_T(a_t)\}^{\alpha - 1} A_t, \tag{17}$$

where F_T is the cumulative distribution function of the t-student distribution and v is the number of degrees of freedom. The notation we use is $\text{PHBS}_T(v, \alpha)$. When $v = 1$ gives the *proportional hazard Cauchy BS*, distribution follows.

5.3. Proportional Hazard Pearson Type VII BS Distribution

The density function for the proportional hazard Pearson type VII BS distribution with parameters (q, r), denoted by $\text{PHBS}_{\text{PVII}}$, is given by

$$\begin{aligned}\varphi(t;\lambda,\beta,\alpha,q,r) &= \alpha\frac{\Gamma(q)}{\sqrt{r\pi}\Gamma(q-1/2)}\left[1+\frac{1}{r\lambda^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right]^{-q}\times\\ &\times\ \ \{1-F_{\text{PVII}}(a_t)\}^{\alpha-1}\,\frac{t^{-3/2}[t+\beta]}{2\lambda\beta^{1/2}},\end{aligned}\tag{18}$$

with $q > 1/2$, $r > 0$, where F_{PVII} is the cumulative distribution function of the Pearson type VII distribution with parameters (q, r), see Nadarajah (2008).

6. Numerical Illustrations

The distribution presented in this paper will be illustrated with the data analyzed by Birnbaum & Saunders (1969*a*). This relates to life cycles $\times\ 10^{(-3)}$ 6061-T6 of parts cut at an angle parallel to the direction of rotation, and with the rate of 18 cycles per second varied at maximum pressure of 21,000 psi.

Descriptive statistics for the data set are: $n = 101$, $\bar{t} = 1400.91$, $S^2 = 1529.10$, $\sqrt{b_1} = 0.142$ and $b_2 = 2.81$ where $\sqrt{b_1}$ and b_2 represent the asymmetry and kurtosis coefficients of the distribution of the data. The results were obtained using a *nlm* function in the statistical package *R*.

There is indication of slight asymmetry and that the kurtosis exceeds that of normality, see Castillo et al. (2011), which might be an indication that the PHBS model can fit the data in a better way. As such we propose the PHBS model as an alternative to analyze the set of data. We also adjust the log-normal model (LN) which has been widely used for this type of situation.

To compare the PHBS model with the LN model, we use the AIC, see Akaike (1974), namely $AIC = -2\hat{\ell}(\cdot) + 2k$, where k is the number of parameters. The best model is the one with the smallest AIC.

For a better justification of using the PHBS model instead of the BS model, we consider the hypothesis test of no difference for the PHBS model with the normal BS model. This is, the hypothesis

$$H_0:\ \alpha = 1\ \text{ vs. }\ H_1:\ \alpha \neq 1,$$

which compares the BS model to PHBS model. To perform this test we use the likelihood ratio statistic based on

$$\Lambda = \frac{L_{\text{BS}}(\hat{\gamma},\hat{\beta})}{L_{\text{PHBS}}(\hat{\gamma},\hat{\beta},\hat{\alpha})}.$$

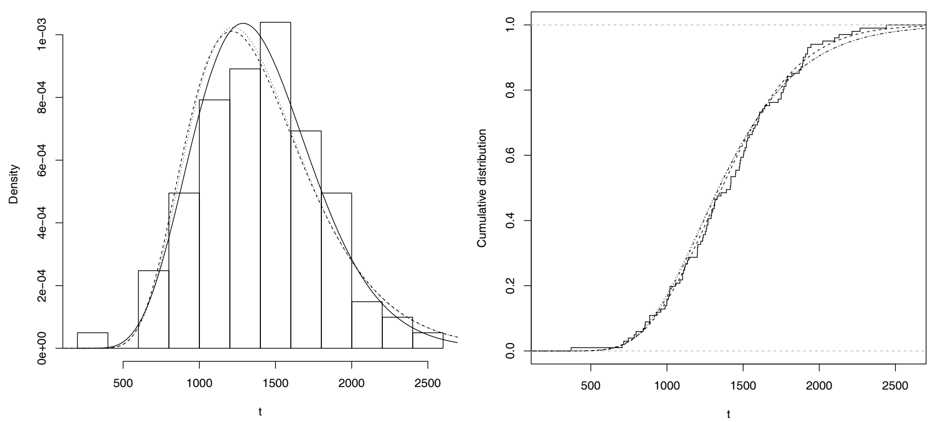
We can therefore obtain that $-2\log(\Lambda) = -2(-751.332 + 747.9702) = 6.723$, which is a greater value than the percentile of the chi-squared distribution with one degree of freedom in 95%, the hose value of which is 3.84. Thus, we conclude that the PHBS model we set is better than the BS model of normality.

Table 1 shows the estimated values for the PHBS model, compared with the classical BS model parameters. According to the AIC criterion, the PHBS model fits better than the BS and LN models.

TABLE 1: MLE for LN, BS and PHBS models.

parameter	LN	BS	PHBS
μ	7.202(0.030)	-	-
σ	0.304(0.021)	-	-
γ	-	0.310(0.021)	0.880(0.001)
β	-	1336.563(40.757)	7443.259(0.201)
α	-	-	45.945(4.593)
AIC	1505.104	1506.664	1501.940

Similarly, Figure 5(a) shows that the PHBS model is much more flexible than the LN and BS models. The empirical cumulative density function of the variable under study that was and obtained from the estimated parameters for each fitted model is shown in Figure 5(b), in which we can see that the model fits PHBS better than the set of observations.



(a) (b)

FIGURE 5: (a) Graphs for PHBS models, (solid line), BS (dashed line), and LN (dotted line) (b) empirical cdf (solid line), cdf to PHBS model (dashed line), BS model (dotted line) and LN model (dashed and dotted line).

7. Application of the LPHBS Model

The following data consists of times to failure (T) in rolling contact fatigue of ten hardened Steel specimens tested at each of the four contact stress points (x) values. The data were obtained using a 4-ball rolling contact test rig at the Princeton Laboratories of the Mobil Research and Development Co.

Chan, Ng, Balakrishnan & Zhou (2008) considers the regression model

$$Y_i = \beta_0 + \beta_1 \log(X_i) + \epsilon_i, \ i = 1, \dots, 40.$$

For this data set we adjust the models log-BS (LBS), log-skewed BS (LSBS) of Lemonte (2012) and log-proportional hazard BS (LPHBS) distributions.

The maximum likelihood estimates of the parameters of the proposed models are given in Table 2.

Table 2: MLE for LBS, LSBS and LPHBS models.

Parameters	LBS	LSBS	LPHBS
γ	1.279(0.143)	2.011(0.313)	0.727(0.002)
β_0	0.097(0.170)	-0.961(0.166)	-1.742(0.002)
β_0^*		0.165	0.228
β_1	-14.116(1.571)	-13.870(1.602)	-13.816(0.016)
α		-0.932(0.174)	0.084(0.013)
AIC	129.235	125.360	122.720

According to the AIC criterion, we can conclude that the regression model with LPHBS error distribution provides a better fit than the regression model with LSBS error distribution.

We also check the hypothesis that there are differences between the LBS and LPHBS with models the test

$$H_0: \ \alpha = 1 \ \text{ Vs } \ H_1: \ \alpha \neq 1$$

using the likelihood ratio statistics (models are nested)

$$\Lambda_1 = \frac{L_{\mathrm{LBS}}(\hat{\boldsymbol{\theta}})}{L_{\mathrm{LPHBS}}(\hat{\boldsymbol{\theta}})}.$$

Numerical evaluations indicate that

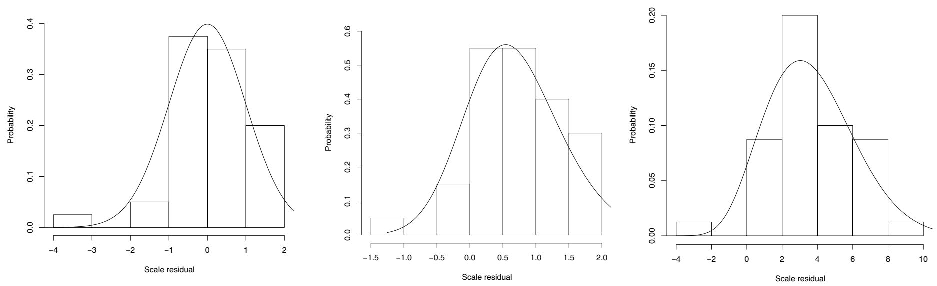
$$-2\log(\Lambda_1) = -2(-59.95 + 57.36) = 5.18,$$

which is greater than the 5% critical value 3.84. We can therefore that the LPHBS model fits the data better than the LBS model; that is, the LBS model fails to adjust the errors of the model proposed by asymmetry or kurtosis outside the range allowed by the sinh-normal distribution.

The good fit of the models studied is verified by plotting the transformed standardized residual scale $Z_i = (2/\gamma)\sinh(Y_i - \boldsymbol{x}^\top \boldsymbol{\theta})/2$ for the distribution of the estimated errors. The transformation performed, takes to the distribution of the random variable Z, which is normal for the LBS model, Skew-normal for the LSBS model and PHN for the LPHBS model.

Figure 6 depicts the distribution for the scaled residuals Z for the set of models with the corresponding theoretical distributions.

It can be seen that the LPHBS model better fits the tails of the distribution of errors, achieving a better fit than the LBS and LSBS models.



(a) (b) (c)

FIGURE 6: Q-qplots for the scaled residuals Z, from the fitted models. (a) LBS, (b) LSBS and (c) LPHBS.

8. Concluding Remarks

In this paper we have defined a new family of distributions. We have discussed several of its properties and an estimation of parameters has been done via maximum likelihood. This is supported with two real data illustrations in which we show that the LPHBS model consistently provides better fits than the LBS and LSBS models. The outcome of this practical demonstration shows that the new family is very flexible and widely applicable.

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