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Statistical Properties and Different Methods of Estimation of Transmuted Rayleigh Distribution

Propiedades estadśticas y diferentes métodos de estimación de la distribución de Rayleigh transmutada

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Abstract

This article addresses various properties and different methods of estimation of the unknown Transmuted Rayleigh (TR) distribution parameters from a frequentist point of view. Although our main focus is on estimation, various mathematical and statistical properties of the TR distribution (such as quantiles, moments, moment generating function, conditional moments, hazard rate, mean residual lifetime, mean past lifetime, mean deviation about mean and median, the stochastic ordering, various entropies, stress-strength parameter, and order statistics) are derived. We briefly describe different methods of estimation such as maximum likelihood, method of moments, percentile based estimation, least squares, method of maximum product of spacings, method of Cramér-von-Mises, methods of Anderson-Darling and right-tail Anderson-Darling, and compare them using extensive simulations studies. Finally, the potentiality of the model is studied using two real data sets. Bias, standard error of the estimates, and bootstrap percentile confidence intervals are obtained by bootstrap resampling.

***Key words*:** Distributional Moments, Order Statistics, Parameter Estimation; Rate Risk Function, Rayleigh Distribution, Transmuted Rayleigh Distribution.

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Resumen

Este artículo se aborda las varias propiedades y diferentes métodos para la estimación de los desconocidos parámetros de Transmuted Rayleigh (TR) distribución desde el punto de vista de un frequentist. Aunque la tema principal de este artículo es estimación, varias propiedades matemáticas y estadísticas de TR distribución (como cuantiles, momentos, una función que genera momentos, momentos condicionales, la tasa de peligro, la media vida residual, media vida pasada, la desviación media por media y mediana, organización stochastic, entropías varias, parámetros de tensión-fuerza y estadisticas de orden) están derivadas. Describimos brevemente los diferentes métodos de estimación, como máxima probabilidad, método de momentos, estimacién basada por percentil, mínimos cuadrados, método de máximos productos de espacios, el método de Cramér-von-Mises, los métodos de Anderson-Darling y right-tail Anderson-Darling, y compararlos con extensos estudios de simulaciones. Por último, la potencialidad del modelo está estudiando con dos conjuntos de datos reales. El margen de error, el promedio de error de las estimaciones y el percentage bootstrap de los confianza intervalos estan derivido por bootstrap remuestro.

***Palabras clave*:** momentos distributivos, estadísticas de orden, la estimación de parámetros, riesgo de tipo de función, rayleigh transmutada.

1. Introduction

Rayleigh distribution was introduced by Rayleigh (1880) and relates to a problem in the field of acoustics. This distribution has been extensively used in various fields such as communication engineering, the life-testing of electro vacuum devices, reliability theory, and survival analysis. An important characteristic of this distribution is that its failure rate is a linear function of time. The reliability function of the Rayleigh distribution decreases at a much higher rate than the reliability function of exponential distribution. This distribution relates to a number of distributions such as generalized extreme value, Weibull and Chi-square distributions and, hence, its applicability in real life situations is significant. Estimations, predictions, and inferential issues for one parameter Rayleigh distributions have been extensively studied by several authors. Interested readers may refer to Johnson, Kotz & Balakrishnan (1995) for on excellent insight into the Rayleigh distribution, and also see Dey & Das (2007), Dey (2009) for some references.

The construction of the transmuted distribution is rather simple and was first proposed by Shaw & Buckley (2007). Since then, transmuted distributions have been widely studied in statistics, and many authors have developed various transmuted-type distributions based on some well known distributions. See, for example, the transmuted extreme value distribution with applications by Alkasasbeh & Raqab (2009*a*); the transmuted Lindley distribution by Merovci (2013*b*); the transmuted generalized Rayleigh distribution by Merovci (2013*a*); the transmuted exponentiated exponential distribution by Merovci (2013*c*); the transmuted Fréchet distribution by Mahmoud & Mandouh (2013); the transmuted generalized inverse Weibull distribution by Merovci, Elbatal & Ahmed (2014); the transmuted

Linear exponential distribution by Tian, Tian & Zhu (2014); and the transmuted Pareto distribution by Merovci & Puka (2014), etc. Hence, efficient estimation of these parameters is extremely important.

We know that the maximum likelihood estimation (MLE) and the method of moments estimation (MME) are traditional methods of estimation. Although MLE is advantageous in terms of its efficiency and has good theoretical properties, there is evidence that it does not perform well, especially in the case of small samples. The method of moments is easily applicable and often gives explicit forms for estimators of unknown parameters. There are, however, cases where the method of moments does not give explicit estimators (e.g., for the parameters of the Weibull and Gompertz distributions). Therefore, other methods have been proposed in the literature as alternatives to the traditional methods of estimation. Among them, the L-moments estimator (LME), least squares estimator (LSE), generalized spacing estimator (GSE), and percentile estimator (PCE) are often suggested. Generally, these methods do not have good theoretical properties, but in some cases they can provide better estimates of the unknown parameters than the MLE and the MME. This paper considers ten different frequentist estimators for the transmuted Rayleigh distribution and evaluates their performance for different sample sizes and different parameter values. Simulations are used to compare the performance as it is not possible to compare all estimators theoretically (see Gupta & Kundu 2001, Gupta & Kundu 2007).

The uniqueness of this study comes from the fact that we provide a comprehensive description of the mathematical and statistical properties of this distribution with the hope that they will attract wider applications in lifetime analysis. Also, to the best of our knowledge thus far, no attempt has been made to compare all these estimators to the two-parameter TR distribution along with mathematical and statistical properties. Comparisons of estimation methods for other distributions have been performed in the literature: Kundu & Raqab (2005) for generalized Rayleigh distributions, Alkasasbeh & Raqab (2009*b*) for generalized logistic distributions, Mazucheli, Louzada & Ghitany (2013) for weighted Lindley distribution, Teimouri, Hoseini & Nadarajah (2013) for the Weibull distribution, Akram & Hayat (2014) for Weibull distribution, and Dey, Dey & Kundu (2014) for two-parameter Rayleigh distribution. There are some works found in the literature on transmuted Rayleigh distribution and its variants such as the slashed Rayleigh distribution Iriarte, Gómez, Varela & Bolfarine (2015) and its exponential variant Salinas, Iriarte & Bolfarine (2015).

The transmuted Rayleigh (TR) distribution was introduced by Merovci (2013*c*). He only studied moments and the maximum likelihood estimation of the unknown parameters.

A random variable X is said to have a transmuted Rayleigh distribution if its cumulative distribution function(cdf) is given by

$$G(x,\alpha,\lambda) = 1 - e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right); \qquad x>0,\ \alpha>0,\ |\lambda|\leq 1 \tag{1}$$

with corresponding pdf

$$g(x,\alpha,\lambda) = 2\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right); \qquad x>0,\ \alpha>0,\ |\lambda|\le 1, \qquad (2)$$

and the survival function is given by

$$S(x,\alpha,\lambda) = e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right); \qquad x>0\,,\alpha>0,\ |\lambda|\le 1. \qquad (3)$$

The distribution studied in this paper is a re-parameterization of the the version proposed by Merovci (2013c) with $\alpha = (2\sigma^2)^{-1}$. Note that the classical Rayleigh distribution is a special case for $\lambda = 0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted Rayleigh distribution for selected values of the parameters λ and α. This distribution has a unimodal pdf and increasing hazard rates. The latter may seem unrealistic at first in real life situation. However, in several situations, only increasing hazard rates are used or observed: Maeda & Nishikawa (2006) state that "Ruling parties in presidential systems face an increasing hazard rate in their survival"; Woosley & Cossman (2007) observe that drugs during clinical development have increasing hazard rates; Saidane, Babai, Aguir & Korbaa (2010) suppose that the demand interval in spare parts inventory systems has increasing hazard rates; Tsarouhas & Arvanitoyannis (2007) show that machines involved in bread production display increasing hazard rates; Koutras (2011) finds that software degradation times have increasing hazard rates; and Lai (2013) investigates the optimum number of minimal repairs for systems under increasing hazard rates; etc.

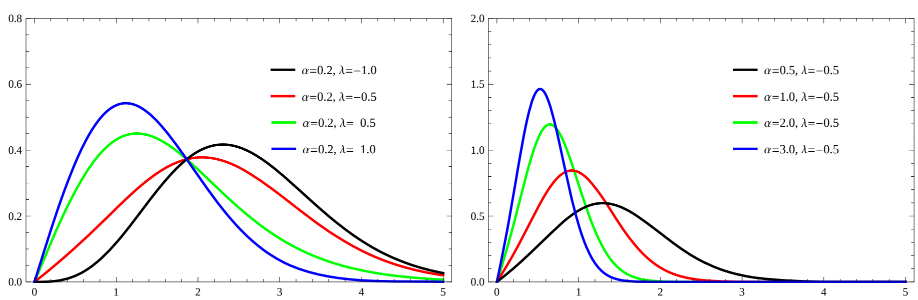


Figure 1: Probability density function of the TR distribution for different values of parameters α and λ.

The principal motivation of the paper is two fold: one is empirical and shows that the studied transmuted Rayleigh distribution outperforms at least two two-parameter distributions with respect to a real data set; the other is to show how different frequentist estimators of this distribution perform for different sample sizes and different parameter values, and to develop a guideline to choose the best estimation method for the transmuted Rayleigh distribution, which we think would be of interest to applied statisticians.

The paper is organized as follows: Various mathematical, statistical and reliability properties of the TR distribution (like shapes, quantiles, moments, moment generating function, hazard rate function, mean residual lifetime, and conditional moments, etc.) are derived in Sections 1 and 2. In Section 3, ten most frequent methods of estimations are discussed. Inference with simulation and two real data applications for the TR distribution are discussed in Section 4 and 5. In the last section, we draw some conclusions about the information contained in this article.

1.1. Shape

The study of shapes is useful to determine if a data set can be modeled by the TR distribution. Here, we discuss the shapes of the pdf g.

Theorem 1. *The probability density function, g, is unimodal.*

Proof. Notice that g is a smooth function. To prove it is unimodal, we shall first show that there is one and only one $x^* \in \mathbb{R}_+$, such that $g'(x^*) = 0$, and $g' > 0$ for $x < x^*$ & $g' < 0$ for $x > x^*$. Letting $u = e^{-\alpha x^2}$, we have

$$g'(x) = 2\alpha e^{-2u}\left[e^u(\lambda-1)(2u-1) + (2-8u)\lambda\right].$$

Since $0 < u < \infty$, $g' >, =, < 0$ depends on whether $e^u(\lambda-1)(2u-1) + (2-8u)\lambda >, =, < 0$, respectively. Now $g'(x) = 0$ implies that

$$e^u(\lambda-1)(2u-1) + (2-8u)\lambda = 0.$$

<u>Case I:</u> ($\lambda = 1$.) This implies $u^* = 1/4$, and hence $x^* = \frac{1}{2\sqrt{\alpha}}$.

Now, for $x > x^*$, i.e. $u^* > 1/4$, we have $e^u(\lambda-1)(2u-1)+(2-8u)\lambda = 2-8u < 0$ and, hence, $g' < 0$. Similarly, for $x < x^*$ we obtain $g' > 0$.

<u>Case II:</u> ($\lambda \neq 1$.) In this case, we have $Q(u) := \frac{(2u-1)e^u}{8u-2} = \frac{\lambda}{\lambda-1}$. Note that Q is a strictly increasing, continuous function of u on $(0, \frac{1}{4})$ and $(\frac{1}{4}, \infty)$. Since $\frac{\lambda}{\lambda-1} \leq \frac{1}{2}$ and Q has range $(\frac{1}{2}, \infty)$ on $(0, \frac{1}{4})$ and range $(-\infty, \infty)$ on $(\frac{1}{2}, \infty)$, there is a unique $u^* \in (\frac{1}{4}, \infty)$ such that $Q(u^*) = \frac{\lambda}{\lambda-1}$. Hence there is a unique $x^* > \frac{1}{2\sqrt{\alpha}}$ such that $g'(x^*) = 0$.

Now, for $u > u^* > 1/4$, $1/4 < u < u^*$ and $0 < u < 1/4 < u^*$, one can easily show that $g' < 0, > 0$ and > 0, respectively. Hence g is unimodal. □

2. Statistical and Mathematical Properties

In this section, we provide some important statistical and mathematical measures for the TR distribution such as quantiles, moment generating functions, moments, hazard rate and mean residual life functions, mean past lifetime, conditional moments, Stochastic ordering, mean deviation about mean and median, Shannon and Rényi entropy, order statistics, and stress strength parameters.

The quantile function $x = Q(p) = G^{-1}(p)$, for $0 < p < 1$, of the TR distribution is obtained from (1); it follows that the quantile function is

$$x = \left[-\frac{1}{\alpha} \ln \left(\frac{(\lambda - 1) + \sqrt{(\lambda - 1)^2 + 4\lambda(1-p)}}{2\lambda} \right) \right]^{\frac{1}{2}}. \tag{4}$$

In particular, the median of the TR distribution can be written as

$$Md(X) = M_d = \left[-\frac{1}{\alpha} \ln \left(\frac{(\lambda - 1) + \sqrt{\lambda^2 + 1}}{2\lambda} \right) \right]^{\frac{1}{2}}. \tag{5}$$

We hardly need to emphasize the necessity and importance of the moments in any statistical analysis, especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness, and kurtosis). If the random variable X is distributed $TR(\alpha, \lambda)$, then its nth moment around zero can be expressed as

$$E(X^n) = 2\alpha \int_0^\infty x^{n+1} \exp\left(-\alpha x^2\right) \left(1 - \lambda + 2\lambda \exp\left(-\alpha x^2\right)\right) \, dx.$$

On simplification, we get

$$E(X^n) = \alpha^{-n/2} \Gamma(1 + \frac{n}{2}) \left(1 - \lambda + \lambda 2^{-n/2}\right) \tag{6}$$

The variance, skewness, and kurtosis measures can now be calculated using the following relations

$$Var(X) = E(X^2) - E^2(X),$$

$$\text{Skewness}(X) = \frac{E(X^3) - 3E(X)E(X^2) + 2E^3(X)}{Var^{3/2}(X)},$$

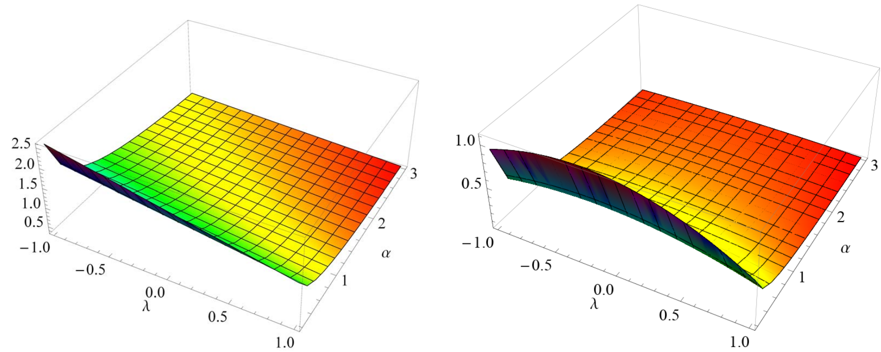
$$\text{Kurtosis}(X) = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)E^2(X) - 3E^4(X)}{Var^2(X)}.$$

Figures 2 and 3 illustrate their variations.

2.1. Moment Generating Function

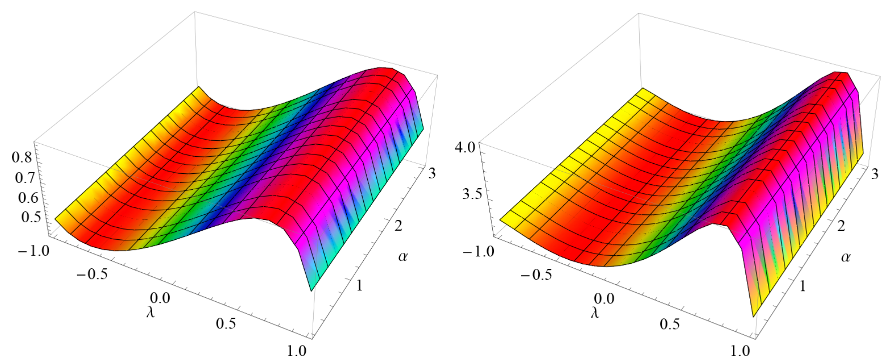
Many of the interesting characteristics and features of a distribution can be obtained via its moment generating function and moments. Let X denote a random variable with the probability density function (2). By definition of the moment generating function of X and using (2), we have

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \int_0^\infty e^{tx} g(x) dx \\ &= \frac{4\sqrt{\alpha} - 2\sqrt{\pi}(\lambda - 1) t e^{\frac{t^2}{4\alpha}} \left(\text{Erf}\left(\frac{t}{2\sqrt{\alpha}}\right) + 1\right) + \sqrt{2\pi}\lambda t e^{\frac{t^2}{8\alpha}} \left(\text{Erf}\left(\frac{t}{2\sqrt{2}\sqrt{\alpha}}\right) + 1\right)}{4\sqrt{\alpha}} \end{aligned} \tag{7}$$



(a) (b)

FIGURE 2: (a) Expectation and (b) Variance of TR distribution for $\lambda = -1, -0.9, -0.8, \cdots, 1$ and $\alpha = 0.2, 0.4, 0.6, \ldots, 3$.



(a) (b)

FIGURE 3: (a) Skewness and (b) Kurtosis of TR distribution for $\lambda = -1, -0.9, -0.8, \ldots, 1$ and $\alpha = 0.2, 0.4, 0.6, \ldots, 3$.

2.2. Stochastic Ordering

Stochastic ordering is a tool used to study structural properties of complex stochastic systems. For example, it is useful to control congestion in information-transfer over the Internet, to define treatment-related trends in clustered binary data, and order expected welfare income under different mechanisms of allocating rewards. There are different types of stochastic orderings, which are useful in ordering random variables in terms of different properties. Here we consider four different stochastic orders, namely, the usual, the hazard rate, the mean residual life, and the likelihood ratio order for two independent TR random variables under a restricted parameter space. If $X$and Y are independent random variables with CDFs F_X and F_Y, respectively, then X is said to be smaller than Y in the

- stochastic order $(X \leq_{st} Y)$ if $\mathrm{F}_X(x) \geq F_Y(x)$ for all x

- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked & Shanthikumar (1994) are well known for establishing stochastic ordering of distributions.

$$\begin{array}{ccc} X \leq_{lr} Y \Longrightarrow X & \leq_{hr} & Y \Longrightarrow X \leq_{mrl} Y \\ & \Downarrow & \\ & X \leq_{st} Y & \end{array} \tag{8}$$

The TR is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem. It shows the flexibility of the two parameter TR distribution.

Theorem 2. *Let* $X\sim\ TR(\alpha_1, \lambda_1)$ *and* $Y\sim\ TR(\alpha_2, \lambda_2)$*. If* $\alpha_1 = \alpha_2 = \alpha$ *and* $\lambda_1 \geq \lambda_2$*, then* $X \leq_{lr} Y$ *and, hence,* $X \leq_{hr} Y$*,* $X \leq_{mrl} Y$ *and* $X \leq_{st} Y$*.*

***Proof*.** The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1 x e^{-\alpha_1 x^2}(1-\lambda_1+2\lambda_1 e^{-\alpha_1 x^2})}{\alpha_2 x e^{-\alpha_2 x^2}(1-\lambda_2+2\lambda_2 e^{-\alpha_2 x^2})}$$

$$\text{Thus,} \frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = -2(\alpha_1-\alpha_2)x - \frac{4\alpha_1 x\lambda_1 e^{-\alpha_1 x^2}}{(1-\lambda_1+2\lambda_1 e^{-\alpha_1 x^2})} + \frac{4\alpha_2 x\lambda_2 e^{-\alpha_2 x^2}}{(1-\lambda_2+2\lambda_2 e^{-\alpha_2 x^2})}.$$

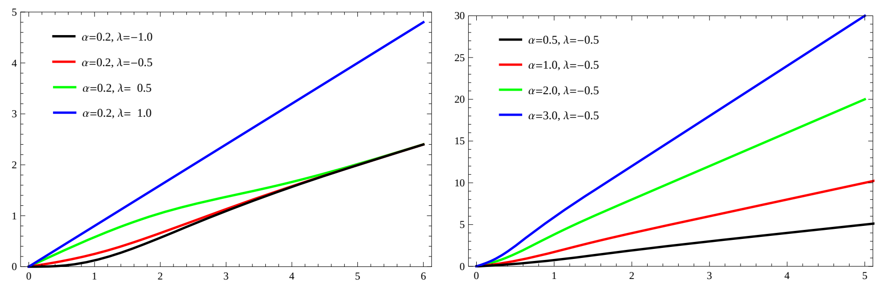
Now, if $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 \geq \lambda_2$, then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0$, which implies that $X \leq_{lr} Y$ and, hence, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

2.3. Hazard Function

The hazard rate function of the transmuted Rayleigh distribution is given by

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{2\alpha x(1-\lambda+2\lambda e^{-\alpha x^2})}{(1-\lambda+\lambda e^{-\alpha x^2})}. \tag{9}$$

Figure 4 illustrates some of the possible shapes of h. Even though, in Figure 4, it apparently looks like that the hazard rate function of the TR distribution is always increasing, it can be noted that for some parameter values, the hazard function decreases. For example if $\alpha = 0.2$ and $\lambda = 0.92$, h decreases in the interval $(4, 4.4)$.



(a) (b)

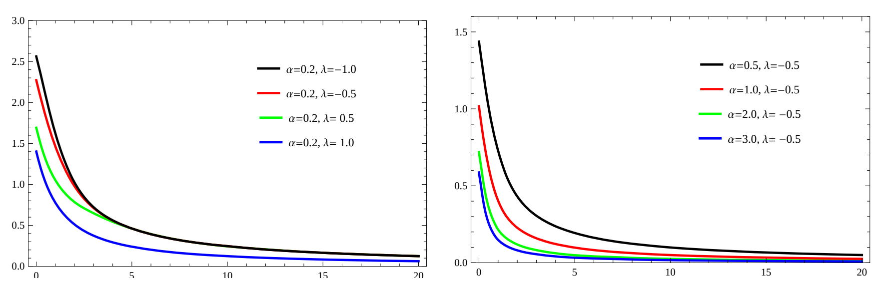
FIGURE 4: Hazard Function of TR distribution for different values of parameters α and λ.

2.4. Mean Residual Life Function

The mean residual life MRL is the expected remaining life, $X - x$, given that the item has survived to time x. Figure 5 illustrates some of the possible shapes of μ.

$$\begin{aligned}\mu(x) &= E(X - x \mid X > x)\\ &= \frac{\int_x^\infty y\, g(y) dy}{1 - G(x)} - x\\ &= \frac{e^{2\alpha x^2}\sqrt{\pi}\left[2(1-\lambda)\ \mathrm{Erfc}(x\sqrt{\alpha})\ +\ \sqrt{2}\lambda\ \mathrm{Erfc}(x\sqrt{2\alpha})\right]}{4\sqrt{\alpha}\left(\lambda + (1-\lambda)e^{\alpha x^2}\right)}.\end{aligned}$$

One can easily show that $\lim_{x\to\infty}\mu(x) = 0$.



(a) (b)

FIGURE 5: MRL function of TR distribution for different values of parameters α and λ.

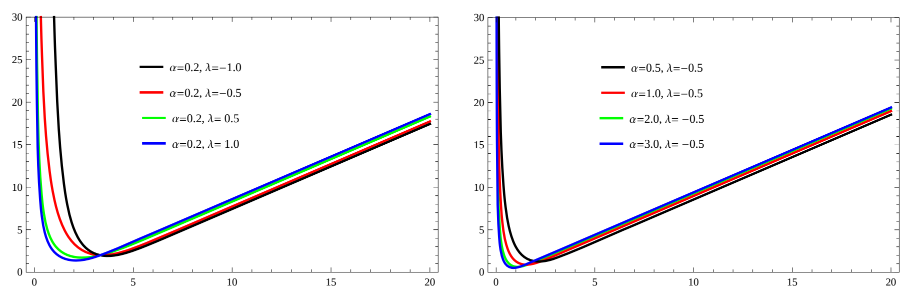
2.5. Mean Past Lifetime (MPL)

In a real life situation, where systems are often not monitored continuously, one might be interested in inferring more about the history of the system e.g.

when the individual components have failed. Now assume that a component with lifetime X has failed at or some time before x, $x \geq 0$. Consider the conditional random variable $x - X \mid X \leq x$. This conditional random variable shows, in fact, the time elapsed from the failure of the component, given that its lifetime is less than or equal to x. Hence, the mean past lifetime (MPL) of the component can be defined as

$$k(x) = E[x - X \mid X \leq x] = \frac{\int_0^x F(t)dt}{F(x)} = x - \frac{\int_0^x tf(t)dt}{F(x)}$$
$$= \frac{-\frac{\sqrt{\pi}\left(\sqrt{2}\lambda\ \text{Erfc}\left(\sqrt{2}\sqrt{\alpha}x\right)-2(\lambda-1)\ \text{Erfc}\left(\sqrt{\alpha}x\right)\right)}{4\sqrt{\alpha}} + (\lambda-1)x\left(-e^{-\alpha x^2}\right) + \lambda x e^{-2\alpha x^2} + x}{\left(1-e^{-\alpha x^2}\right)\left(1+\lambda e^{-\alpha x^2}\right)}.$$

One can easily show that $k(x) \to \infty$ as $x \to 0, \infty$. Figure 6 illustrates some of the possible shapes of k.



(a) (b)

FIGURE 6: MPL function of TR distribution for different values of parameters α and λ.

2.6. Conditional Moments

For lifetime models, it is also of interest to know what $E(X^n \mid X > x)$ is. It can be easily seen that

$$E(X^n \mid X > x) = \frac{1}{S(x)} \int_x^{\infty} x^n f(x) dx$$

In particular

$$E(X \mid X > x) = \frac{e^{\alpha x^2}\left(\frac{\sqrt{\pi}\left(\sqrt{2}\lambda\text{Erfc}\left(\sqrt{2}\sqrt{\alpha}x\right)-2(\lambda-1)\text{Erfc}\left(\sqrt{\alpha}x\right)\right)}{4\sqrt{\alpha}} + xe^{-2\alpha x^2}\left(\lambda-(\lambda-1)e^{\alpha x^2}\right)\right)}{\lambda\left(e^{-\alpha x^2}-1\right)+1}$$

$$E(X^2 \mid X > x) = \frac{2(\lambda-1)e^{\alpha x^2}\left(\alpha x^2+1\right) - \lambda\left(2\alpha x^2+1\right)}{2\alpha\left((\lambda-1)e^{\alpha x^2}-\lambda\right)}$$

$$E(X^3 \mid X > x) = \frac{12\sqrt{\pi}(\lambda-1)e^{2\alpha x^2}\operatorname{Erfc}\left(\sqrt{\alpha}x\right) - 3\sqrt{2\pi}\lambda e^{2\alpha x^2}\operatorname{Erfc}\left(\sqrt{2}\sqrt{\alpha}x\right)}{16\alpha^{3/2}\left((\lambda-1)e^{\alpha x^2}-\lambda\right)}$$

$$+ \quad \frac{4\sqrt{\alpha}x\left(2(\lambda-1)e^{\alpha x^2}\left(2\alpha x^2+3\right) - \lambda\left(4\alpha x^2+3\right)\right)}{16\alpha^{3/2}\left((\lambda-1)e^{\alpha x^2}-\lambda\right)}$$

and

$$E(X^4 \mid X > x) = \frac{2(\lambda-1)e^{\alpha x^2}\left(\alpha^2x^4+2\alpha x^2+2\right) - \lambda\left(2\alpha^2x^4+2\alpha x^2+1\right)}{2\alpha^2\left((\lambda-1)e^{\alpha x^2}-\lambda\right)}$$

The mean residual lifetime function is $E(X \mid X > x) - x$.

2.7. Mean Deviation

The mean deviations of the mean and the median can be used as measures of spread in a population. Let $\mu = E(X)$ and M be the mean and the median of the transmuted Rayleigh distribution, respectively. The mean deviations about the mean and of the median can be calculated as

$$\delta_1(X) = \int_0^\infty |x-\mu| g(x)\, dx \qquad \delta_2(X) = \int_0^\infty |x-M| g(x)\, dx, \tag{10}$$

respectively.

$$\begin{aligned} E\left(|X-m|\right) &= \int_0^\infty |x-m| g(x)\, dx \\ &= \int_0^m (m-x)g(x)dx + \int_m^\infty (x-m)g(x)dx \\ &= 2m\int_0^m g(x)dx - m - E(X) + 2\int_m^\infty xg(x)\, dx \\ &= 2mF(m) + 2\int_m^\infty xg(x)\, dx - E(X) - m \end{aligned}$$

Therefore, since we have $\mu = \frac{\sqrt{\pi}((\sqrt{2}-2)\lambda+2)}{4\sqrt{\alpha}}$ and $M = \sqrt{\frac{1}{\alpha}\ \log\left(\frac{2\lambda}{\sqrt{\lambda^2+1}+\lambda-1}\right)}$,

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} xg(x)\ dx \\ &= \frac{1}{2\sqrt{\alpha}}\sqrt{\pi}\left[\sqrt{2}\lambda\ \text{Erfc}\left(\frac{1}{2}\sqrt{\pi}\left(-\sqrt{2}\lambda+\lambda+\sqrt{2}\right)\right)\right. \\ &\left. +2(1-\lambda)\ \text{Erfc}\left(\frac{1}{4}\sqrt{\pi}\left(\left(\sqrt{2}-2\right)\lambda+2\right)\right)\right],\end{aligned} \tag{11}$$

$$\begin{aligned}\delta_2(X) &= 2\int_{M}^{\infty} xg(x)dx - \mu \\ &= \frac{1}{4\alpha}\left[4\sqrt{\alpha\log\left(\frac{2\lambda}{\sqrt{\lambda^2+1}+\lambda-1}\right)} + 2(1-\lambda)\sqrt{\alpha\pi}\lambda + \lambda\sqrt{2\alpha\pi}\right. \\ &-2\lambda\sqrt{2\alpha\pi}\text{Erf}\left(\sqrt{2\log\left(\frac{2\lambda}{\sqrt{\lambda^2+1}+\lambda-1}\right)}\right) \\ &\left.-4\sqrt{\alpha\pi}(1-\lambda)\text{Erf}\left(\sqrt{\log\left(\frac{2\lambda}{\sqrt{\lambda^2+1}+\lambda-1}\right)}\right)\right].\end{aligned} \tag{12}$$

2.8. Entropies

An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies greater uncertainty in the data. The concept of entropy is important in different areas such as physics, probability and statistics, communication theory, and economics, etc. Several measures of entropy have been studied and compared in the literature. If X has the probability, distribution function $f(\cdot)$ then the Shannon entropy Shannon (1948) is defined by

$$\begin{aligned}H(x) &= -E\left(\ln g(x)\right) \\ &= -\int_0^{\infty} g(x)\ln g(x)dx \\ &= -\int_0^{\infty} g(x)\ln\left[2\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right)\right]dx \\ &= -\frac{1}{4}\Big(-2(\log(\alpha)+\lambda\log(2)+\gamma)+4\log(2\alpha)+2(\lambda-2) \\ &+ \frac{-2\lambda+(\lambda-1)^2(-\log(1-\lambda))+(\lambda+1)^2\log(\lambda+1)}{\lambda}\Big).\end{aligned} \tag{13}$$

Rényi entropy (Renyi 1961) can be expressed as

$$H_{\beta}(x) = \frac{1}{1-\beta}\ln\left(\int_0^{\infty} g^{\beta}(x)dx\right), \qquad \beta > 0, \beta \neq 1$$

Then $H_\beta(x)$ can as simplified as follows: for $-1 \leq \lambda < \frac{1}{3}$,

$$-\ln 2 - \ln\sqrt{\alpha} + \frac{1}{1-\beta}\ln\Gamma\left(\frac{\beta+1}{2}\right) + \frac{1}{1-\beta}\ln\sum_{k=0}^{\infty}\binom{\beta}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k(\beta+k)^{-\frac{\beta+1}{2}},$$

for $\frac{1}{3} < \lambda < 1$, $-\ln 2 - \ln\sqrt{\alpha} + \frac{1}{1-\beta}\ln\Big[\sum_{k=0}^{\infty}\binom{\beta}{k}\Big\{2^k\lambda^k(1-\lambda)^{\beta-k}\alpha^{-\frac{\beta+1}{2}}(\beta+k)^{-\frac{\beta+1}{2}}$ $Ga\left(\frac{\beta+1}{2}, (k+\beta)\ln\frac{2\lambda}{1-\lambda}\right) + 2^{\beta-k}\lambda^{\beta-k}(1-\lambda)^k(2\beta-k)^{-\frac{\beta+1}{2}}\Big(\Gamma\left(\frac{\beta+1}{2}\right) -$ $Ga\left(\frac{\beta+1}{2}, (2\beta-k)\ln\frac{2\lambda}{1-\lambda}\right)\Big)\Big\}\Big]$, and for $\lambda = 1$, $-\frac{3}{2}\ln 2 - \ln\sqrt{\alpha} - \frac{\beta+1}{2}\ln\beta$ $+\ln\Gamma\left(\frac{\beta+1}{2}\right)$, where $Ga(a,x) = \int_x^{\infty} t^{a-1}e^{-t}dt$. It can be noted that, when $\beta \to 1$, the Rényi entropy converges to the Shannon entropy. For further details, see Song (2001).

2.9. Order Statistics

Moments of order statistics play an important role in quality control testing and reliability to predict the failure of future items based on the times of few early failures. Thus, the kth order statistic of a sample is its kth smallest value. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$\text{range}\{X_1, \ldots, X_n\} = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)} \leq \cdots \leq X_{(n)}$ denotes the order statistic of a random sample $X_1, \ldots, X_n$ from a continuous population with cdf $G_X(x)$ and pdf $g_X(x)$, then the pdf of $X_{(j)}$ is given by

$$g_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}g_X(x)\left(G_X(x)\right)^{j-1}\left(1-G_X(x)\right)^{n-j},$$

for $j = 1, \ldots, n$. The pdf of the jth order statistic for a transmuted Rayleigh distribution is given by

$$\begin{aligned} g_{X_{(j)}}(x) =& \frac{n!}{(j-1)!(n-j)!}2\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right) \\ &\times\left[1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^{j-1} \\ &\times\left[e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^{n-j}. \end{aligned}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is

$$g_{X_{(n)}}(x) = 2n\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right) \times \left[1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^{n-1},$$

and the pdf of the smallest order statistic $X_{(1)}$ is

$$g_{X_{(1)}}(x) = 2n\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right).$$

2.10. Distribution of Minimum, Maximum and Median

Let $X_1, X_2, \ldots, X_n$ be an independently identically distributed order random variables from the TR distribution having first, last, and median order probability density functions, which are given by the following

$$\begin{aligned} g_{1:n}(x) &= n\left[1-G(x,\Phi)\right]^{n-1} g(x,\Phi) \\ &= 2n\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right) \\ &\times \left[e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^{n-1}, \end{aligned} \tag{14}$$

$$\begin{aligned} g_{n:n}(x) &= n\left[G(x_{(n)},\Phi)\right]^{n-1} g(x_{(n)},\Phi) \\ &= 2n\alpha x\left[1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^{n-1} \\ &\times e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right), \end{aligned} \tag{15}$$

and

$$\begin{aligned} g_{m+1:n}(\widetilde{x}) &= \frac{(2m+1)!}{m!\,m!}(G(x))^m(1-G(x))^m g(x) \\ &= \frac{(2m+1)!}{m!\,m!}\left[1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^m \\ &\times \left[e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]^m \\ &\times 2\alpha x e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right), \end{aligned} \tag{16}$$

where $\Phi = (\alpha, \lambda)$.

2.11. Stress Strength Parameter

In reliability theory, the stress-strength model describes the life of a component or item which has a random strength X_1 that is subjected to a random stress X_2. The component fails instantaneously when the stress applied to it surpasses the strength, and the component will function satisfactorily/acceptably whenever $X_1 > X_2$. So, $R = Pr\,(X_1 > X_2)$ is a measure of component reliability. Its applicability is found in many spheres especially in engineering fields such as structures, deterioration of rocket motors, fatigue failure of aircraft structures, and the aging of concrete pressure vessels, etc. Extensive work on estimation of reliability of stress-strength models has been undertaken for the well known standard distributions. However, there are still some distributions (including generalizations of the well-known distributions) for which the form of R has not been investigated. Here, we derive the reliability R when X_1 and X_2 are independent random variables distributed with parameters (α_1, λ_1) and (α_2, λ_2), then

$$\begin{aligned} R &= \int_0^\infty g_1(x) G_2(x)\ dx \\ &= \int_0^\infty 2\alpha_1 x e^{-\alpha_1 x^2}(1-\lambda_1+2\lambda_1 e^{-\alpha_1 x^2})[1-e^{-\alpha_2 x^2}(1-\lambda_2+\lambda_2 e^{-\alpha_2 x^2})]\ dx \\ &= \frac{\alpha_2\left(\alpha_1^2(2-\lambda_1)(1+\lambda_2)+\alpha_1\alpha_2(5+\lambda_2-\lambda_1(2-\lambda_2))+2\alpha_2^2\right)}{(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)} \end{aligned} \tag{17}$$

If $\alpha_1 = \alpha_2 = \alpha$, then

$$R = \frac{3-\lambda_1+\lambda_2}{6} \tag{18}$$

and if $\lambda_1 = \lambda_2 = \lambda$, then

$$R = \frac{\alpha_2\left(\alpha_1^2\left(-\lambda^2+\lambda+2\right)+\alpha_1\alpha_2\left(\lambda^2-\lambda+5\right)+2\alpha_2^2\right)}{(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}. \tag{19}$$

3. Methods of Estimation

In this section, we describe ten methods to estimate the parameters, α and λ, the TR distribution. We assume throughout that $x = (x_1, \cdots x_n)$ is a random sample of size n from the TR distribution; both parameters α and λ are unknown.

3.1. Method of Maximum Likelihood Estimation

The ML method is the most frequently used method of parameter estimation (Casella & Berger 1990). Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. Let $x_1, \ldots, x_n$ be a random sample of size n from (2), the likelihood function of the density (2) is given by

$$
\begin{aligned}
L(\alpha, \lambda; x) &= \prod_{i=1}^{n} g(x_i, \alpha, \lambda) \\
&= 2^n \alpha^n \exp\left(-\alpha \sum_{i=1}^{n} x_i^2\right) \prod_{i=1}^{n} x_i \prod_{i=1}^{n} \left(1 - \lambda + 2\lambda e^{-\alpha x_i^2}\right).
\end{aligned} \tag{20}
$$

The log-likelihood function without constant terms is given by

$$
\begin{aligned}
\ell(\alpha, \lambda; x) &= \log L(\alpha, \lambda; x) \\
&= \sum_{i=1}^{n} \log(x_i) + n \log \alpha - \alpha \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \log\left(1 - \lambda + 2\lambda e^{-\alpha x_i^2}\right).
\end{aligned}
$$

For ease of notation, we will denote the first partial derivatives of any function $f(x, y)$ by f_x, and f_y, and its second partial derivatives by f_{xx}, f_{yy}, f_{xy}, and f_{yx}. Now setting

$$
\ell_\alpha = 0 \qquad \text{and} \qquad \ell_\lambda = 0\,,
$$

we have

$$
\ell_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} \frac{x_i^2 e^{-\alpha x_i^2}}{(1 - \lambda + 2\lambda e^{-\alpha x_i^2})} = 0, \tag{21}
$$

and

$$
\ell_\lambda = \sum_{i=1}^{n} \frac{2e^{-\alpha x_i^2} - 1}{1 - \lambda + 2\lambda e^{-\alpha x_i^2}} = 0. \tag{22}
$$

The MLE $\hat{\theta} = (\hat{\alpha}, \hat{\lambda})$ of $\theta = (\alpha, \lambda)$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the sample likelihood function given in (20).

3.2. Method of Moment Estimation

The MMEs of the two-parameter TR distribution can be obtained by equating the first two theoretical moments of (2) with the sample moments $\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\frac{1}{n}\sum_{i=1}^{n} x_i^2$ respectively,

$$
\frac{1}{n} \sum_{i=1}^{n} x_i = \alpha^{-1/2} \Gamma\left(1 + \frac{1}{2}\right) \left(1 - \lambda + \lambda 2^{-1/2}\right), \tag{23}
$$

and

$$
\frac{1}{n} \sum_{i=1}^{n} x_i^2 = \alpha^{-1} \left(1 - \lambda + \lambda 2^{-1}\right). \tag{24}
$$

3.3. Method of Least-Square Estimation

The least square estimators and weighted least square estimators were proposed by Swain, Venkatraman & Wilson (1988) to estimate the parameters of Beta distributions. In this paper, we apply the same technique for the TR distribution. The least square estimators of the unknown parameters α and λ of TR distribution can be obtained by minimizing

$$\sum_{j=1}^{n}\left[F(X_{(j)})-\frac{j}{n+1}\right]^2$$

with respect to unknown parameters α and λ.

Suppose $F(X_{(j)})$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$, where $\{X_1, X_2, \cdots, X_n\}$ is a random sample of size n from a distribution function $F(\cdot)$. Therefore, in this case, the least square estimators of α and λ, say $\hat{\alpha}_{LSE}$ and $\hat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left[1-e^{-\alpha x_{(i)}^2}\left(1-\lambda+\lambda e^{-\alpha x_{(i)}^2}\right)-\frac{j}{n+1}\right]^2$$

with respect to α and λ.

The weighted least square estimators of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[F(X_{(j)})-\frac{j}{n+1}\right]^2$$

with respect to α and λ. The weights w_j are equal to $\frac{1}{V(X_{(j)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$. Therefore, in this case, the weighted least square estimators of α and λ, say $\hat{\alpha}_{WLSE}$ and $\hat{\lambda}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{n-j+1}\left[1-e^{-\alpha x_{(i)}^2}\left(1-\lambda+\lambda e^{-\alpha x_{(i)}^2}\right)-\frac{j}{n+1}\right]^2$$

with respect to α and λ.

3.4. Method of Percentile Estimation

If the data comes from a distribution function which has a closed form, then we can estimate the unknown parameters by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. This method was originally suggested by Kao (1958, 1959) and it has been used for Weibull distribution and for generalized exponential distribution. In this paper, we apply the same technique for the TR distribution. Since,

$$G(x,\alpha,\lambda)=1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right);$$

therefore,

$$x = \left[-\frac{1}{\alpha} \ln \left(\frac{(\lambda - 1) + \sqrt{(\lambda - 1)^2 + 4\lambda(1 - p)}}{2\lambda} \right) \right]^{\frac{1}{2}}.$$

Let $X_{(j)}$ be the jth order statistic, i.e, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. If p_j denotes an estimate of $G(x_{(j)}; \alpha, \lambda)$, then the estimate of α and λ can be obtained by minimizing

$$\sum_{j=1}^{n} \left(x_{(j)} - \left[-\frac{1}{\alpha} \ln \left(\frac{(\lambda - 1) + \sqrt{(\lambda - 1)^2 + 4\lambda(1 - p_j)}}{2\lambda} \right) \right]^{\frac{1}{2}} \right)^2$$

with respect to α and λ. The estimates of α and λ can be obtained by solving the following nonlinear equations

$$\sum_{j=1}^{n} \frac{\log\left(\frac{\lambda-1+\sqrt{\lambda^2+2\lambda-4\lambda p_j+1}}{2\lambda}\right) + \sqrt{\alpha}\, x_{(j)} \sqrt{\log\left(\frac{2\lambda}{\lambda-1+\sqrt{\lambda^2+2\lambda-4\lambda p_j+1}}\right)}}{\alpha^2} = 0$$

$$\sum_{j=1}^{n} \frac{\left(\sqrt{\lambda^2 + 2\lambda - 4\lambda p_j + 1} + \lambda(2p_j - 1) - 1\right) \left(\sqrt{-\frac{\log\left(\frac{\lambda+\sqrt{\lambda^2+2\lambda-4\lambda p_j+1}-1}{2\lambda}\right)}{\alpha}} - x_{(j)} \right)}{\sqrt{\alpha}\lambda\sqrt{\lambda^2 + \lambda(2 - 4p_j) + 1}\left(\lambda + \sqrt{\lambda^2 + 2\lambda - 4\lambda p_j + 1} - 1\right)\sqrt{\log\left(\frac{2\lambda}{\lambda+\sqrt{\lambda^2+2\lambda-4\lambda p_j+1}-1}\right)}} = 0,$$

respectively. We call the corresponding estimators the percentile estimators or PCE's. Several estimators of p_j can be used in this case, see, for example, Mann, Schafer & Singpurwalla (1974). In this paper, we consider $p_j = \frac{j}{n+1}$.

3.5. Method of L-Moments Estimation

In this section, we provide the L-moments estimators, which can be obtained as the linear combinations of order statistics. The L-moments estimators were originally proposed by Hosking (1990), and it is observed that the L-moments estimators are more robust than the usual moment estimators. The L-moment estimators are also obtained in the same way as the ordinary moment estimators, i.e. by equating the sample L-moments with the population L-moments. L-moment estimation provides an alternative method of estimation that is analogous to conventional moments. It has the advantage that it exists whenever the mean of the distribution exists, even though some higher moments may not exist, and it is relatively robust to the effects of outliers (Hosking 1994). Hosking (1990) states that the L-moment estimators are reasonably efficient when compared to the maximum likelihood estimators for distributions such as the normal distribution, the Gumbel distribution, and the GEV distribution.

In this case, the L-moments estimators can be obtained by equating the first two sample L-moments with the corresponding population L-moments. The first two sample L-moments are

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}, \qquad l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} - l_1,$$

and the first two population L-moments are

$$\begin{aligned}\lambda_1 &= E(X_{1:1}) = E(X) \qquad (25)\\ &= \alpha^{-1/2}\Gamma\left(1+\frac{1}{2}\right)\left(1-\lambda+\lambda 2^{-1/2}\right)\\ \lambda_2 &= \frac{1}{2}\left[E(X_{2:2}) - E(X_{2:1})\right], \qquad (26)\end{aligned}$$

where

$$\begin{aligned}E(X_{2:2}) &= 4\alpha\int_0^\infty x^2 e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right)\left[1-e^{-\alpha x^2}\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\right]\,dx\\ &= -\frac{\sqrt{\pi}\left(\left(3+3\sqrt{2}-4\sqrt{3}\right)\lambda^2+4\left(3-3\sqrt{2}+\sqrt{3}\right)\lambda+3\left(\sqrt{2}-4\right)\right)}{12\sqrt{\alpha}},\end{aligned}$$

and

$$\begin{aligned}E(X_{1:2}) &= 4\alpha\int_0^\infty x^2 e^{-\alpha x^2}\left(1-\lambda+2\lambda e^{-\alpha x^2}\right)\left(1-\lambda+\lambda e^{-\alpha x^2}\right)\,dx\\ &= \frac{\sqrt{\pi}\left(\lambda\left(\left(3+3\sqrt{2}-4\sqrt{3}\right)\lambda-6\sqrt{2}+4\sqrt{3}\right)+3\sqrt{2}\right)}{12\sqrt{\alpha}}.\end{aligned}$$

The L-moments estimators $\widehat{\alpha}_{LME}$ and $\widehat{\lambda}_{LME}$ of the parameters α and λ can be obtained by numerically solving the following equations:

$$\lambda_1\left(\widehat{\alpha}_{LME}, \widehat{\lambda}_{LME}\right) = l_1\ ,\ \lambda_2\left(\widehat{\alpha}_{LME}, \widehat{\lambda}_{LME}\right) = l_2. \qquad (27)$$

3.6. Method of Maximum Product of Spacings

Cheng & Amin (1979, 1983) introduced the maximum product of spacings (MPS) method as an alternative to MLE for the estimation of parameters of continuous univariate distributions. Ranneby (1984) independently developed the same method as an approximation for the Kullback-Leibler measure of information.

Using the same notations in subsection 3.4, the uniform spacings of a random sample from the TR distribution are defined as:

$$D_i(\alpha,\lambda) = F\left(x_{i:n} \mid \alpha,\lambda\right) - F\left(x_{i-1:n} \mid \alpha,\lambda\right), \qquad i = 1, 2, \ldots, n,$$

where $F(x_{0:n} \mid \alpha,\lambda) = 0$ and $F(x_{n+1:n} \mid \alpha,\lambda) = 1$. Clearly $\sum_{i=1}^{n+1} D_i(\alpha,\lambda) = 1$.

Following (Cheng & Amin 1983), the maximum product of spacings estimators $\widehat{\alpha}_{MPS}$ and $\widehat{\lambda}_{MPS}$, of the parameters α, and λ, are obtained by maximizing, with respect to α and λ, the geometric mean of the spacings:

$$G\left(\alpha,\lambda\right)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\lambda)\right]^{\frac{1}{n+1}}, \tag{28}$$

or, equivalently, by maximizing the function

$$H\left(\alpha,\lambda\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\lambda). \tag{29}$$

The estimators $\widehat{\alpha}_{MPS}$ and $\widehat{\lambda}_{MPS}$ of the parameters α and λ can be obtained by solving the nonlinear equations

$$\frac{\partial H\left(\alpha,\lambda\right)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda)}\left[\Delta_1(x_{i:n}\mid\alpha,\lambda)-\Delta_1(x_{i-1:n}\mid\alpha,\lambda)\right]=0, \tag{30}$$

$$\frac{\partial}{\partial\lambda}H\left(\alpha,\lambda\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda)}\left[\Delta_2(x_{i:n}\mid\alpha,\lambda)-\Delta_2(x_{i-1:n}\mid\alpha,\lambda)\right]=0, \tag{31}$$

where

$$\Delta_1\left(x_{i:n}\mid\alpha,\lambda\right)=x_{i:n}^2e^{-\alpha x_{i:n}^2}-\lambda x_{i:n}^2e^{-\alpha x_{i:n}^2}+2\lambda x_{i:n}^2e^{-\alpha x_{i:n}^2}. \tag{32}$$

and

$$\Delta_2\left(x_{i:n}\mid\alpha,\lambda\right)=e^{-\alpha x_{i:n}^2}-e^{-2\alpha x_{i:n}^2}. \tag{33}$$

Cheng & Amin (1983) showed that maximizing H as a method of parameter estimation is as efficient as MLE estimation and the MPS estimators are consistent under more general conditions than the MLE estimators.

3.6.1. Method of Cramér-Von-Mises

To motivate our choice of Cramér-von-Mises type minimum distance estimators, (Macdonald 1971) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, The Cramér-von-Mises estimates that $\widehat{\alpha}_{CME}$ and $\widehat{\lambda}_{CME}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$C(\alpha,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)^2. \tag{34}$$

These estimators can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)\Delta_1\left(x_{i:n}\mid\alpha,\lambda\right)=0,$$

$$\sum_{i=1}^{n}\left(F\left(x_{i:n}\mid\alpha,\lambda\right)-\frac{2i-1}{2n}\right)\Delta_2\left(x_{i:n}\mid\alpha,\lambda\right)=0,$$

where $\Delta_1\left(.\mid\alpha,\lambda\right)$ and $\Delta_2\left(.\mid\alpha,\lambda\right)$ are given by (32) and (33), respectively.

3.6.2. Methods of Anderson-Darling and Right-tail Anderson-Darling

The Anderson-Darling test was developed in 1952 by Anderson & Darling (1952) as an alternative to other statistical tests for detecting sample distributions departure from normality. Specifically, the AD test converges very quickly towards the asymptote (Anderson & Darling 1954, Pettitt 1976, Stephens 2013).

The Anderson-Darling estimators $\widehat{\alpha}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$A(\alpha, \lambda) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log F\left(x_{i:n} \mid \alpha, \lambda\right) + \log \overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right) \right\}. \quad (35)$$

These estimators can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n} (2i-1) \left[\frac{\Delta_1\left(x_{i:n} \mid \alpha, \lambda\right)}{F\left(x_{i:n} \mid \alpha, \lambda\right)} - \frac{\Delta_1\left(x_{n+1-i:n} \mid \alpha, \lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right)} \right] = 0,$$

$$\sum_{i=1}^{n} (2i-1) \left[\frac{\Delta_2\left(x_{i:n} \mid \alpha, \lambda\right)}{F\left(x_{i:n} \mid \alpha, \lambda\right)} - \frac{\Delta_2\left(x_{n+1-i:n} \mid \alpha, \lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right)} \right] = 0,$$

where $\Delta_1(\cdot \mid \alpha, \lambda)$ and $\Delta_2(\cdot \mid \alpha, \lambda)$ are given by (32) and (33), respectively.

The Right-tail Anderson-Darling estimates $\widehat{\alpha}_{RTADE}$ and $\widehat{\lambda}_{RTADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$R(\alpha, \lambda) = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{i:n} \mid \alpha, \lambda\right) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right). \quad (36)$$

These estimators can also be obtained by solving the non-linear equations:

$$-2\sum_{i=1}^{n} \frac{\Delta_1\left(x_{i:n} \mid \alpha, \lambda\right)}{F\left(x_{i:n} \mid \alpha, \lambda\right)} + \frac{1}{n} \sum_{i=1}^{n} (2i-1) \frac{\Delta_1\left(x_{n+1-i:n} \mid \alpha, \lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right)} = 0,$$

$$-2\sum_{i=1}^{n} \frac{\Delta_2\left(x_{i:n} \mid \alpha, \lambda\right)}{F\left(x_{i:n} \mid \lambda, \sigma\right)} + \frac{1}{n} \sum_{i=1}^{n} (2i-1) \frac{\Delta_2\left(x_{n+1-i:n} \mid \alpha, \lambda\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha, \lambda\right)} = 0,$$

where $\Delta_1(\cdot \mid \alpha, \lambda)$ and $\Delta_2(\cdot \mid \alpha, \lambda)$ are given by (32) and (33), respectively.

4. Simulation

We conduct Monte Carlo simulation studies to compare the performance of the estimators discussed in the previous sections. We evaluate the performance of the estimators based on bias, root mean squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. Methods are compared for sample sizes $n = 20$, 50, 100, and

200. 10,000 independent samples of size n are first generated from the transmuted Rayleigh (TR) distribution with parameters $\alpha = 0.5, 1, 3, 5, 10$ and $\lambda = 0.7$. We observed that 10,000 repetitions were sufficiently large provide have stable results. For all the methods considered in this study, we first estimated the parameters using the method of maximum likelihood. For all other methods, the maximum likelihood estimates were used as the initial values. Also, the same randomly generated samples were used to compare all the estimation methods. The results of the simulation studies are reported in Tables 1-5.

The pseudo-random numbers were generated from the Transmuted Rayleigh distribution using the following function:

$$x = \left[-\frac{1}{\alpha} \ln \left(\frac{(\lambda - 1) + \sqrt{(\lambda - 1)^2 + 4\lambda(1 - u)}}{2\lambda} \right) \right]^{\frac{1}{2}}, \tag{37}$$

where $u \sim UNIF(0, 1)$

For each estimate we calculate the bias, root mean-squared error, and the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. The statistics are obtained using the following formulae.

$$\text{Bias}(\hat{\alpha}) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\alpha}_i - \alpha), \quad \text{Bias}(\hat{\lambda}) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\lambda}_i - \lambda) \tag{38}$$

$$\text{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{R} \sum_{i=1}^{R} (\hat{\alpha}_i - \alpha)^2}, \quad \text{RMSE}(\hat{\lambda}) = \sqrt{\frac{1}{R} \sum_{i=1}^{R} (\hat{\lambda}_i - \lambda)^2} \tag{39}$$

$$D_{\text{abs}}(\hat{\alpha}) = \frac{1}{(nR)} \sum_{i=1}^{R} \sum_{j=1}^{n} \left| F(x_{ij} | \alpha, \lambda) - F(x_{ij} | \hat{\alpha}, \hat{\lambda}) \right| \tag{40}$$

$$D_{\max}(\hat{\alpha}) = \frac{1}{R} \sum_{i=1}^{R} \max_j \left| F(x_{ij} | \alpha, \lambda) - F(x_{ij} | \hat{\alpha}, \hat{\lambda}) \right| \tag{41}$$

Simulated bias, RMSE, and D_{abs}, $D_{\max}$ for the estimates are presented in Tables 1-5. The row indicating $\sum$ Ranks shows the partial sum of the ranks. A superscript indicates the rank of each of the estimators among all the estimators for that metric. For example, Table 1 shows the bias of MLE($\hat{\alpha}$) as 0.295^{10} for $n = 20$. This indicates, that the bias of $\hat{\alpha}$, obtained using the method of maximum likelihood ranks was 10^{th} among all other estimators. Table 6 shows the partial and overall rank of the estimators; it is used to find the overall performance of estimation techniques.

The following observations can be drawn from the Tables 1-5.

1. All the estimators show the property of consistency, i.e. the RMSE decreases as the sample size increases.
2. The bias of $\hat{\alpha}$ decreases with an increasing n for all the method of estimations.
3. The bias of $\hat{\lambda}$ decreases with an increasing n for all the method of estimations.
4. The bias of $\hat{\alpha}$ generally increases with an increasing alpha for any given alpha and n and for all methods of estimation.
5. In terms of RMSE, all the methods of estimation produce smaller RMSE for $\hat{\alpha}$ compared to that of $\hat{\lambda}$.
6. D_{abs} is smaller than $D_{\max}$ for all the estimation techniques. Again, these statistics get smaller with a a sample size increase.
7. In terms of performance of the methods of estimation, we found that maximum product spacing (MPS)estimators is the best method as it produces the least estimate biases with the least RMSE for most of the configurations considered in our studies. The next best method is the percentile estimators (PCE), followed by right tailed Anderson-Darling estimators. The weighted lest squared estimation (WLS) method ranked 4th while MLE ranked 5th. Method of Cramér-von-Mises ranked 10th among the ten method of estimation.
8. While MPS uniformly performed the best for all values of n and α, the PCE and WLS method performed the best for $\alpha \leq 5$. Their performance was degraded for $\alpha = 10$.

The overall positions of the estimators are presented in Table 6, from which we can confirm the superiority of MPS and PCE.

Table 1: Simulation results for $\alpha = 0.5$ and $\lambda = 0.7$.

n	Est.	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	0.295^{10}	0.283^{9}	0.269^{8}	0.202^{3}	0.207^{5}	0.155^{2}	0.153^{1}	0.260^{7}	0.205^{4}	0.232^{6}
	RMSE($\hat{\alpha}$)	0.412^{9}	0.416^{10}	0.401^{8}	0.305^{3}	0.336^{4}	0.291^{2}	0.290^{1}	0.363^{6}	0.350^{5}	0.377^{7}
	Bias($\hat{\lambda}$)	-0.618^{9}	-0.628^{10}	-0.577^{8}	-0.429^{4}	-0.443^{5}	-0.416^{3}	-0.399^{2}	-0.533^{7}	-0.364^{1}	-0.507^{6}
	RMSE($\hat{\lambda}$)	0.757^{6}	0.849^{8}	0.764^{7}	0.611^{3}	0.618^{4}	0.572^{2}	0.568^{1}	0.722^{5}	1.366^{10}	0.987^{9}
	D_{abs}	0.050^{1}	0.054^{9}	0.052^{5}	0.053^{7}	0.051^{3}	0.051^{4}	0.051^{2}	0.054^{10}	0.053^{6}	0.053^{8}
	D_{max}	0.081^{4}	0.088^{10}	0.083^{6}	0.082^{5}	0.080^{3}	0.079^{2}	0.078^{1}	0.086^{8}	0.084^{7}	0.087^{9}
	$\sum$Ranks	39^{6}	56^{10}	42^{7}	25^{4}	24^{3}	15^{2}	8^{1}	43^{8}	33^{5}	45^{9}
50	Bias($\hat{\alpha}$)	0.208^{8}	0.208^{7}	0.210^{9}	0.206^{6}	0.182^{3}	0.120^{2}	0.115^{1}	0.245^{10}	0.184^{4}	0.184^{5}
	RMSE($\hat{\alpha}$)	0.293^{9}	0.293^{8}	0.287^{7}	0.282^{6}	0.267^{3}	0.216^{1}	0.217^{2}	0.322^{10}	0.278^{4}	0.279^{5}
	Bias($\hat{\lambda}$)	-0.476^{7}	-0.493^{9}	-0.486^{8}	-0.452^{6}	-0.417^{4}	-0.329^{2}	-0.305^{1}	-0.525^{10}	-0.398^{3}	-0.423^{5}
	RMSE($\hat{\lambda}$)	0.626^{8}	0.646^{9}	0.619^{7}	0.600^{5}	0.569^{3}	0.479^{1}	0.480^{2}	0.672^{10}	0.593^{4}	0.605^{6}
	D_{abs}	0.033^{3}	0.033^{7}	0.033^{5}	0.035^{9}	0.033^{4}	0.032^{2}	0.032^{1}	0.035^{10}	0.034^{8}	0.033^{6}
	D_{max}	0.052^{3}	0.054^{7}	0.053^{5}	0.055^{9}	0.052^{4}	0.050^{2}	0.050^{1}	0.057^{10}	0.054^{8}	0.053^{6}
	$\sum$Ranks	38^{6}	47^{9}	41^{7}	41^{7}	21^{3}	10^{2}	8^{1}	60^{10}	31^{4}	33^{5}
100	Bias($\hat{\alpha}$)	0.157^{3}	0.172^{7}	0.181^{8}	0.199^{9}	0.165^{6}	0.095^{2}	0.089^{1}	0.222^{10}	0.165^{5}	0.158^{4}
	RMSE($\hat{\alpha}$)	0.229^{4}	0.234^{6}	0.236^{7}	0.260^{9}	0.230^{5}	0.174^{1}	0.176^{2}	0.284^{10}	0.237^{8}	0.229^{3}
	Bias($\hat{\lambda}$)	-0.374^{5}	-0.423^{7}	-0.436^{8}	-0.447^{9}	-0.387^{6}	-0.265^{2}	-0.238^{1}	-0.492^{10}	-0.373^{3}	-0.374^{4}
	RMSE($\hat{\lambda}$)	0.522^{5}	0.538^{7}	0.539^{8}	0.568^{9}	0.517^{3}	0.407^{1}	0.410^{2}	0.614^{10}	0.525^{6}	0.521^{4}
	D_{abs}	0.023^{3}	0.024^{5}	0.023^{4}	0.025^{9}	0.024^{7}	0.023^{2}	0.023^{1}	0.025^{10}	0.024^{8}	0.024^{6}
	D_{max}	0.037^{3}	0.037^{5}	0.037^{4}	0.040^{9}	0.037^{6}	0.036^{2}	0.035^{1}	0.041^{10}	0.038^{8}	0.037^{7}
	$\sum$Ranks	23^{3}	37^{6}	39^{8}	54^{9}	33^{5}	10^{2}	8^{1}	60^{10}	38^{7}	28^{4}
200	Bias($\hat{\alpha}$)	0.119^{3}	0.146^{5}	0.161^{8}	0.188^{9}	0.148^{6}	0.076^{2}	0.069^{1}	0.203^{10}	0.149^{7}	0.138^{4}
	RMSE($\hat{\alpha}$)	0.181^{3}	0.191^{4}	0.200^{7}	0.235^{9}	0.198^{6}	0.142^{1}	0.144^{2}	0.250^{10}	0.204^{8}	0.192^{5}
	Bias($\hat{\lambda}$)	-0.291^{3}	-0.367^{7}	-0.398^{8}	-0.431^{9}	-0.355^{6}	-0.209^{2}	-0.182^{1}	-0.462^{10}	-0.350^{5}	-0.333^{4}
	RMSE($\hat{\lambda}$)	0.429^{3}	0.455^{5}	0.478^{8}	0.529^{9}	0.461^{6}	0.341^{1}	0.344^{2}	0.558^{10}	0.472^{7}	0.452^{4}
	D_{abs}	0.016^{3}	0.016^{4}	0.017^{7}	0.017^{9}	0.016^{6}	0.016^{2}	0.016^{1}	0.018^{10}	0.017^{8}	0.016^{5}
	D_{max}	0.025^{3}	0.026^{4}	0.026^{7}	0.028^{9}	0.026^{6}	0.025^{1}	0.025^{2}	0.029^{10}	0.027^{8}	0.026^{5}
	$\sum$Ranks	18^{3}	29^{5}	45^{8}	54^{9}	36^{6}	9^{1}	9^{1}	60^{10}	43^{7}	27^{4}

TABLE 2: Simulation results for $\alpha = 1$ and $\lambda = 0.7$.

n	Est.	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	0.597^{10}	0.581^{9}	0.547^{8}	0.360^{5}	0.425^{6}	0.278^{4}	0.221^{2}	0.435^{7}	0.167^{1}	0.233^{3}
	RMSE($\hat{\alpha}$)	0.812^{8}	0.842^{10}	0.813^{9}	0.536^{3}	0.687^{7}	0.526^{2}	0.445^{1}	0.602^{6}	0.548^{4}	0.597^{5}
	Bias($\hat{\lambda}$)	-0.627^{9}	-0.636^{10}	-0.581^{8}	-0.390^{5}	-0.451^{6}	-0.384^{4}	-0.313^{3}	-0.460^{7}	0.042^{1}	-0.120^{2}
	RMSE($\hat{\lambda}$)	0.764^{6}	0.853^{8}	0.767^{7}	0.591^{3}	0.624^{4}	0.551^{2}	0.507^{1}	0.684^{5}	2.383^{10}	1.676^{9}
	D_{abs}	0.050^{1}	0.053^{9}	0.052^{5}	0.053^{8}	0.051^{3}	0.051^{4}	0.051^{2}	0.054^{10}	0.052^{6}	0.052^{7}
	D_{max}	0.081^{4}	0.087^{10}	0.083^{8}	0.082^{5}	0.080^{3}	0.078^{2}	0.077^{1}	0.084^{9}	0.082^{6}	0.082^{7}
	$\sum$Ranks	38^{7}	56^{10}	45^{9}	29^{4}	29^{4}	18^{2}	10^{1}	44^{8}	28^{3}	33^{6}
50	Bias($\hat{\alpha}$)	0.428^{10}	0.425^{9}	0.421^{8}	0.362^{5}	0.369^{6}	0.237^{3}	0.187^{2}	0.416^{7}	0.174^{1}	0.240^{3}
	RMSE($\hat{\alpha}$)	0.596^{10}	0.595^{9}	0.577^{8}	0.496^{5}	0.540^{6}	0.431^{2}	0.379^{1}	0.552^{7}	0.441^{3}	0.486^{4}
	Bias($\hat{\lambda}$)	-0.490^{9}	-0.501^{10}	-0.489^{8}	-0.407^{5}	-0.422^{6}	-0.328^{4}	-0.257^{3}	-0.458^{7}	-0.151^{1}	-0.255^{2}
	RMSE($\hat{\lambda}$)	0.636^{8}	0.650^{10}	0.621^{7}	0.574^{6}	0.574^{5}	0.480^{3}	0.443^{1}	0.637^{9}	0.450^{2}	0.516^{4}
	D_{abs}	0.032^{4}	0.033^{8}	0.033^{7}	0.034^{9}	0.033^{6}	0.032^{3}	0.032^{1}	0.035^{10}	0.033^{5}	0.032^{2}
	D_{max}	0.052^{5}	0.053^{8}	0.052^{7}	0.054^{9}	0.052^{6}	0.050^{2}	0.049^{1}	0.056^{10}	0.051^{4}	0.051^{3}
	$\sum$Ranks	46^{8}	54^{10}	45^{7}	39^{6}	35^{5}	17^{3}	9^{1}	50^{9}	16^{2}	19^{4}
100	Bias($\hat{\alpha}$)	0.327^{5}	0.349^{7}	0.364^{8}	0.378^{9}	0.332^{6}	0.193^{3}	0.158^{1}	0.423^{10}	0.173^{2}	0.230^{4}
	RMSE($\hat{\alpha}$)	0.468^{6}	0.473^{7}	0.475^{8}	0.496^{9}	0.462^{5}	0.351^{2}	0.330^{1}	0.543^{10}	0.385^{3}	0.420^{4}
	Bias($\hat{\lambda}$)	-0.388^{5}	-0.426^{7}	-0.438^{9}	-0.428^{8}	-0.389^{6}	-0.265^{4}	-0.213^{2}	-0.470^{10}	-0.173^{1}	-0.258^{3}
	RMSE($\hat{\lambda}$)	0.531^{6}	0.540^{7}	0.540^{8}	0.558^{9}	0.518^{5}	0.409^{3}	0.391^{1}	0.603^{10}	0.402^{2}	0.454^{4}
	D_{abs}	0.023^{4}	0.023^{7}	0.023^{6}	0.025^{9}	0.023^{8}	0.023^{3}	0.023^{1}	0.025^{10}	0.023^{5}	0.023^{2}
	D_{max}	0.037^{5}	0.037^{7}	0.037^{6}	0.039^{9}	0.037^{8}	0.035^{2}	0.035^{1}	0.041^{10}	0.036^{4}	0.036^{3}
	$\sum$Ranks	31^{5}	42^{7}	45^{8}	53^{9}	38^{6}	17^{2}	7^{1}	60^{10}	17^{2}	20^{4}
200	Bias($\hat{\alpha}$)	0.249^{5}	0.295^{6}	0.325^{8}	0.375^{9}	0.299^{7}	0.153^{2}	0.126^{1}	0.405^{10}	0.161^{3}	0.214^{4}
	RMSE($\hat{\alpha}$)	0.369^{5}	0.384^{6}	0.404^{8}	0.469^{9}	0.398^{7}	0.284^{2}	0.275^{1}	0.500^{10}	0.333^{3}	0.359^{4}
	Bias($\hat{\lambda}$)	-0.303^{5}	-0.369^{7}	-0.400^{8}	-0.430^{9}	-0.357^{6}	-0.209^{3}	-0.166^{1}	-0.459^{10}	-0.173^{2}	-0.250^{4}
	RMSE($\hat{\lambda}$)	0.436^{5}	0.457^{6}	0.480^{8}	0.530^{9}	0.463^{7}	0.341^{2}	0.333^{1}	0.559^{10}	0.364^{3}	0.406^{4}
	D_{abs}	0.016^{3}	0.016^{5}	0.017^{8}	0.018^{9}	0.017^{7}	0.016^{2}	0.016^{1}	0.018^{10}	0.016^{6}	0.016^{4}
	D_{max}	0.026^{3}	0.026^{5}	0.026^{8}	0.029^{9}	0.026^{7}	0.025^{2}	0.025^{1}	0.029^{10}	0.026^{6}	0.026^{4}
	$\sum$Ranks	26^{5}	35^{6}	48^{8}	54^{9}	41^{7}	13^{2}	6^{1}	60^{10}	23^{3}	24^{4}

TABLE 3: Simulation results for $\alpha = 3$ and $\lambda = 0.7$.

n	Est.	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	0.954^{7}	1.742^{10}	1.611^{9}	0.873^{4}	0.817^{3}	0.463^{2}	0.449^{1}	0.974^{8}	0.933^{6}	0.817^{4}
	RMSE($\hat{\alpha}$)	1.448^{4}	2.522^{10}	2.409^{9}	1.417^{3}	1.584^{7}	1.055^{1}	1.063^{2}	1.514^{5}	1.618^{8}	1.557^{6}
	Bias($\hat{\lambda}$)	-0.356^{8}	-0.630^{10}	-0.573^{9}	-0.316^{6}	-0.291^{4}	-0.257^{2}	-0.236^{1}	-0.342^{7}	-0.292^{5}	-0.275^{3}
	RMSE($\hat{\lambda}$)	0.610^{5}	0.852^{8}	0.765^{7}	0.543^{4}	0.510^{3}	0.464^{2}	0.453^{1}	0.616^{6}	1.478^{9}	1.535^{10}
	D_{abs}	0.051^{4}	0.053^{10}	0.052^{5}	0.053^{7}	0.051^{2}	0.051^{3}	0.050^{1}	0.053^{9}	0.052^{6}	0.053^{8}
	D_{max}	0.079^{4}	0.087^{10}	0.083^{7}	0.080^{5}	0.078^{3}	0.077^{2}	0.076^{1}	0.082^{6}	0.085^{8}	0.086^{9}
	$\sum$Ranks	32^{5}	58^{10}	46^{9}	30^{5}	22^{3}	12^{2}	7^{1}	41^{7}	42^{8}	40^{6}
50	Bias($\hat{\alpha}$)	0.918^{8}	1.266^{10}	1.245^{9}	0.787^{4}	0.764^{3}	0.416^{2}	0.368^{1}	0.853^{6}	0.876^{7}	0.787^{5}
	RMSE($\hat{\alpha}$)	1.391^{8}	1.770^{10}	1.714^{9}	1.205^{3}	1.260^{4}	0.899^{2}	0.893^{1}	1.300^{6}	1.345^{7}	1.282^{5}
	Bias($\hat{\lambda}$)	-0.358^{8}	-0.493^{10}	-0.484^{9}	-0.298^{4}	-0.297^{3}	-0.221^{2}	-0.185^{1}	-0.315^{5}	-0.334^{7}	-0.319^{6}
	RMSE($\hat{\lambda}$)	0.552^{5}	0.648^{10}	0.620^{9}	0.508^{4}	0.491^{3}	0.407^{2}	0.391^{1}	0.556^{6}	0.566^{8}	0.558^{7}
	D_{abs}	0.033^{3}	0.033^{7}	0.033^{5}	0.034^{9}	0.033^{4}	0.032^{1}	0.032^{1}	0.034^{10}	0.033^{8}	0.033^{6}
	D_{max}	0.051^{4}	0.053^{9}	0.052^{5}	0.053^{6}	0.051^{3}	0.049^{2}	0.048^{1}	0.054^{10}	0.053^{8}	0.053^{7}
	$\sum$Ranks	36^{5}	56^{10}	46^{9}	30^{5}	20^{3}	12^{2}	6^{1}	43^{7}	45^{8}	36^{5}
100	Bias($\hat{\alpha}$)	0.796^{7}	1.043^{9}	1.084^{10}	0.719^{3}	0.729^{4}	0.395^{2}	0.327^{1}	0.760^{5}	0.829^{8}	0.770^{6}
	RMSE($\hat{\alpha}$)	1.241^{8}	1.409^{9}	1.417^{10}	1.103^{3}	1.146^{4}	0.827^{2}	0.806^{1}	1.176^{6}	1.209^{7}	1.165^{5}
	Bias($\hat{\lambda}$)	-0.316^{7}	-0.421^{9}	-0.435^{10}	-0.275^{3}	-0.288^{5}	-0.197^{2}	-0.156^{1}	-0.285^{4}	-0.322^{8}	-0.311^{6}
	RMSE($\hat{\lambda}$)	0.486^{5}	0.538^{9}	0.539^{10}	0.469^{4}	0.456^{3}	0.360^{2}	0.347^{1}	0.500^{8}	0.500^{7}	0.487^{6}
	D_{abs}	0.023^{2}	0.023^{6}	0.023^{5}	0.024^{9}	0.023^{4}	0.023^{3}	0.023^{1}	0.024^{10}	0.024^{8}	0.023^{7}
	D_{max}	0.036^{3}	0.037^{7}	0.037^{5}	0.038^{9}	0.036^{4}	0.035^{2}	0.035^{1}	0.039^{10}	0.038^{8}	0.037^{6}
	$\sum$Ranks	32^{5}	49^{9}	50^{10}	31^{4}	24^{3}	13^{2}	6^{1}	43^{7}	46^{8}	36^{6}
200	Bias($\hat{\alpha}$)	0.631^{3}	0.884^{9}	0.974^{10}	0.649^{4}	0.680^{6}	0.372^{2}	0.273^{1}	0.669^{5}	0.785^{8}	0.738^{6}
	RMSE($\hat{\alpha}$)	1.023^{4}	1.147^{9}	1.210^{10}	1.009^{3}	1.024^{5}	0.759^{2}	0.709^{1}	1.054^{6}	1.100^{8}	1.059^{7}
	Bias($\hat{\lambda}$)	-0.255^{5}	-0.366^{9}	-0.400^{10}	-0.250^{3}	-0.272^{6}	-0.175^{2}	-0.125^{1}	-0.253^{4}	-0.310^{8}	-0.299^{7}
	RMSE($\hat{\lambda}$)	0.408^{3}	0.456^{9}	0.480^{10}	0.429^{5}	0.414^{4}	0.319^{2}	0.302^{1}	0.448^{7}	0.452^{8}	0.435^{6}
	D_{abs}	0.016^{2}	0.016^{5}	0.017^{7}	0.017^{9}	0.016^{4}	0.016^{3}	0.016^{1}	0.017^{10}	0.017^{8}	0.016^{6}
	D_{max}	0.025^{3}	0.026^{4}	0.026^{7}	0.027^{9}	0.026^{5}	0.025^{2}	0.025^{1}	0.028^{10}	0.027^{8}	0.026^{6}
	$\sum$Ranks	20^{3}	45^{8}	54^{10}	33^{5}	30^{4}	13^{2}	6^{1}	42^{7}	48^{9}	39^{6}

Table 4: Simulation results for $\alpha = 5$ and $\lambda = 0.7$.

n	Est.	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	1.569^{6}	2.929^{10}	2.649^{9}	1.546^{5}	1.299^{3}	0.822^{2}	0.801^{1}	1.767^{8}	1.675^{7}	1.442^{4}
	RMSE($\hat{\alpha}$)	2.375^{4}	4.231^{10}	3.960^{9}	2.438^{5}	2.349^{3}	1.810^{1}	1.815^{2}	2.635^{6}	2.825^{8}	2.674^{7}
	Bias($\hat{\lambda}$)	-0.359^{7}	-0.634^{10}	-0.570^{9}	-0.339^{6}	-0.291^{3}	-0.274^{2}	-0.254^{1}	-0.378^{8}	-0.329^{5}	-0.309^{4}
	RMSE($\hat{\lambda}$)	0.617^{5}	0.852^{8}	0.762^{7}	0.558^{4}	0.509^{3}	0.475^{2}	0.466^{1}	0.637^{6}	1.394^{10}	1.263^{9}
	D_{abs}	0.052^{3}	0.054^{10}	0.052^{5}	0.053^{7}	0.051^{2}	0.052^{4}	0.051^{1}	0.053^{9}	0.053^{6}	0.053^{8}
	D_{max}	0.080^{4}	0.088^{10}	0.083^{7}	0.081^{5}	0.078^{3}	0.078^{2}	0.077^{1}	0.083^{6}	0.085^{8}	0.087^{9}
	$\sum$Ranks	29^{4}	58^{10}	46^{9}	32^{5}	17^{3}	13^{2}	7^{1}	43^{7}	44^{8}	41^{6}
50	Bias($\hat{\alpha}$)	1.481^{6}	2.142^{10}	2.077^{9}	1.473^{5}	1.320^{3}	0.775^{2}	0.696^{1}	1.620^{8}	1.587^{7}	1.415^{4}
	RMSE($\hat{\alpha}$)	2.223^{6}	2.968^{10}	2.852^{9}	2.146^{4}	2.124^{3}	1.568^{2}	1.562^{1}	2.331^{7}	2.362^{8}	2.218^{5}
	Bias($\hat{\lambda}$)	-0.352^{6}	-0.496^{10}	-0.483^{9}	-0.335^{4}	-0.308^{3}	-0.240^{2}	-0.205^{1}	-0.359^{7}	-0.361^{8}	-0.341^{5}
	RMSE($\hat{\lambda}$)	0.554^{5}	0.648^{10}	0.618^{9}	0.530^{4}	0.497^{3}	0.419^{2}	0.407^{1}	0.581^{8}	0.579^{7}	0.567^{6}
	D_{abs}	0.033^{4}	0.033^{7}	0.033^{5}	0.034^{9}	0.033^{3}	0.033^{2}	0.032^{1}	0.035^{10}	0.034^{8}	0.033^{6}
	D_{max}	0.051^{4}	0.054^{9}	0.053^{5}	0.053^{7}	0.051^{3}	0.050^{2}	0.049^{1}	0.055^{10}	0.054^{8}	0.053^{6}
	$\sum$Ranks	31^{4}	56^{10}	46^{7}	33^{6}	18^{3}	12^{2}	6^{1}	50^{9}	46^{7}	32^{5}
100	Bias($\hat{\alpha}$)	1.288^{4}	1.752^{9}	1.799^{10}	1.377^{5}	1.260^{3}	0.711^{2}	0.601^{1}	1.487^{7}	1.495^{8}	1.381^{6}
	RMSE($\hat{\alpha}$)	1.982^{4}	2.349^{9}	2.352^{10}	1.983^{5}	1.929^{3}	1.418^{2}	1.387^{1}	2.130^{8}	2.117^{7}	2.028^{6}
	Bias($\hat{\lambda}$)	-0.311^{4}	-0.423^{9}	-0.434^{10}	-0.318^{5}	-0.301^{3}	-0.211^{2}	-0.171^{1}	-0.337^{7}	-0.349^{8}	-0.335^{6}
	RMSE($\hat{\lambda}$)	0.482^{4}	0.538^{10}	0.538^{9}	0.495^{5}	0.464^{3}	0.369^{2}	0.358^{1}	0.530^{8}	0.514^{7}	0.499^{6}
	D_{abs}	0.023^{3}	0.024^{6}	0.023^{5}	0.024^{9}	0.023^{4}	0.023^{2}	0.023^{1}	0.025^{10}	0.024^{8}	0.024^{7}
	D_{max}	0.036^{3}	0.037^{7}	0.037^{5}	0.039^{9}	0.037^{4}	0.036^{2}	0.035^{1}	0.039^{10}	0.038^{8}	0.037^{6}
	$\sum$Ranks	22^{4}	50^{9}	49^{8}	38^{6}	20^{3}	12^{2}	6^{1}	50^{9}	46^{7}	37^{5}
200	Bias($\hat{\alpha}$)	1.079^{3}	1.477^{9}	1.612^{10}	1.325^{6}	1.186^{4}	0.645^{2}	0.525^{1}	1.382^{7}	1.448^{8}	1.309^{5}
	RMSE($\hat{\alpha}$)	1.717^{3}	1.910^{7}	2.004^{10}	1.862^{6}	1.736^{4}	1.289^{2}	1.250^{1}	1.953^{9}	1.942^{8}	1.829^{5}
	Bias($\hat{\lambda}$)	-0.264^{3}	-0.366^{9}	-0.398^{10}	-0.308^{5}	-0.286^{4}	-0.182^{2}	-0.143^{1}	-0.317^{6}	-0.343^{8}	-0.318^{7}
	RMSE($\hat{\lambda}$)	0.413^{3}	0.455^{6}	0.478^{9}	0.463^{7}	0.422^{4}	0.324^{2}	0.316^{1}	0.483^{10}	0.469^{8}	0.444^{5}
	D_{abs}	0.016^{3}	0.016^{4}	0.017^{7}	0.017^{9}	0.016^{5}	0.016^{2}	0.016^{1}	0.017^{10}	0.017^{8}	0.016^{6}
	D_{max}	0.025^{3}	0.026^{4}	0.026^{7}	0.028^{9}	0.026^{5}	0.025^{2}	0.025^{1}	0.028^{10}	0.027^{8}	0.026^{6}
	$\sum$Ranks	18^{3}	39^{6}	53^{10}	42^{7}	26^{4}	12^{2}	6^{1}	52^{9}	48^{8}	34^{5}

TABLE 5: Simulation results for $\alpha = 10$ and $\lambda = 0.7$.

n	Est.	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
20	Bias($\hat{\alpha}$)	5.044^{10}	-0.995^{7}	-0.541^{4}	0.662^{5}	3.883^{9}	2.813^{8}	-0.281^{1}	0.305^{2}	-0.332^{3}	-0.891^{6}
	RMSE($\hat{\alpha}$)	6.450^{10}	3.473^{2}	3.721^{5}	3.714^{4}	5.571^{9}	4.890^{8}	2.902^{1}	3.676^{3}	4.288^{7}	3.916^{6}
	Bias($\hat{\lambda}$)	-0.579^{10}	0.209^{5}	0.146^{4}	-0.008^{1}	-0.438^{9}	-0.402^{8}	0.018^{2}	0.057^{3}	0.340^{6}	0.351^{7}
	RMSE($\hat{\lambda}$)	0.711^{8}	0.352^{4}	0.394^{5}	0.252^{2}	0.587^{7}	0.542^{6}	0.219^{1}	0.267^{3}	2.513^{10}	1.743^{9}
	D_{abs}	0.049^{2}	0.051^{3}	0.052^{10}	0.052^{9}	0.051^{5}	0.052^{6}	0.049^{1}	0.052^{7}	0.052^{8}	0.051^{4}
	$D_{\max}$	0.078^{3}	0.078^{5}	0.080^{8}	0.077^{2}	0.079^{7}	0.078^{6}	0.073^{1}	0.078^{4}	0.083^{10}	0.082^{9}
	$\sum$Ranks	43^{8}	26^{4}	36^{5}	23^{3}	46^{10}	42^{7}	7^{1}	22^{2}	44^{9}	41^{6}
50	Bias($\hat{\alpha}$)	4.060^{10}	-0.779^{7}	-0.457^{5}	0.457^{6}	3.692^{9}	2.377^{8}	-0.334^{4}	0.197^{2}	0.119^{1}	-0.258^{3}
	RMSE($\hat{\alpha}$)	5.464^{10}	2.503^{2}	2.715^{3}	2.825^{4}	5.048^{9}	4.117^{8}	2.113^{1}	2.841^{5}	3.318^{7}	2.861^{6}
	Bias($\hat{\lambda}$)	-0.476^{10}	0.140^{7}	0.097^{6}	-0.010^{1}	-0.428^{9}	-0.331^{8}	0.031^{2}	0.034^{3}	0.050^{4}	0.081^{5}
	RMSE($\hat{\lambda}$)	0.615^{10}	0.280^{4}	0.315^{5}	0.254^{2}	0.558^{9}	0.472^{8}	0.198^{1}	0.265^{3}	0.350^{7}	0.324^{6}
	D_{abs}	0.032^{3}	0.032^{4}	0.033^{9}	0.033^{7}	0.033^{6}	0.032^{5}	0.031^{1}	0.033^{8}	0.034^{10}	0.032^{2}
	$D_{\max}$	0.051^{7}	0.050^{4}	0.051^{8}	0.050^{3}	0.052^{9}	0.050^{2}	0.047^{1}	0.051^{6}	0.053^{10}	0.050^{5}
	$\sum$Ranks	50^{9}	28^{5}	36^{6}	23^{2}	51^{10}	39^{7}	10^{1}	27^{3}	39^{7}	27^{3}
100	Bias($\hat{\alpha}$)	3.209^{9}	-0.624^{5}	-0.379^{4}	1.772^{6}	3.400^{10}	1.939^{8}	-0.340^{3}	1.875^{7}	0.145^{2}	-0.125^{1}
	RMSE($\hat{\alpha}$)	4.549^{9}	2.020^{2}	2.199^{3}	3.157^{6}	4.559^{10}	3.456^{8}	1.771^{1}	3.377^{7}	3.054^{5}	2.583^{4}
	Bias($\hat{\lambda}$)	-0.385^{9}	0.102^{5}	0.070^{4}	-0.201^{6}	-0.402^{10}	-0.269^{8}	0.035^{2}	-0.207^{7}	0.009^{1}	0.037^{3}
	RMSE($\hat{\lambda}$)	0.524^{10}	0.235^{2}	0.264^{3}	0.418^{7}	0.518^{9}	0.407^{6}	0.186^{1}	0.450^{8}	0.310^{5}	0.276^{4}
	D_{abs}	0.023^{3}	0.023^{2}	0.023^{5}	0.024^{7}	0.024^{6}	0.023^{4}	0.022^{1}	0.024^{8}	0.028^{10}	0.026^{9}
	$D_{\max}$	0.037^{4}	0.036^{3}	0.037^{5}	0.038^{7}	0.037^{6}	0.036^{2}	0.034^{1}	0.038^{8}	0.044^{10}	0.040^{9}
	$\sum$Ranks	44^{8}	19^{2}	24^{3}	39^{7}	51^{10}	36^{6}	9^{1}	45^{9}	33^{5}	30^{4}
200	Bias($\hat{\alpha}$)	2.449^{7}	-0.457^{5}	-0.271^{3}	3.962^{9}	3.063^{8}	1.533^{6}	-0.296^{4}	4.256^{10}	-0.080^{1}	-0.245^{2}
	RMSE($\hat{\alpha}$)	3.658^{7}	1.649^{2}	1.813^{3}	4.787^{9}	4.008^{8}	2.851^{6}	1.517^{1}	5.068^{10}	2.825^{5}	2.578^{4}
	Bias($\hat{\lambda}$)	-0.299^{7}	0.072^{5}	0.048^{4}	-0.455^{9}	-0.367^{8}	-0.211^{6}	0.033^{3}	-0.484^{10}	0.011^{1}	0.028^{2}
	RMSE($\hat{\lambda}$)	0.433^{7}	0.198^{2}	0.221^{3}	0.538^{9}	0.466^{8}	0.341^{6}	0.176^{1}	0.565^{10}	0.288^{5}	0.265^{4}
	D_{abs}	0.016^{3}	0.016^{4}	0.017^{6}	0.017^{7}	0.016^{5}	0.016^{2}	0.016^{1}	0.018^{8}	0.027^{10}	0.025^{9}
	$D_{\max}$	0.025^{3}	0.026^{4}	0.026^{5}	0.028^{7}	0.026^{6}	0.025^{2}	0.024^{1}	0.029^{8}	0.041^{10}	0.038^{9}
	$\sum$Ranks	34^{7}	22^{2}	24^{3}	50^{9}	43^{8}	28^{4}	11^{1}	56^{10}	32^{6}	30^{5}

Table 6: Partial and overall ranks of all the methods of estimation for $\lambda = 0.7$ and various α.

α	n	MLE	MME	LME	LSE	WLS	PCE	MPS	CVM	AD	RAD
0.5	20	6	10	7	4	3	2	1	8	5	9
	50	6	9	7	7	3	2	1	10	4	5
	100	3	6	8	9	5	2	1	10	7	4
	200	3	5	8	9	6	1	1	10	7	4
1	20	7	10	9	4	4	2	1	8	3	6
	50	8	10	7	6	5	3	1	9	2	4
	100	5	7	8	9	6	2	1	10	2	4
	200	5	6	8	9	7	2	1	10	3	4
3	20	5	10	9	4	3	2	1	7	8	6
	50	5	10	9	4	3	2	1	7	8	5
	100	5	9	10	4	3	2	1	7	8	6
	200	3	8	10	5	4	2	1	7	9	6
5	20	4	10	9	5	3	2	1	7	8	6
	50	4	10	7	6	3	2	1	9	7	5
	100	4	9	8	6	3	2	1	9	7	5
	200	3	6	10	7	4	2	1	9	8	5
10	20	8	4	5	3	10	7	1	2	9	6
	50	9	5	6	2	10	7	1	3	7	3
	100	8	2	3	7	10	6	1	9	5	4
	200	7	2	3	9	8	4	1	10	6	5
$\sum$Ranks		108	148	151	119	103	56	20	161	123	102
Overall Rank		5	8	9	6	4	2	1	10	7	3

5. Real Data Analysis

In the following section, for illustrative purposes we present two applications for the proposed TR distribution to real data for illustrative purposes. These applications will show the flexibility of the TR distribution in modeling positive data. For the purpose of comparison, we also fit one Rayleigh parameter, Weibull, and Gamma distributions for the same data. We use the `fitdistrplus` R package to fit the distributions. The fits of these four distributions are presented in the subsequent sections. We present a comparative density plot, plots of the distribution functions, and Q-Q and P-P plots for all four distributions. For example, Figure 7 shows the comparative density, distribution, q-q, and p-p plots for the guinea pig data which we discussed in Section 5.1.

Classical goodness of fit of a given distribution can be assessed via the density plot and the CDF plot. The Q-Q an P-P plots may provide additional information in some cases. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit at the center of the Muller and Dutang distribution (Delignette-Muller & Dutang 2014).

To evaluate the performance of the candidate distributions that best fit the data, we calculate the log-likelihood of the fitted models based on numerically obtained maximum likelihood estimations via the `fitdist()` function, available in the `fitdistrplus` R package. We then obtain the values of AIC and BIC. Additionally, we test for goodness of fit for the candidate models using the Kolmogorov-Smirnov (K-S) test, Anderson-Darling's (AD) test, and the Cramér-von-Mises (CVM) test. Note that, the K-S test requires unique items for us to obtain exact p-values. Therefore, we made sure to break the ties by adding a tiny random noise, which is uniform between 0.001 and 0.01, to each duplicate observation. Adding this small noise did not affect the statistical properties of the data. The two data sets considered in this study include tied observations. As such, we removed ties for both data sets using the process outlined above. We present a descriptive summary of original data as well as for the data after ties were broken. This shows a minor difference between the original and modified data sets. It should also be mentioned that removal of duplicate observations is necessary for the MPS method of estimation.

5.1. Example 1: Guinea Pig Data

Our first data set contains survival times (in days) of guinea pigs injected with different doses of tubercle bacilli and is given in Table 7. The data has been analyzed by Kundu & Howlader (2010) and by Singh, Singh & Sharma (2013). We present the data after breaking the ties.

Table 7: Survival times (in days) of guinea pigs injected with different doses of tubercle bacilli.

0.012	0.015	0.022	0.024	0.032	0.033	0.034	0.038	0.043	0.044	0.048	0.052
0.053	0.054	0.055	0.056	0.057	0.058	0.059	0.060	0.061	0.062	0.063	0.065
0.067	0.068	0.070	0.072	0.073	0.075	0.076	0.081	0.083	0.084	0.085	0.087
0.091	0.095	0.096	0.098	0.099	0.109	0.110	0.121	0.127	0.129	0.131	0.143
0.146	0.175	0.211	0.233	0.258	0.263	0.297	0.341	0.376	0.030	0.036	0.043
0.061	0.060	0.063	0.063	0.063	0.072	0.081	0.153	0.181	0.260	0.347	0.074

5.1.1. Finding a suitable distribution for guinea pig data

We compare the fit of Rayleigh, Weibull, Gamma, and Transmuted Rayleigh distribution for the guinea pig data set. The data contains tied observations, and, therefore, we modified the data as discussed above to remove the ties. The summary of original data set, as well as the modified one, is presented in Table 8. We do not observe any noticeable difference in the calculated statistics for the modified data set compared to the original one.

Table 8: Summary of Guinea Pig data: Original vs Modified.

Data	Min	Q1	Median	Mean	Q3	Max
Original	0.01200	0.05475	0.07000	0.09982	0.11280	0.37600
Ties broken	0.01200	0.05575	0.07177	0.10080	0.11280	0.37600

Next, we present the fit statistics for four distributions for the guinea pig data. The log-likelihood statistic, the AIC, the BIC, and the tests statistics along with the p-values for K-S, AD, and CVM goodness of fit tests are presented in Table 9. Results show that the transmuted Rayleigh distribution best fits the guinea pig data having the largest log-likelihood, smallest AIC, and smallest BIC among all four candidate distributions. Graphical check for the model fitting is shown in Figure 7. From the histogram and theoretical density plots, transmuted Rayleigh distribution shows reasonably good fit especially in the right tail area. Fitting right tail area is particularly important for many life testing reliability problems. From the Q-Q plot, we find that the TR model is reasonably a good fit at the lower tail of the distribution as well. However, Weibull and gamma seems to be fitting the data in the middle of the distribution. We made our decision of finding the best fitting model based on the largest log-likelihood value and smallest AIC and BIC values. Based on these criteria, TR fits the guinea pig data best.

Table 9: (Guinea pig data) Fitting distributions to find the best fit model.

Distribution	LogLik	AIC	BIC	K-S(p)	AD(p)	CVM(p)
Rayleigh	91.48	−178.96	−174.41	0.25 (.000)	6.15 (.000)	1.28 (.001)
Weibull	99.83	−195.66	−191.10	0.15 (.088)	2.36 (.059)	0.41 (.067)
Gamma	102.83	−201.66	−197.11	0.99 (.000)	781.60 (.000)	23.99 (.000)
TRayleigh	111.37	−218.73	−214.18	0.48 (.000)	46.29 (.000)	6.52 (.000)

5.1.2. Bootstrap Estimates (Guinea Pig Data): Bias and Standard Errors

Since the transmuted Rayleigh distribution best fits the data, we carry out ordinary nonparametric bootstrap resampling to obtain bias, standard error, and 95% bootstrap percentile confidence intervals for the parameters α and λ of the transmuted Rayleigh distribution. In ordinary nonparametric resampling, repeated samples are drawn from the original data with replacements. In addition, to the original data set, we draw 999 bootstrap samples to obtain the estimators bias and standard error of the estimators. Results are shown in Table 10. We note that moments based methods have estimates for α that are around 53 while the estimates for λ are around 0.58. For $\hat{\alpha}$, WLS produces the least biased estimate while MPS a has higher bias. For $\hat{\lambda}$, all the methods produce a small bias and standard error.

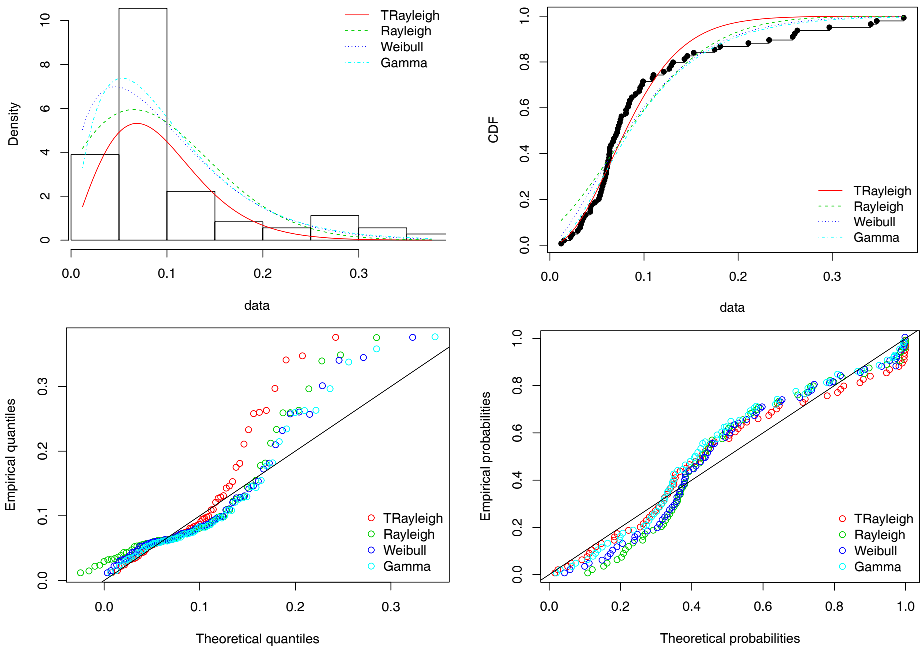


FIGURE 7: (Guinea pig data) Comparison of fit for the four distributions of the guinea pig survival time data.

TABLE 10: (Guinea pig data) Estimated bias and standard error of the estimates $\hat{\alpha}$ and $\hat{\lambda}$ and their 95% percentiles of the lower and upper confidence limits based on 999 bootstrap samples.

Method	$\hat{\alpha}$					$\hat{\lambda}$				
	Est.	Bias	SE	LCL	UCL	Est.	Bias	SE	LCL	UCL
MLE	46.642	1.703	8.889	33.72	69.84	0.647	−0.002	0.053	0.544	0.751
LSE	77.284	−2.983	10.602	55.22	96.51	0.587	0.068	0.144	0.508	1.000
WLS	75.952	−0.030	12.727	52.65	101.29	0.570	0.020	0.059	0.532	0.757
PCE	43.992	1.398	9.374	30.62	66.27	0.582	0.000	0.020	0.544	0.622
MPS	44.364	−37.897	6.020	3.76	25.76	0.654	0.285	0.151	0.387	1.000
CVM	76.546	−3.044	10.610	54.99	96.46	0.600	0.074	0.153	0.509	1.000
RAD	55.178	0.867	12.452	33.80	81.31	0.657	−0.001	0.047	0.565	0.751
LME	51.856	1.964	10.834	36.23	76.86	0.552	−0.006	0.034	0.535	0.552
MME	52.467	1.185	10.417	35.60	77.82	0.578	0.008	0.020	0.578	0.595
MMM	52.566	0.959	10.429	36.74	77.25	0.593	−0.006	0.006	0.577	0.594

5.2. Example 2: Fibre Strength Data

The second data set consists of 100 measurements on breaking stress of carbon fibres (in Gba). The data in Table 11 is taken from Cordeiro, Ortega & Popović (2014). The data was also used in Nadarajah, Cordeiro & Ortega (2013). We present the data after the ties broker.

Table 11: Breaking strengths of carbon fibres (in Gba).

0.39	0.81	0.85	0.98	1.08	1.12	1.17	1.18	1.22	1.25
1.36	1.41	1.47	1.57	1.59	1.61	1.69	1.71	1.73	1.80
1.84	1.87	1.89	1.92	2.00	2.03	2.05	2.12	2.17	2.35
2.38	2.41	2.43	2.48	2.50	2.53	2.55	2.56	2.59	2.67
2.73	2.74	2.76	2.77	2.79	2.81	2.82	2.83	2.85	2.87
2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.15	3.19	3.22
3.27	3.28	3.31	3.33	3.39	3.51	3.56	3.60	3.65	3.68
3.70	3.75	4.20	4.38	4.42	4.70	4.90	4.91	5.08	5.56
1.58	1.60	1.61	1.70	1.85	2.04	2.17	2.17	2.48	2.55
2.82	2.98	3.11	3.16	3.19	3.23	3.31	3.39	3.68	3.69

5.2.1. Finding a suitable distribution for fibre data

Similar to the guinea pig data, we compare the fits of Rayleigh, Weibull, Gamma, and transmuted Rayleigh distribution for the fibres data set. There were 20 tied observations in the data set. We prepared the data by breaking the ties following the procedure discussed earlier. The summary of the original and the modified data (after breaking ties) is shown in Table 12. We notice that the modified data has nearly the same descriptive characteristics as those in the original data set.

Table 12: Summary of Fibres data: Original vs Modified.

Data	Min	Q1	Median	Mean	Q3	Max
Original	0.390	1.840	2.700	2.621	3.220	5.560
Ties broken	0.390	1.847	2.700	2.622	3.221	5.560

For the fibres data, we present fit statistics of Rayleigh, Weibull, Gamma, and transmuted Rayleigh distribution in Table 13. these show that the transmuted Rayleigh distribution best fits the data ans has the largest log−likelihood (−105.39), the smallest AIC (214.77), and the smallest BIC (219.99) among all four candidate distributions. A histogram with theoretical density curves superimposed over the data is shown in panel (a) of Figure 8. The cdf, Q-Q, and P-P plots also show evidence that the TR is a good candidate model.

TABLE 13: (Fibres strength data) Fitting distributions to find the best fit model.

Distribution	LogLik	AIC	BIC	K-S(p)	AD(p)	CVM(p)
Rayleigh	−144.50	293.00	298.21	0.11 (.157)	1.66 (.143)	0.314 (.123)
Weibull	−141.53	287.05	292.26	0.06 (.868)	.0.41 (.836)	0.06 (.797)
Gamma	−143.24	290.49	295.70	0.96 (.000)	490.87 (.000)	32.74 (.000)
TRayleigh	−105.39	214.77	219.99	0.17 (.014)	4.82 (.003)	0.854 (.005)

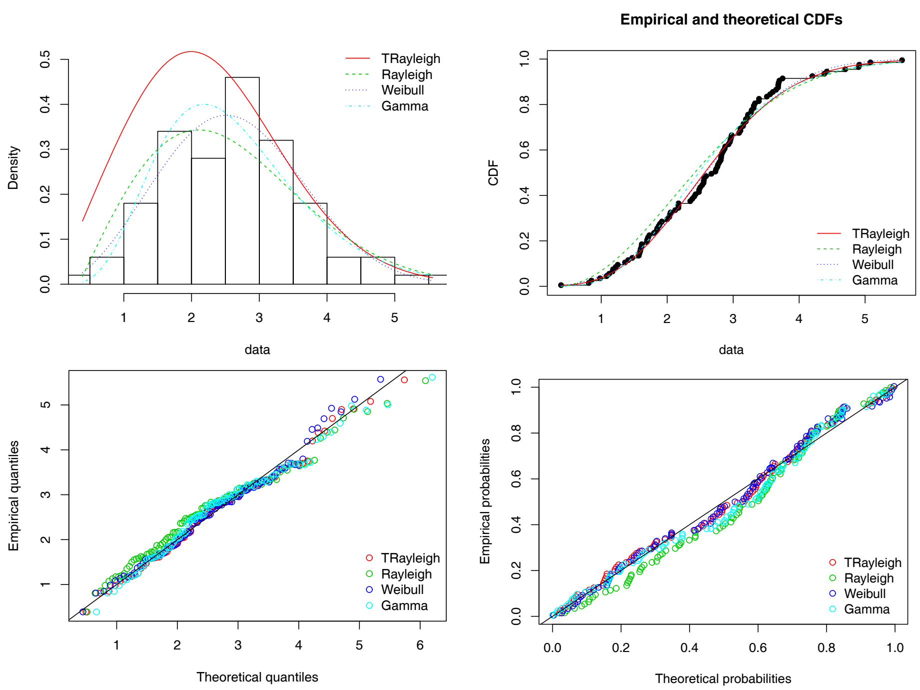


FIGURE 8: Comparison of fit of several distributions for the fibres strength data.

5.2.2. Bootstrap Estimates (fibres data): Bias and Standard Errors

Since transmuted Rayleigh distribution best fits the fibres data, were obtain bootstrap biases and standard errors of the estimates of the parameters of the TR distribution using different methods of estimation. Results are shown in Table 14. We find that the MLE and all the moment-based methods are nearly unbiased for the parameter α. The worst performing method for this data set is the MPS, which produces a relatively large bias (0.325) for the estimate. The bootstrap percentile confidence intervals for λ shows that the lower limit is actually at the lower bound for the parameter. Also, note that the moment-based methods produce confidence intervals for λ, the lower limit of which falls outside the bound of the parameter, which is -1.

TABLE 14: (Fibres Data) Estimated bias and standard error of the estimates $\hat{\alpha}$ and $\hat{\lambda}$ and their 95% percentiles of the lower and upper confidence limits based on 999 bootstrap samples.

Method	$\hat{\alpha}$					$\hat{\lambda}$				
	Est.	Bias	SE	LCL	UCL	Est.	Bias	SE	LCL	UCL
MLE	0.184	−0.000	0.015	0.153	0.216	−0.919	0.009	0.110	−1.000	−0.626
LSE	0.178	−0.001	0.014	0.151	0.205	−0.895	0.032	0.153	−1.000	−0.476
WLS	0.189	−0.002	0.015	0.154	0.216	−1.000	0.075	0.125	−1.000	−0.588
PCE	0.180	0.001	0.156	0.151	0.213	−0.884	0.004	0.130	−1.000	−0.546
MPS	0.178	0.325	3.535	0.130	0.238	−0.851	0.101	0.204	−0.852	−0.246
CVM	0.181	−0.002	0.014	0.152	0.208	−0.923	0.035	0.129	−1.000	−0.572
RAD	0.188	−0.005	0.017	0.151	0.221	−1.000	0.089	0.135	−1.000	−0.525
LME	0.186	0.001	0.017	0.157	0.221	−0.945	−0.011	0.159	−1.261	−0.641
MME	0.188	0.001	0.016	0.157	0.222	−0.961	−0.009	0.161	−1.263	−0.653
MMM	0.187	0.001	0.018	0.156	0.227	−0.950	−0.002	0.161	−1.234	−0.623

6. Conclusion

In this article, we have provide explicit expressions for the quantiles, moments, moment generating function, conditional moments, hazard rates, mean residual lifetime, mean past lifetime, mean deviation about mean and median, the stochastic ordering, various entropies, stress-strength parameter, and order statistics. The model parameters are estimated by ten methods of estimation, namely, maximum likelihood, moments, L-moments, percentile, least squares, weighted least squares, maximum product of spacing, Cramer-von Mises, Anderson-Darling, and right tailed Anderson-Darling. We have performed an extensive simulation study to compare these methods. We have compared estimators with respect to bias, root mean-squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. We have also compared estimators wing two real data applications.

The simulation results show that maximum product spacing (MPS) estimators is the best performing estimator in terms of biases and RMSE. The next best performing estimator is the percentile estimator (PCE), followed by the right tailed Anderson-Darling estimators. The real data applications show that the maximum product of spacing estimator gives the shortest confidence intervals for the alpha for Guinea Pig data set and the LSE for Fibre strength data set. Hence, we can argue that the percentile estimators, least squares estimators, right tailed Anderson-Darling estimators and the maximum product spacing estimators are among the best performing estimators for TR distribution.

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